Asynchronous Games
Innocence without Alternation

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Groupe de travail concurrence
June 20, 2007

Part I

Game semantics

Game semantics

- An *interactive* and *trace* semantics for proofs and programs
- A successful series of models:
  - PCF
  - PCF + control
  - references (Idealized Algol)
  - linear logic
  - ...

Can we reflect the
concurrency of proofs in games?

Mixing different points of view

Mixing ideas from

- game semantics
- concurrency theory
- linear logic

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- game semantics
- concurrency theory
- linear logic

We relate here

1. sequential games (traces)
2. event structures
3. relational model
4. concurrent games (closure operators)

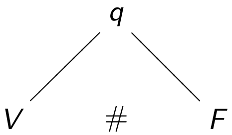
Mixing points of view

sequential games	traces
causal games	event structures
relational games	relations
concurrent games	closure operators

Game semantics

- Formulas are interpreted as **games**

$[\![\mathbb{B}]\!] \quad =$



consisting of and event structure $(M, \leq, \#)$ where

- M is a set of **moves**
- causal dependencies $(\leq)$ and incompatibilities $(\#)$ between these moves
- a **polarization** of moves $\lambda : M \to \{O, P\}$

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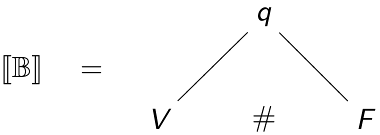

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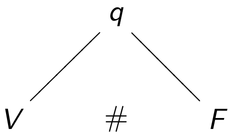

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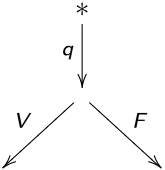
$[\![\mathbb{B}]\!] \quad =$ 

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Game semantics

- Proofs are interpreted as **strategies**

$$[\![\texttt{true}]\!] \quad = \quad \{\varepsilon, q, q \cdot V\}$$

The strategy `not`

$$\mathbb{B} \xrightarrow{\texttt{not}} \mathbb{B}$$

	q
q	
V	
	F

The strategy `not`

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	q
q	
F	
	V

Game semantics

Here,

- we only consider formulas of MALL:

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}(\parr) \qquad \frac{\vdash \Gamma_1, A \qquad \vdash \Gamma_2, B}{\vdash \Gamma_1, \Gamma_2, A \otimes B}(\otimes)$$

$$\frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B}(\&) \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}(\oplus)$$

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$$\frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B}(\&) \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}(\oplus)$$

- with explicit moves:

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, \uparrow A}(\uparrow) \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, \downarrow A}(\downarrow)$$

Game semantics

Game semantics are usually:

- alternating
- sequential

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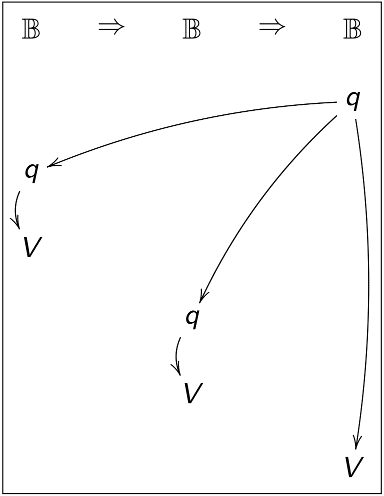
Game semantics are usually:

- alternating
  - does not reflect the derivations!
- sequential
  - conceals the concurrency of proofs!

Alternating game semantics

Left and



Alternating game semantics

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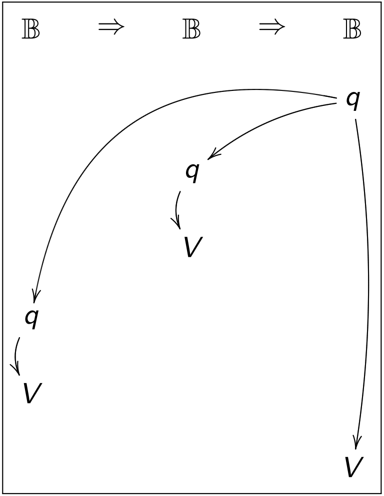
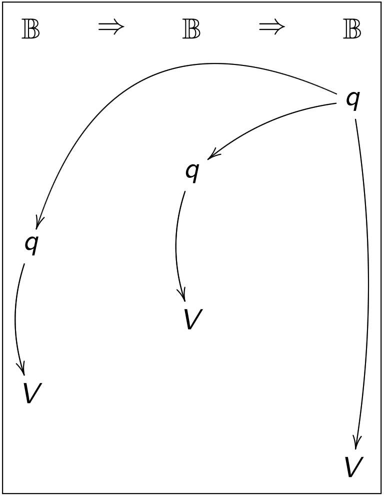


