TD3 – λ-calculus

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1 Reduction graphs

The *reduction graph* of a λ-term M is the graph, whose vertices are λ-terms, defined as the smallest graph such that M is a vertex and there is an arrow between two vertices M and M' whenever $M \to_\beta M'$.

1. Write the respective reduction graphs of
$$(\lambda x.xx)(\lambda y.y)z \qquad \text{and} \qquad (\lambda xy.x)((\lambda x.xx)(\lambda xy.xy))$$
2. Can a reduction graph have loops?

2 Booleans

We encode the booleans $\top$ (true) and $\bot$ (false) into λ-terms respectively as
$$[\![\top]\!] = \lambda x.\lambda y.x \qquad \text{and} \qquad [\![\bot]\!] = \lambda x.\lambda y.y$$

1. Define a λ-term if such that
$$if[\![\top]\!]MN \xrightarrow{*}_\beta M \qquad \text{and} \qquad if[\![\bot]\!]MN \xrightarrow{*}_\beta N$$
2. Define λ-terms *and*, *or* and *not* such that for every booleans b and b',
$$and[\![b]\!][\![b']\!] \to_\beta [\![b \wedge b']\!] \qquad or[\![b]\!][\![b']\!] \to_\beta [\![b \vee b']\!] \qquad not[\![b]\!] \to_\beta [\![\neg b]\!]$$

3 Weak normalization of the λ-calculus

An *abstract rewriting system* (ARS) is a graph whose vertices are called *terms* and whose edges are called *rewriting rules*. We often write $x \to y$ when there exists an edge from x to y and $x \xrightarrow{*} y$ when there exists a directed path from x to y (in the latest case, we say that x rewrites to y). For instance, we can consider the ARS of λ-terms and β-reduction. An ARS is

- *locally confluent* when $y_1 \leftarrow x \to y_2$ implies that there exists z such that $y_1 \xrightarrow{*} z \xleftarrow{*} y_2$,
- *confluent* when $y_1 \xleftarrow{*} x \xrightarrow{*} y_2$ implies that there exists z such that $y_1 \xrightarrow{*} z \xleftarrow{*} y_2$,
- *strongly confluent* when $y_1 \leftarrow x \to y_2$ implies that there exists z such that $y_1 \to z \leftarrow y_2$.

1. Which properties imply another? Give counter-examples for implications which fail.
2. A *normal form* is a term x such that there is no y for which $x \to y$. Show that in a confluent rewriting system a term reduces to at most one normal form.
3. [Well-founded induction] Given a property $P(n)$ on integers, the induction principle is
$$(P(0) \quad \wedge \quad \forall n \in \mathbb{N}.\ P(n) \Rightarrow P(n+1)) \quad \Rightarrow \quad \forall n \in \mathbb{N}.\ P(n)$$
A partial ordered set $(E, \leq)$ is *well-founded* if there is no infinite strictly decreasing sequence of elements of E. Formulate an induction principle for a property on the elements of E, and show that this principle is valid.
4. [Newman's lemma] An ARS is *terminating* if it does not contain any infinite path. Show that an ARS which is terminating and locally confluent is confluent.
5. Show that in a terminating and (locally) confluent rewriting system, normal forms are in bijection with connected components of the graph.
6. A λ-term is *strongly terminating* when it can only be reduced a finite number of times, *divergent* when it does not reduce to a normal form and *weakly terminating* when it can reduce to a normal form. Give example of λ-terms with such properties.

7. The *parallel reduction* $M \Rightarrow N$ on λ-terms is defined by:
   (a) $x \Rightarrow x$
   (b) $M \Rightarrow M'$ and $N \Rightarrow N'$ implies $MN \Rightarrow M'N'$
   (c) $M \Rightarrow M'$ implies $\lambda x.M \Rightarrow \lambda x.M'$
   (d) $M \Rightarrow M'$ and $N \Rightarrow N'$ implies $(\lambda x.M)N \Rightarrow M'[N'/x]$

   Show that $\Rightarrow$ is strongly confluent.

8. Show that $\to_\beta \subseteq \Rightarrow \subseteq \to_\beta^*$. Provide counter-examples showing that these inclusions are strict.

9. Conclude that $\to_\beta$ is confluent.

4 Cartesian closed categories

Recall that a category $\mathcal{C}$ is *cartesian closed* when it has finite products and for every object B of $\mathcal{C}$ the functor $- \times B : \mathcal{C} \to \mathcal{C}$ admits a right adjoint, written $(-)^B$, i.e. there exists a bijection $\mathcal{C}(A \times B, C) \cong \mathcal{C}(A, C^B)$ natural in A and C.

1. Show that the category **Set** is cartesian closed.

2. Describe the unit and the counit of the adjunction defining the closure. Check that the laws between the unit and counit in an adjunction are satisfied.

3. Show that the category **Cat** is cartesian closed.

5 Simply typed λ-calculus

Recall the syntax of λ-terms with products:

$$M \quad ::= \quad x \quad | \quad \lambda x.M \quad | \quad MM \quad | \quad (M, M) \quad | \quad \pi_1 \quad | \quad \pi_2 \quad | \quad ()$$

and the syntax of types:

$$A \quad ::= \quad a \quad | \quad A \times A \quad | \quad 1$$

The λ-terms will be considered modulo α-conversion (renaming of bound variables). The β-conversion rules are defined by

$$(\lambda x.M)N \equiv_\beta M[N/x] \qquad\qquad \pi_i(M_1, M_2) \equiv_\beta M_i$$

and those of η-conversion by

$$\lambda x.Mx \equiv_\eta M \qquad\qquad (\pi_1 M, \pi_2 M) \equiv_\eta M \qquad\qquad M \equiv_\eta () \text{ if } M \text{ has type } 1$$

1. Recall the typing rules.

2. Define the substitution operation $M[N/x]$ by induction on the structure of the term M. Show that typing is preserved under β-reduction.

3. We want to make explicit context manipulations. Which rules do we have to add if we want to obtain an equivalent deduction system where the axiom and unit rule have been replaced by

$$\frac{}{x : A \vdash x : A} \qquad \text{and} \qquad \frac{}{\vdash () : 1}$$

   and where contexts are lists (and not sets)?

4. Let $\mathcal{C}$ be a cartesian closed category. We suppose fixed a function $[\![-]\!]$ which to every type A associates an objects $[\![A]\!]$ such that $[\![A \times B]\!] = [\![A]\!] \times [\![B]\!]$ and $[\![A \Rightarrow B]\!] = [\![B]\!]^{[\![A]\!]}$. Define the interpretation of a sequent

$$x_1 : A_1, \ldots, x_n : A_n \vdash M : A$$

   as a morphism

$$[\![M]\!] : [\![A_1]\!] \times \ldots \times [\![A_n]\!] \to [\![A]\!]$$

   such that this interpretation is invariant by β- and η-equivalences (i.e. if $M \equiv_{\beta\eta} N$ then $[\![M]\!] = [\![N]\!]$).

5. Conversely, explain how to build a category Λ whose objects are types and morphisms are simply typed λ-terms. Check that this category is cartesian closed.