# TD5 – Algebras

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## 1 Algebras for an endofunctor

An algebra for an endofunctor  $F : \mathcal{C} \to \mathcal{C}$  is a pair (A, f) where A is an object of  $\mathcal{C}$  and  $f : FA \to A$  a morphism of  $\mathcal{C}$ . A morphism  $h : (A, f) \to (B, g)$  between two such algebras consists of a morphism  $h : A \to B$  such that



In the following, we mostly consider algebras in **Set**.

- 1. Define inductively the functions length : 'a list -> int (giving the length of a list), map : ('a -> 'b) -> 'a list -> 'b list (applying a function to all elements of a list), double : 'a list -> 'a list (which duplicates every successive element, for instance double [1;2;3] = [1;1;2;2;3;3]).
- 2. We suppose given a type 'a ilist of infinite lists with elements of type 'a. Define coinductively odd : 'a ilist -> 'a ilist (keeping elements of a list at odd positions), merge : 'a ilist -> 'a ilist -> 'a ilist (taking alternatively elements from one of two lists).
- 3. Show that  $[0, S] : 1 + \mathbb{N} \to \mathbb{N}$  is an initial algebra for the endofunctor T(X) = 1 + X of **Set**.
- 4. Use this fact to define the function  $f : \mathbb{N} \to \mathbb{Q}$  such that  $f(n) = 2^{-n}$ .
- 5. Show that two initial algebras of an endofunctor are isomorphic (via morphisms of algebras).
- 6. Show that an initial algebra  $f: FA \to A$  of an endofunctor F is an isomorphism.
- 7. Show that the set  $A^* = \biguplus_{n \in \mathbb{N}} A^n$ , which can be seen as the set of lists of elements of A, is an initial algebra for  $T(X) = 1 + A \times X$ .
- 8. Use this fact to define the length function  $\ell : A^* \to \mathbb{N}$  and the double function  $d : A^* \to A^*$ . Show that  $\ell \circ d(l) = 2\ell(l)$  for every  $l \in A^*$ .
- 9. Explain briefly how we could interpret simple inductive types of OCaml by using initial algebras.
- 10. What is the initial algebra for  $T(X) = 1 + X \times X$ ? For  $T(X) = X^*$ ?

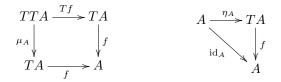
### 2 Coalgebras for an endofunctor

A coalgebra for  $F : \mathcal{C} \to \mathcal{C}$  is a pair (A, f) with  $f : A \to FA$ . Morphisms are defined similarly as previously.

- 1. Show that the set  $A^{\mathbb{N}}$  of *streams* is a final coalgebra for the endofunctor  $T(X) = A \times X$ .
- 2. Use this to define, given  $a \in A$ , the constant stream equal to a. Define the function  $\mathbb{N} \to \mathbb{N}^{\mathbb{N}}$ which to n associates the stream (n, n + 1, n + 2, ...). Define the function  $A^{\mathbb{N}} \times A^{\mathbb{N}} \to A^{\mathbb{N}}$ which merges two streams. Define the functions  $A^{\mathbb{N}} \to A^{\mathbb{N}}$  keeping even and odd elements.
- 3. Show that final coalgebras are unique up to isomorphism and are isomorphisms.
- 4. Show that merge(even(l), odd(l)) = l for every  $l \in A^{\mathbb{N}}$ .
- 5. A bisimulation on  $A^{\mathbb{N}}$  is a relation  $R \subseteq A^{\mathbb{N}} \times A^{\mathbb{N}}$  such that R(x :: l, x' :: l') implies x = x'and R(l, l'). The coinductive proof principle says that if R(l, l') for some bisimulation R then l = l'. Assuming this principle, show again the result of previous question.
- 6. Show the coinductive proof principle.
- 7. What is the final coalgebra of  $T(X) = 1 + A \times X$ ? of T(X) = 1 + X?

# 3 Algebras for a monad

An algebra for a monad  $(T, \mu, \eta)$  is a pair (A, f) with  $f: TA \to A$  such that



A morphism of T-algebras is defined as a morphism between algebras for a functor. Given a category  $\mathcal{C}$  and T a monad on  $\mathcal{C}$ , we write  $\mathcal{C}^T$  for the category of T-algebras.

- 1. Consider the monad T on **Set** induced by the adjunction whose right adjoint is the forgetful functor  $U : \mathbf{Mon} \to \mathbf{Set}$  where **Mon** is the category of monoids and morphisms of monoids. Describe this monad. What is an algebra for this monad?
- 2. Show that the forgetful functor  $\mathcal{C}^T \to \mathcal{C}$  has a left adjoint.
- 3. Fix a monad T on C and consider the category whose objects are triples  $(\mathcal{D}, F, G)$  with  $F: \mathcal{C} \to \mathcal{D}$  left adjoint to  $G: \mathcal{D} \to \mathcal{C}$ , and whose morphisms  $H: (\mathcal{D}, F, G) \to (\mathcal{D}', F', G')$  are functors  $H: \mathcal{D} \to \mathcal{D}'$  such that  $H \circ F = F'$  and  $G' \circ H = G$ . Show that the above adjunction is a terminal object in this category.
- 4. Show that the Kleisli category  $C_T$  is an initial object in this category.