Alternating game semantics

Parallel and



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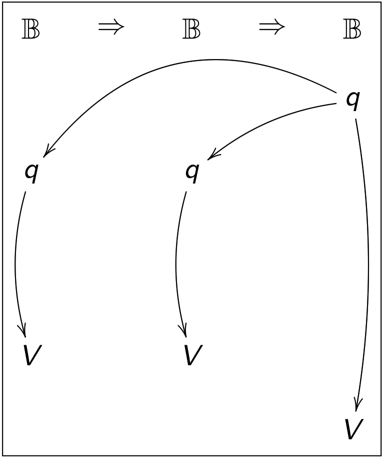

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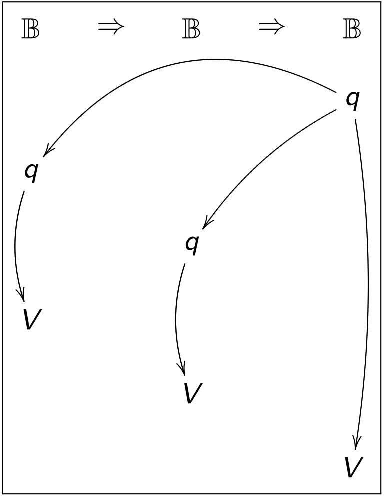

Innocence

Can we characterize the *definable* strategies?

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We have to restrict the space of strategies.

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We have to restrict the space of strategies.

innocent strategy	$=$	strategy behaving like a proof

Reformulating innocence

An **innocent** strategy is a strategy with partial memory which plays according to its *view*.

The original definition by Hyland and Ong

- is technical (pointers)
- relies on the fact that plays are alternating

From formulas to games

In linear logic, the formula corresponding to booleans is

$$\mathbb{B} \quad = \quad \uparrow(\downarrow 1 \oplus \downarrow 1)$$

which is like of $1 \oplus 1$ with explicit changes of polarities.

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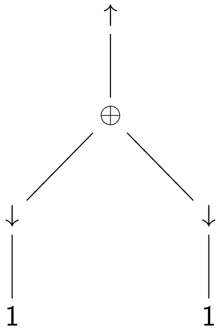

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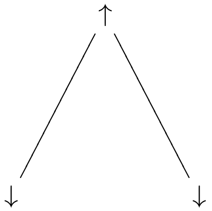

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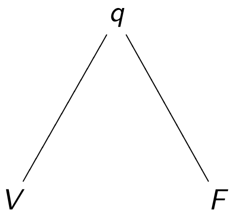

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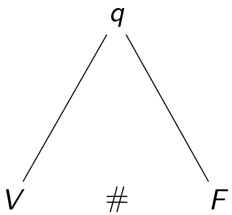

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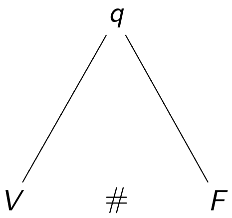
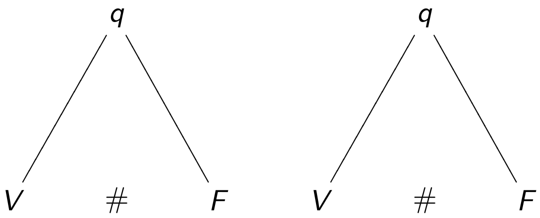


From formulas to games

So, the game $\mathbb{B} \otimes \mathbb{B}$ is

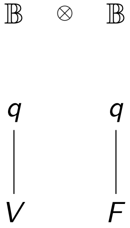


From formulas to games

So, the game $\mathbb{B} \otimes \mathbb{B}$ is

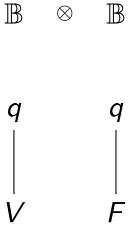
Let's consider strategies associated to the state $\texttt{true} \otimes \texttt{false}$.

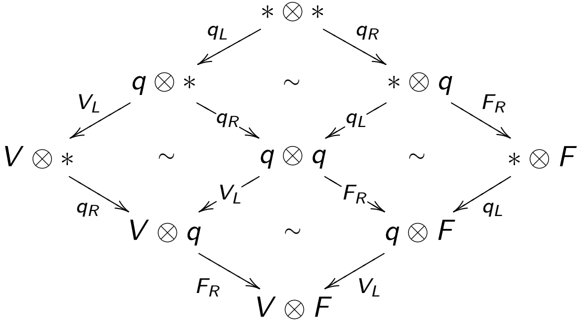
The strategy $\texttt{true} \otimes \texttt{false}$.



