TD5 – Algebras

Samuel Mimram

7 November 2013

1 Algebras for an endofunctor

An *algebra* for an endofunctor $F : \mathcal{C} \to \mathcal{C}$ is a pair (A, f) where A is an object of $\mathcal{C}$ and $f : FA \to A$ a morphism of $\mathcal{C}$. A morphism $h : (A, f) \to (B, g)$ between two such algebras consists of a morphism $h : A \to B$ such that

$$\begin{tikzcd} FA \arrow[r, "Fh"] \arrow[d, "f"'] & FB \arrow[d, "g"] \\ A \arrow[r, "h"'] & B \end{tikzcd}$$

In the following, we mostly consider algebras in **Set**.

1. Define inductively the functions `length : 'a list -> int` (giving the length of a list), `map : ('a -> 'b) -> 'a list -> 'b list` (applying a function to all elements of a list), `double : 'a list -> 'a list` (which duplicates every successive element, for instance `double [1;2;3] = [1;1;2;2;3;3]`).

2. We suppose given a type `'a ilist` of infinite lists with elements of type `'a`. Define coinductively `odd : 'a ilist -> 'a ilist` (keeping elements of a list at odd positions), `merge : 'a ilist -> 'a ilist -> 'a ilist` (taking alternatively elements from one of two lists).

3. Show that $[0, S] : 1 + \mathbb{N} \to \mathbb{N}$ is an initial algebra for the endofunctor $T(X) = 1 + X$ of **Set**.

4. Use this fact to define the function $f : \mathbb{N} \to \mathbb{Q}$ such that $f(n) = 2^{-n}$.

5. Show that two initial algebras of an endofunctor are isomorphic (via morphisms of algebras).

6. Show that an initial algebra $f : FA \to A$ of an endofunctor F is an isomorphism.

7. Show that the set $A^* = \biguplus_{n \in \mathbb{N}} A^n$, which can be seen as the set of lists of elements of A, is an initial algebra for $T(X) = 1 + A \times X$.

8. Use this fact to define the length function $\ell : A^* \to \mathbb{N}$ and the double function $d : A^* \to A^*$. Show that $\ell \circ d(l) = 2\ell(l)$ for every $l \in A^*$.

9. Explain briefly how we could interpret simple inductive types of OCaml by using initial algebras.

10. What is the initial algebra for $T(X) = 1 + X \times X$? For $T(X) = X^*$?

2 Coalgebras for an endofunctor

A *coalgebra* for $F : \mathcal{C} \to \mathcal{C}$ is a pair (A, f) with $f : A \to FA$. Morphisms are defined similarly as previously.

1. Show that the set $A^{\mathbb{N}}$ of *streams* is a final coalgebra for the endofunctor $T(X) = A \times X$.
2. Use this to define, given $a \in A$, the constant stream equal to a. Define the function $\mathbb{N} \to \mathbb{N}^{\mathbb{N}}$ which to n associates the stream $(n, n+1, n+2, \ldots)$. Define the function $A^{\mathbb{N}} \times A^{\mathbb{N}} \to A^{\mathbb{N}}$ which merges two streams. Define the functions $A^{\mathbb{N}} \to A^{\mathbb{N}}$ keeping even and odd elements.
3. Show that final coalgebras are unique up to isomorphism and are isomorphisms.
4. Show that $merge(even(l), odd(l)) = l$ for every $l \in A^{\mathbb{N}}$.
5. A *bisimulation* on $A^{\mathbb{N}}$ is a relation $R \subseteq A^{\mathbb{N}} \times A^{\mathbb{N}}$ such that $R(x :: l, x' :: l')$ implies $x = x'$ and $R(l, l')$. The *coinductive proof principle* says that if $R(l, l')$ for some bisimulation R then $l = l'$. Assuming this principle, show again the result of previous question.
6. Show the coinductive proof principle.
7. What is the final coalgebra of $T(X) = 1 + A \times X$? of $T(X) = 1 + X$?

3 Algebras for a monad

An *algebra* for a monad (T, μ, η) is a pair (A, f) with $f : TA \to A$ such that

$$\begin{tikzcd} TTA \arrow[r, "Tf"] \arrow[d, "\mu_A"'] & TA \arrow[d, "f"] \\ TA \arrow[r, "f"'] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "\mathrm{id}_A"'] & TA \arrow[d, "f"] \\ & A \end{tikzcd}$$

A morphism of T-algebras is defined as a morphism between algebras for a functor. Given a category $\mathcal{C}$ and T a monad on $\mathcal{C}$, we write $\mathcal{C}^T$ for the category of T-algebras.

1. Consider the monad T on **Set** induced by the adjunction whose right adjoint is the forgetful functor $U : \mathbf{Mon} \to \mathbf{Set}$ where **Mon** is the category of monoids and morphisms of monoids. Describe this monad. What is an algebra for this monad?
2. Show that the forgetful functor $\mathcal{C}^T \to \mathcal{C}$ has a left adjoint.
3. Fix a monad T on $\mathcal{C}$ and consider the category whose objects are triples $(\mathcal{D}, F, G)$ with $F : \mathcal{C} \to \mathcal{D}$ left adjoint to $G : \mathcal{D} \to \mathcal{C}$, and whose morphisms $H : (\mathcal{D}, F, G) \to (\mathcal{D}', F', G')$ are functors $H : \mathcal{D} \to \mathcal{D}'$ such that $H \circ F = F'$ and $G' \circ H = G$. Show that the above adjunction is a terminal object in this category.
4. Show that the Kleisli category $\mathcal{C}_T$ is an initial object in this category.