TD4 – Cartesian and monoidal categories

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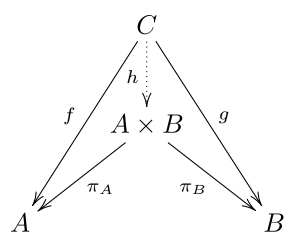
1 Cartesian categories

Suppose given a category $\mathcal{C}$. A *cartesian product* of two objects A and B is given by an object $A \times B$ together with two morphisms

$$\pi_1 : A \times B \to A \qquad \text{and} \qquad \pi_2 : A \times B \to B$$

such that for every object C and morphisms $f : C \to A$ and $g : C \to B$, there exists a unique morphism $h : C \to A \times B$ making the diagram



commute. We also recall that a *terminal object* in a category is an object 1 such that for every object A there exists a unique morphism $f_A : A \to 1$. A category is *cartesian* when it has finite products, i.e. has a terminal object and every pair of objects admits a product.

1. Suppose that $(E, \leq)$ is a poset. We associate to it category whose objects are elements of E and such that there exists a unique morphism between object a and b iff $a \leq b$. What is an initial object and a product in this category?
2. Show that the category **Set** of sets and functions is cartesian.
3. Show that two terminal objects in a category are necessarily isomorphic.
4. Similarly, show that the cartesian product of two objects is defined up to isomorphism.
5. Show that for every object A of a cartesian category, the objects $1 \times A$, A and $A \times 1$ are isomorphic.
6. Show that for every objects A and B, $A \times B$ and $B \times A$ are isomorphic.
7. Show that for every objects A, B and C, $(A \times B) \times C$ and $A \times (B \times C)$ are isomorphic.
8. The notion of *coproduct* is dual to the notion of product. Show that **Set** has all coproducts and an initial object.
9. Show that the category **Rel** of sets and relations is cartesian.
10. Show that the category **Cat** is cartesian.
11. Given a cartesian category $\mathcal{C}$, show that the cartesian product induces a functor $A, B \mapsto A \times B : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$.

12. Given a category $\mathcal{C}$, show that $\mathrm{Hom}_{\mathcal{C}}(-,-)$ induces a functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$.
13. Given an adjunction $F \dashv G : \mathcal{C} \to \mathcal{D}$, we have a natural bijection $\mathcal{D}(F-,-) \cong \mathcal{C}(-,G-)$. Elaborating on previous question, make the naturality condition explicit.

2 Monoidal categories

A *monoidal category* $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ is a category $\mathcal{C}$ equipped with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, an object $I \in \mathcal{C}$, and three natural bijections of components

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C) \qquad \lambda_A : I \otimes A \to A \qquad \rho_A : A \otimes I \to A$$

such that the diagrams

$$\begin{array}{ccccc} ((A\otimes B)\otimes C)\otimes D & \xrightarrow{\alpha_{A,B,C}\otimes D} & (A\otimes(B\otimes C))\otimes D & \xrightarrow{\alpha_{A,B\otimes C,D}} & A\otimes((B\otimes C)\otimes D) \\ {\scriptstyle \alpha_{A\otimes B,C,D}}\downarrow & & & & \downarrow{\scriptstyle A\otimes\alpha_{B,C,D}} \\ (A\otimes B)\otimes(C\otimes D) & & \xrightarrow[\alpha_{A,B,C\otimes D}]{} & & A\otimes(B\otimes(C\otimes D)) \end{array}$$

and

$$\begin{array}{ccc} (A\otimes I)\otimes B & \xrightarrow{\alpha_{A,I,B}} & A\otimes(I\otimes B) \\ & {\scriptstyle \rho_A\otimes B}\searrow \quad \swarrow{\scriptstyle A\otimes\lambda_B} & \\ & A\otimes B & \end{array}$$

commute.

A *braided monoidal category* $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \gamma)$ is a monoidal category equipped with a natural bijection γ of components

$$\gamma_{A,B} : A \otimes B \to B \otimes A$$

such that suitable diagrams commute. A *symmetric monoidal category* is a braided monoidal category such that $\gamma_{B,A} \circ \gamma_{A,B} = \mathrm{id}_{A \otimes B}$ for every objects A and B.

1. Show that every cartesian category $\mathcal{C}$ can be equipped with a structure of symmetric monoidal category.
2. Define a notion of *monoid* in a monoidal category (so that a monoid in the cartesian category **Set** corresponds to the usual notion). Define similarly morphisms of monoids.
3. What is a monoid in $(\mathbf{Cat}, \times, 1)$?
4. Show that a monoidal category is cartesian (with tensor as product) if and only if every object is equipped with a natural structure of comonoid.
5. We write **Vect** for the category of $\Bbbk$-vector spaces (where $\Bbbk$ is a fixed field) and linear functions. Show that this category is cartesian.
6. Given a basis for A and B, describe a basis for $A \times B$.
7. Show that the forgetful functor $U : \mathbf{Vect} \to \mathbf{Set}$ admits a left adjoint $F : \mathbf{Set} \to \mathbf{Vect}$.
8. Given vector spaces A, B and C, we write $\mathrm{Bilin}(A, B; C)$ for the set of bilinear applications from $A \times B$ to C. Show that there exists a vector space, written $A \otimes B$ such that we have a (natural) bijection

$$\mathrm{Bilin}(A, B; C) \qquad \cong \qquad \mathbf{Vect}(A \otimes B; C)$$

   It can be helpful to write $A \otimes B$ as a quotient of the vector space $F(U(A) \times U(B))$.
9. Given a basis for A and B, describe a basis for $A \otimes B$.