The strategy true ⊗ false

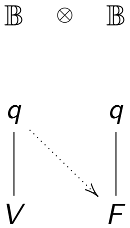
The strategy `true` ⊗ `false`.

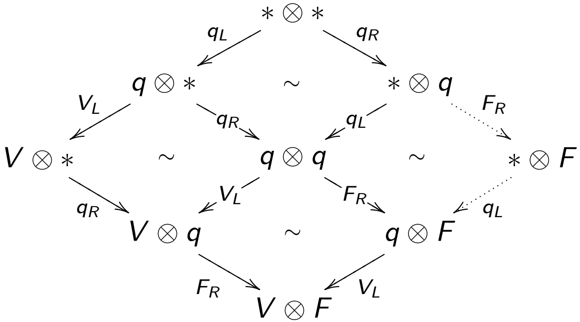




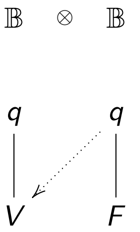
The strategy true $\otimes$ false

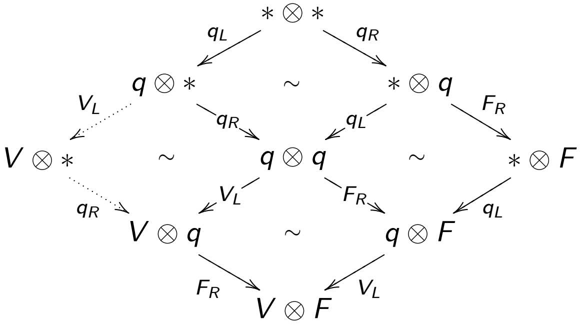
A biased variant.





Another biased variant.

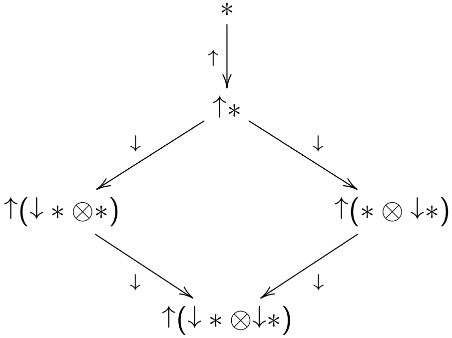




A **game** is an *asynchronous graph* $\mathcal{G}$:

- vertices are **positions** (+ initial position $*$),
- edges are **moves**,

$[\![\uparrow(\downarrow 1 \otimes \downarrow 1)]\!] \quad =$



Games

A **game** is an *asynchronous graph* $\mathcal{G}$:

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- 2-dimensional tiles generate **homotopy** between paths.

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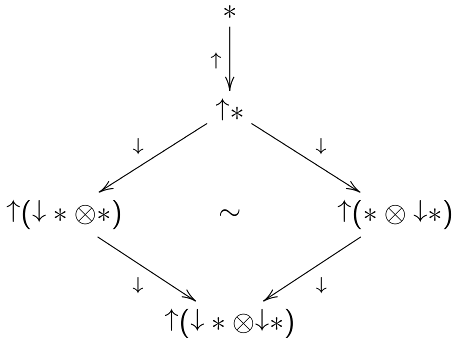
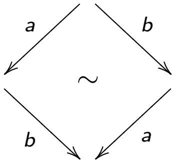
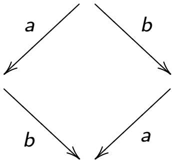


An approach to interferences

The Mazurkiewicz approach to *true concurrency*.

$$a \| b \qquad \text{vs.} \qquad a \cdot b + b \cdot a$$





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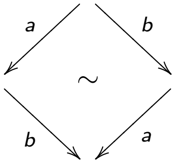

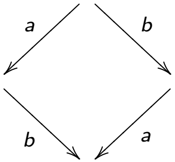

$x := 4 \| y := 5$

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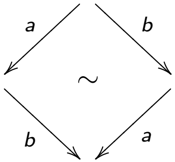

