TD7 – Monoidal Theories

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1 Monoidal categories

A *monoidal category* $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ is a category $\mathcal{C}$ equipped with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, an object $I \in \mathcal{C}$, and three natural bijections of components

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C) \qquad \lambda_A : I \otimes A \to A \qquad \rho_A : A \otimes I \to A$$

such that the diagrams

$$\begin{CD}
((A \otimes B) \otimes C) \otimes D @>{\alpha_{A,B,C}\otimes D}>> (A \otimes (B \otimes C)) \otimes D @>{\alpha_{A,B\otimes C,D}}>> A \otimes ((B \otimes C) \otimes D) \\
@V{\alpha_{A\otimes B,C,D}}VV @. @VV{A\otimes\alpha_{B,C,D}}V \\
(A \otimes B) \otimes (C \otimes D) @>>{\alpha_{A,B,C\otimes D}}> \cdots @>>> A \otimes (B \otimes (C \otimes D))
\end{CD}$$

and

$$\begin{array}{ccc}
(A \otimes I) \otimes B & \xrightarrow{\alpha_{A,I,B}} & A \otimes (I \otimes B) \\
& {}_{\rho_A \otimes B}\searrow \quad \swarrow_{A \otimes \lambda_B} & \\
& A \otimes B &
\end{array}$$

commute.

A *braided monoidal category* $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho, \gamma)$ is a monoidal category equipped with a natural bijection γ of components

$$\gamma_{A,B} : A \otimes B \to B \otimes A$$

such that suitable diagrams commute. A *symmetric monoidal category* is a braided monoidal category such that $\gamma_{B,A} \circ \gamma_{A,B} = \mathrm{id}_{A\otimes B}$ for every objects A and B.

1. Show that every cartesian category $\mathcal{C}$ can be equipped with a structure of symmetric monoidal category.
2. We write **Vect** for the category of $\Bbbk$-vector spaces (where $\Bbbk$ is a fixed field) and linear functions. Show that this category is cartesian.
3. Given a basis for A and B, describe a basis for $A \times B$.
4. Show that the forgetful functor $U : \mathbf{Vect} \to \mathbf{Set}$ admits a left adjoint $F : \mathbf{Set} \to \mathbf{Vect}$.
5. Given vector spaces A, B and C, we write $\mathrm{Bilin}(A, B; C)$ for the set of bilinear applications from $A \times B$ to C. Show that there exists a vector space, written $A \otimes B$ such that we have a (natural) bijection

   $$\mathrm{Bilin}(A, B; C) \qquad \cong \qquad \mathbf{Vect}(A \otimes B; C)$$

   It can be helpful to write $A \otimes B$ as a quotient of the vector space $F(U(A) \times U(B))$.
6. Given a basis for A and B, describe a basis for $A \otimes B$.
7. Define a notion of *monoid* in a monoidal category (so that a monoid in the cartesian category **Set** corresponds to the usual notion). Define similarly morphisms of monoids.
8. What is a monoid in $(\mathbf{Cat}, \times, 1)$?
9. Show that a monoidal category is cartesian (with tensor as product) if and only if every object is equipped with a natural structure of comonoid.

2 Monoidal theories

In the following, we will assume that all the monoidal categories we consider are *strict* (i.e. α, λ and ρ) are identities. It can namely be shown that every monoidal category is equivalent to a strict one. A *monoidal functor* between two monoidal categories is a functor commuting to the tensor and units.

We write Δ for the category whose objects are strictly positive natural numbers $n \in \mathbb{N}$, and morphisms $f : m \to n$ are (weakly) increasing functions $f : [m] \to [n]$ where $[n] = \{0, \ldots, n-1\}$.

1. Equip Δ with a structure of (strict) monoidal category, given by addition on objects.
2. Show that the object 1 is terminal in this category.
3. Show that the object 1 is canonically equipped with a structure of monoid.
4. Recall how to show that $\mathbb{N} \times \mathbb{N}/2\mathbb{N}$ admits the presentation $\langle a, b \mid ba = ab, bb = 1\rangle$ using rewriting theory.
5. Show that Δ is the free monoidal category containing a monoid by showing that morphisms are in bijection with canonical forms of composites of morphisms generated by the operations of monoids. In which sense can this be thought as providing a presentation for Δ?
6. Show that, given a monoidal category $\mathcal{C}$, monoidal functors $\Delta \to \mathcal{C}$ are in bijection with monoids in $\mathcal{C}$.
7. Which category should play the role of Δ if we were interested in the theory of commutative monoids in a symmetric monoidal category?
8. Which theory do we obtain if we restrict the morphisms to injective (resp. surjective) functions?
9. What is the Lawvere theory of commutative monoids?

3 Simplicial sets

We write Δ_+ for the category with $\mathbb{N}$ as objects and whose morphisms $f : m \to n$ are weakly increasing functions $f : [m+1] \to [n+1]$.

1. Show that Δ_+ is isomorphic to a full subcategory of Δ.
2. Show that Δ_+ is generated, as a category, by the morphisms $\delta_i^n : [n] \to [n+1]$ and $\varepsilon_i^n : [n+2] \to [n+1]$ (with $0 \le i \le n$) defined by

$$\delta_i^m(k) = \begin{cases} k & \text{si } k < i \\ k+1 & \text{sinon} \end{cases} \qquad \text{and} \qquad \varepsilon_i^n = \begin{cases} k & \text{si } k \le i \\ k-1 & \text{sinon} \end{cases}$$

   How are these generators linked with the presentation given in previous section?
3. Find out the right members of the following equations satisfied by those morphisms:

$$\delta_j^{n+1} \circ \delta_i^n = ? \qquad \varepsilon_j^{n+1} \circ \varepsilon_i^{n+2} = ? \qquad \varepsilon_j^{n+1} \circ \delta_i^{n+1} = ? \tag{1}$$

4. Deduce a presentation of Δ_+ as a category.
5. The *standard n-simplex* Δ_n is the subspace of the euclidian space $\mathbb{R}^n$ whose points are

$$\Delta_n \quad = \quad \{\ (x_1, \ldots, x_n) \mid x_i \ge 0 \quad \text{and} \quad \sum_i x_i = 1\ \}$$

   Provide a geometrical interpretation of functors $\phi : \Delta^{\mathrm{op}} \to \mathbf{Set}$, which are called *simplicial sets*, by considering elements of $\phi(n)$ as standard n-simplices and morphisms $\phi(\varepsilon_i^n)$ as describing faces. What is the geometrical interpretation of $\phi(\delta_i^n)$? What is the geometrical interpretation of the above equations?
6. Give a description as simplicial sets of an empty square, a filled square, a (empty or filled) cube, a torus, a Möbius strip, etc.