TD1 – λ-calculus

Samuel Mimram

26 September 2013

1 Reduction graphs

The *reduction graph* of a λ-term M is the graph, whose vertices are λ-terms, defined as the smallest graph such that M is a vertex and there is an arrow between two vertices M and M' whenever $M \to_\beta M'$.

1. Write the respective reduction graphs of

$$(\lambda x.xx)(\lambda y.y)z \qquad \text{and} \qquad (\lambda xy.x)((\lambda x.xx)(\lambda xy.xy))$$

2. Can a reduction graph have loops?

2 Booleans

We encode the booleans $\top$ (true) and $\bot$ (false) into λ-terms respectively as

$$[\![\top]\!] = \lambda x.\lambda y.x \qquad \text{and} \qquad [\![\bot]\!] = \lambda x.\lambda y.y$$

1. Define a λ-term if such that

$$if[\![\top]\!]MN \stackrel{*}{\to}_\beta M \qquad \text{and} \qquad if[\![\bot]\!]MN \stackrel{*}{\to}_\beta N$$

2. Define λ-terms *and*, *or* and *not* such that for every booleans b and b',

$$and[\![b]\!][\![b']\!] \to_\beta [\![b \wedge b']\!] \qquad or[\![b]\!][\![b']\!] \to_\beta [\![b \vee b']\!] \qquad not[\![b]\!] \to_\beta [\![\neg b]\!]$$

3 Weak normalization of the λ-calculus

An *abstract rewriting system* (ARS) is a graph whose vertices are called *terms* and whose edges are called *rewriting rules*. We often write $x \to y$ when there exists an edge from x to y and $x \stackrel{*}{\to} y$ when there exists a directed path from x to y (in the latest case, we say that x rewrites to y). For instance, we can consider the ARS of λ-terms and β-reduction. An ARS is

- *locally confluent* when $y_1 \leftarrow x \to y_2$ implies that there exists z such that $y_1 \stackrel{*}{\to} z \stackrel{*}{\leftarrow} y_2$,

- *confluent* when $y_1 \stackrel{*}{\leftarrow} x \stackrel{*}{\to} y_2$ implies that there exists z such that $y_1 \stackrel{*}{\to} z \stackrel{*}{\leftarrow} y_2$,

- *strongly confluent* when $y_1 \leftarrow x \to y_2$ implies that there exists z such that $y_1 \to z \leftarrow y_2$.

1. Which properties imply another? Give counter-examples for implications which fail.

2. A *normal form* is a term x such that there is no y for which $x \to y$. Show that in a confluent rewriting system a term reduces to at most one normal form.

3. [Newman's lemma] An ARS is *terminating* if it does not contain any infinite path. Show that an ARS which is terminating and locally confluent is confluent.

4. Show that in a terminating and (locally) confluent rewriting system, normal forms are in bijection with connected components of the graph.

5. A λ-term is *strongly terminating* when it can only be reduced a finite number of times, *divergent* when it does not reduce to a normal form and *weakly terminating* when it can reduce to a normal form. Give example of λ-terms with such properties.

6. The *parallel reduction* $M \Rightarrow N$ on λ-terms is defined by:

   - $M \Rightarrow M$
   - $M \Rightarrow M'$ and $N \Rightarrow N'$ implies $MN \Rightarrow M'N'$
   - $M \Rightarrow M'$ implies $\lambda x.M \Rightarrow \lambda x.M'$
   - $M \Rightarrow M'$ and $N \Rightarrow N'$ implies $(\lambda x.M)N \Rightarrow M'[N'/x]$

   Show that $\Rightarrow$ is strongly confluent.

7. Show that $\rightarrow_\beta \subseteq \Rightarrow \subseteq \rightarrow_\beta^*$. Provide counter-examples showing that these inclusions are strict.

8. Conclude that $\rightarrow_\beta$ is confluent.