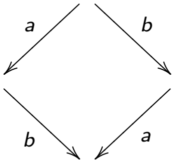

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multiplicatives   additives

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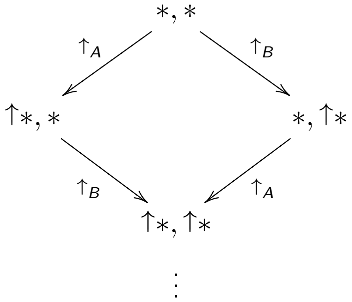
The corresponding asynchronous graph contains



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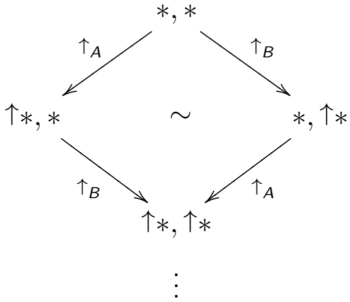

Non-alternation and asynchrony

Three proofs of $\uparrow A \parr \uparrow B$:

$$*, *$$

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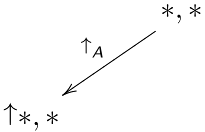

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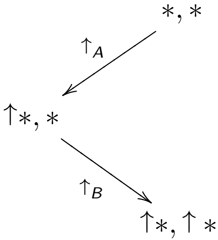

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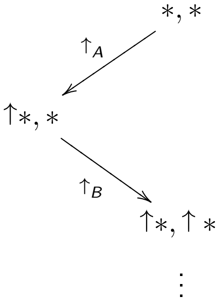

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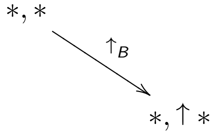

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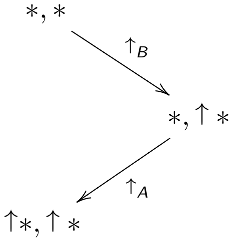

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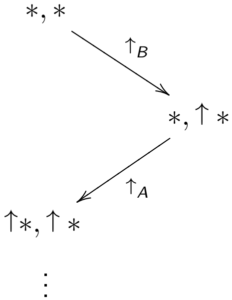

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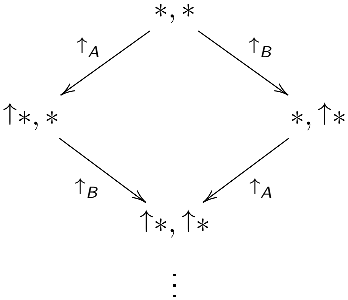

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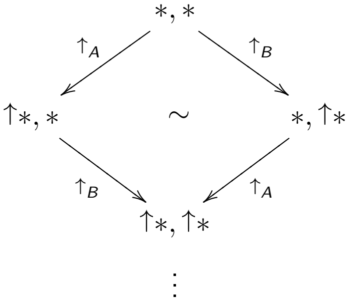

Non-alternation and asynchrony

play	=	exploration of the formula
proof	=	strategy of exploration

Proofs correspond to refinements of the partial order of the game.

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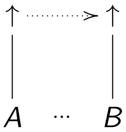

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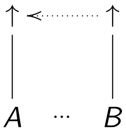

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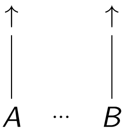


Part II

Traces vs. partial orders

Traces vs. partial orders

formula = partial order on the moves

proof = refinement of the partial order of the formula

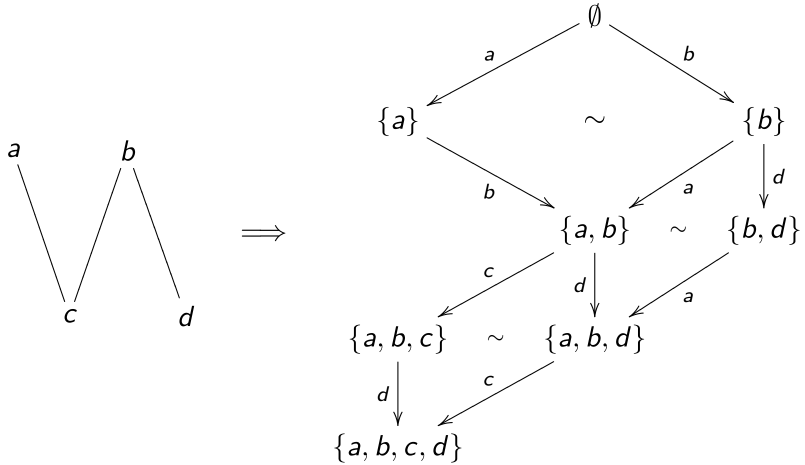
How do we relate *sequential* and *causal* strategies?

Mixing points of view

sequential games	traces
causal games	event structures
relational games	relations
concurrent games	closure operators

From causality to sequentiality

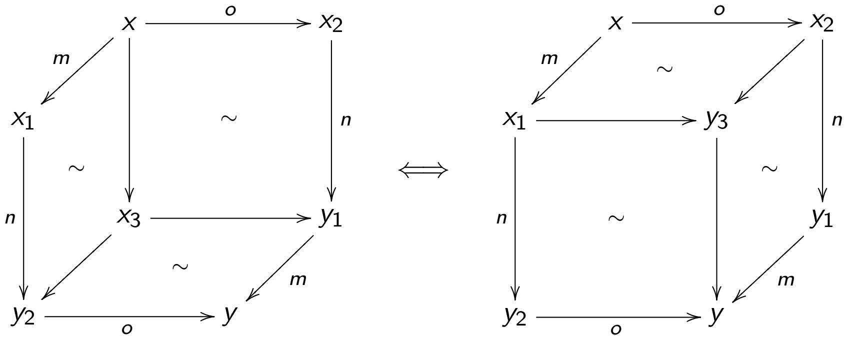
Every partial order defines an asynchronous graph.



Extracting causality from sequentiality

Conversely, one needs the 2-dimensional structure.

The Cube Property



Theorem

Paths modulo homotopy are characterized by partial order on their moves.

Games

Definition

A **game** is an asynchronous graph satisfying the Cube Property.

Definition

A **play** is a path in A starting from the root.

Strategies

Definition

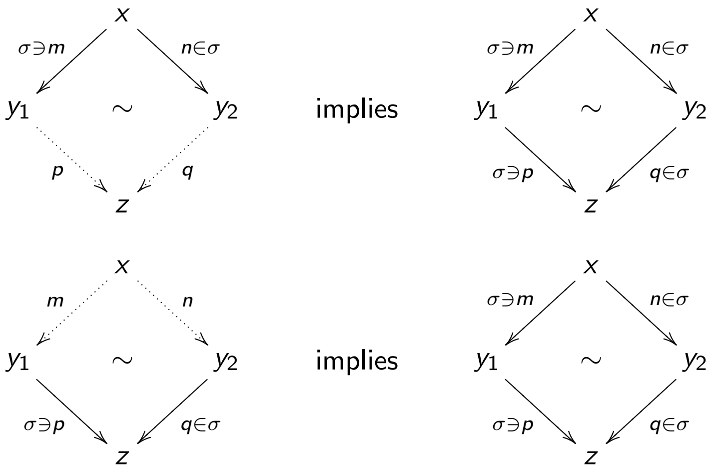
A **strategy** $\sigma : A$ is a set of plays A.

Definition

A strategy $\sigma : A$ is **positional** when its paths form a subgraph of the game A.

Strategies

We consider positional strategies which satisfy



(this implies the Cube Property)

Composition

Unfortunately, the Cube Property is not compositional.

Closure

$$A \multimap B \quad = \quad A^* \parr B \quad = \quad A^* \otimes B$$

The strategy **not**:

$$\begin{array}{ccc} \mathbb{B} & \xrightarrow{\texttt{not}} & \mathbb{B} \\ & & q \\ q & & \\ V & & \\ & & F \end{array}$$

Closure

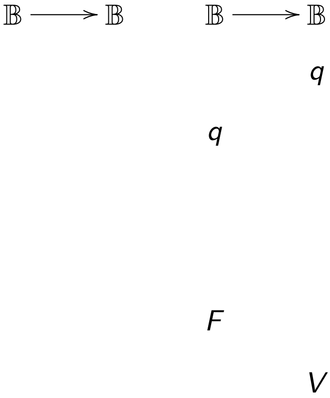
$$A \multimap B \quad = \quad A^* \parr B \quad = \quad A^* \otimes B$$

The strategy **not**:

$$\begin{array}{ccc}
\mathbb{B} & \xrightarrow{\texttt{not}} & \mathbb{B} \\
 & & q \\
q & & \\
F & & \\
 & & V
\end{array}$$

Composition

Traces compose by *parallel composition*



Traces compose by *parallel composition*

$$\mathbb{B} \longrightarrow \mathbb{B} \qquad \mathbb{B} \longrightarrow \mathbb{B}$$

q

q

V

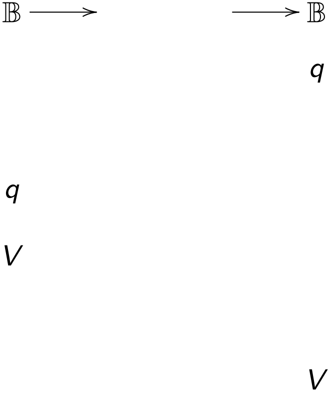
F

Traces compose by *parallel composition*

$$
\begin{array}{ccccccc}
\mathbb{B} & \longrightarrow & \mathbb{B} & \qquad & \mathbb{B} & \longrightarrow & \mathbb{B} \\
 & & & & & & q \\
 & & q & & q & & \\
q & & & & & & \\
V & & & & & & \\
 & & F & & F & & \\
 & & & & & & V
\end{array}
$$

Composition

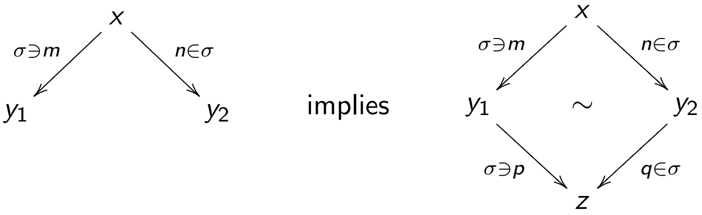
Traces compose by *parallel composition* + *hiding.*



Determinism

Definition

A strategy $\sigma : A$ is **deterministic** when



where m is a Proponent move.

Deterministic strategies do compose!

Deterministic strategies do compose!

| sequential strategies $\iff$ causal strategies |

Part III

Partial orders vs. concurrent games

Mixing points of view

sequential games	traces
causal games	event structures
relational games	relations
concurrent games	closure operators

Concurrent strategies

Abramsky and Melliès introduced the notion of concurrent games.

Definition

A **closure operator** σ on a complete lattice A is a function $\sigma : A \to A$ such that

$$\begin{array}{llll} (1) & \sigma \text{ is increasing:} & \forall x \in D, & x \leq \sigma(x), \\ (2) & \sigma \text{ is idempotent:} & \forall x \in D, & \sigma(x) = \sigma(\sigma(x)), \\ (3) & \sigma \text{ is monotone:} & \forall x, y \in D, & x \leq y \Rightarrow \sigma(x) \leq \sigma(y). \end{array}$$

A closure operator is **continuous** when

$$\sigma(\overrightarrow{\bigvee_i} x_i) = \overrightarrow{\bigvee_i} \sigma(x_i)$$

Composing concurrent strategies

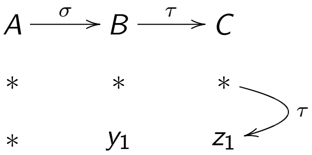
Composing $\sigma : A \to B$ and $\tau : B \to C$.

$$A \xrightarrow{\sigma} B \xrightarrow{\tau} C$$

$$* \qquad\qquad * \qquad\qquad *$$

Composing concurrent strategies

Composing $\sigma : A \to B$ and $\tau : B \to C$.



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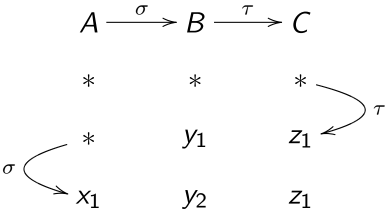

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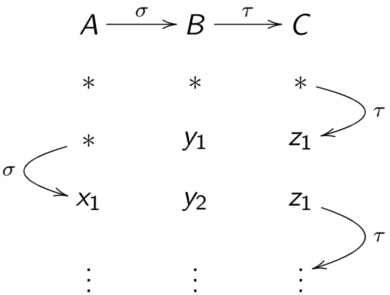

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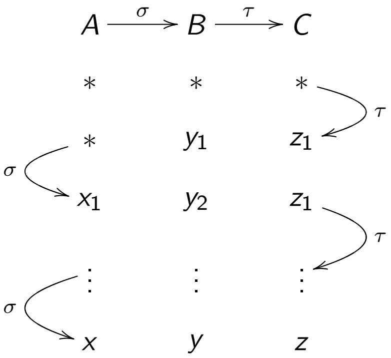

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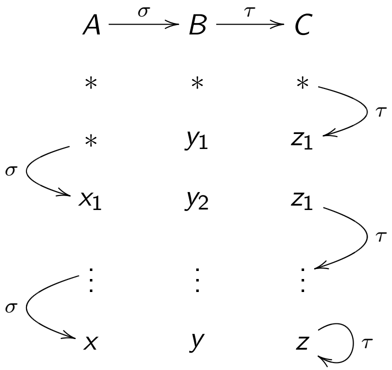

Closure operators as relations

closure operator $\iff$ set of positions closed under meets

Closure operators as relations

closure operator $\Longleftrightarrow$ set of positions closed under meets

$$\sigma \quad \Rightarrow \quad \mathrm{fix}(\sigma) = \{x \mid \sigma(x) = x\}$$

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$$x \mapsto \bigwedge\{y \in X \mid x \leq y\} \quad \Leftarrow \quad X$$

Closure operators as relations

closure operator $\iff$ set of positions closed under meets

$$\sigma \quad \Rightarrow \quad \mathrm{fix}(\sigma) = \{x \mid \sigma(x) = x\}$$

$$x \mapsto \bigwedge\{y \in X \mid x \leq y\} \quad \Leftarrow \quad X$$

Extends to continuous closure operators.

Closure operators as relations

closure operator $\iff$ set of positions closed under meets

$$\sigma \quad \Rightarrow \quad \mathrm{fix}(\sigma) = \{x \mid \sigma(x) = x\}$$

$$x \mapsto \bigwedge\{y \in X \mid x \leq y\} \quad \Leftarrow \quad X$$

A strategy $\sigma : A \to B$ can be seen as a relation on $A \times B$.

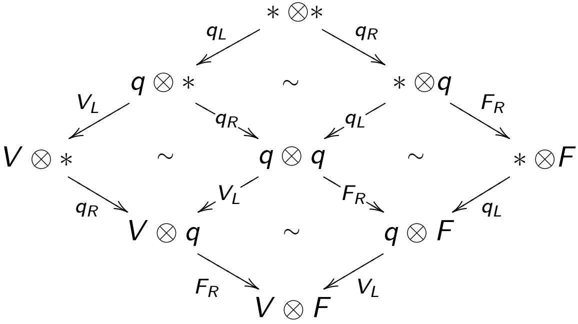
Halting positions

Definition

A position of a strategy $\sigma : A$ is **halting** when there is no Proponent move $m : x \longrightarrow y$ in σ.

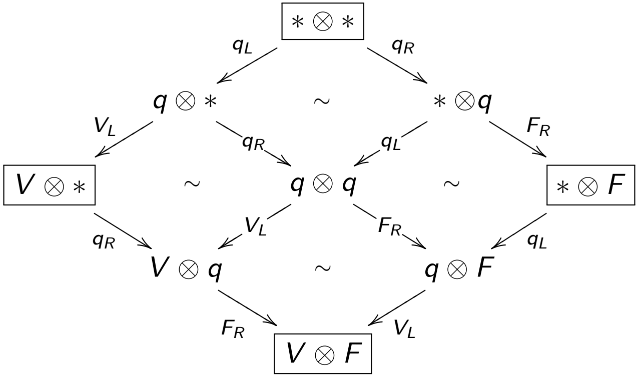
Halting positions

The game $\mathbb{B} \otimes \mathbb{B}$.



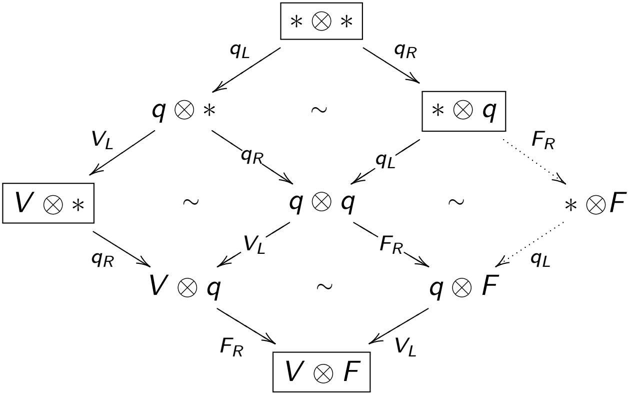
Halting positions

The *parallel* implementation of `true` and `false`.



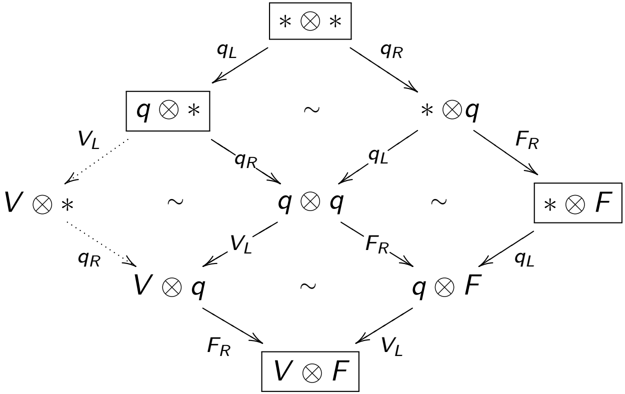
Halting positions

The *left* implementation of `true` and `false`.



Halting positions

The *right* implementation of `true` and `false`.



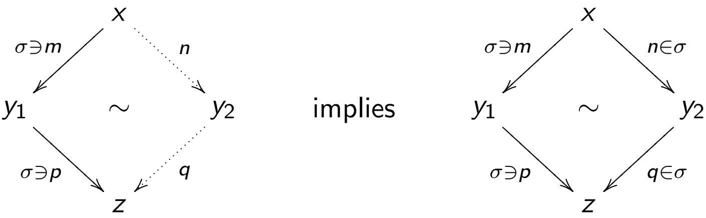
Ingenuous strategies

We would like strategies to be characterized by their *halting positions*.

Definition

A strategy $\sigma : A$ is **ingenuous** when it is

1. positional,
2. deterministic,
3. *courteous*:



where m is a Proponent move.

Ingenuous strategies as closure operators

Theorem

Under suitable conditions, we have:

$$\sigma \quad \Longleftrightarrow \quad \sigma^\circ \quad \Longleftrightarrow \quad \mathrm{Cl}(\sigma^\circ)$$

ingenuous strategies	$\Longleftrightarrow$	ingenuous concurrent strategies

Part IV

Innocence

Functoriality

There is a mismatch between sequential and concurrent games: we don't have

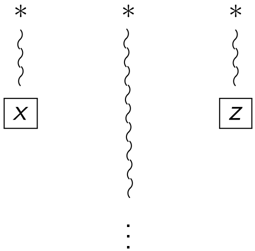
$$(\sigma; \tau)^\circ \quad = \quad \sigma^\circ; \tau^\circ$$

Functoriality

The *livelock*:

$$(\sigma;\tau)^\circ \subseteq \sigma^\circ;\tau^\circ$$

$$A \xrightarrow{\sigma} B \xrightarrow{\tau} C$$



Functoriality

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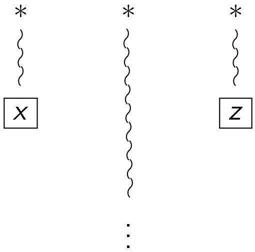


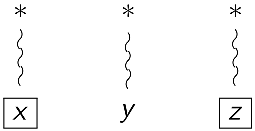
Solution: handle infinite positions

Functoriality

The *deadlock*:

$$(\sigma;\tau)^\circ \supseteq \sigma^\circ;\tau^\circ$$

$$A \xrightarrow{\sigma} B \xrightarrow{\tau} C$$



Functoriality

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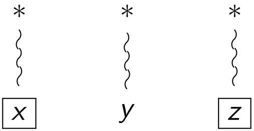
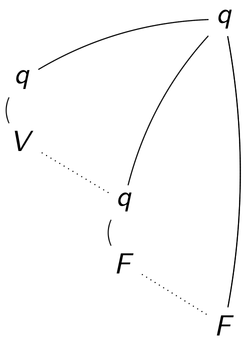


Solution: add a scheduling criterion

The scheduling criterion

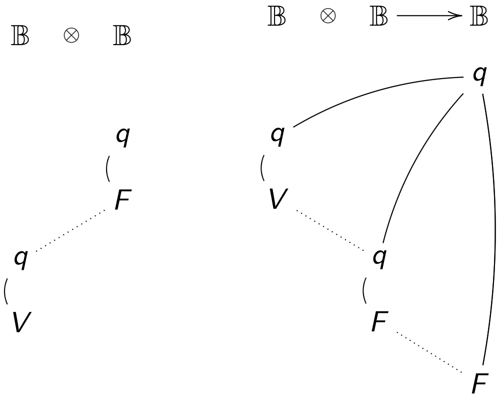
the left conjunction:

$$\mathbb{B} \quad \otimes \quad \mathbb{B} \longrightarrow \mathbb{B}$$



The scheduling criterion

The right boolean composed with the left conjunction:



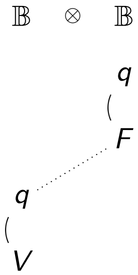
The scheduling criterion

Two kinds of tensors: $\otimes$ and $\parr$.

$$\mathbb{B} \otimes \mathbb{B} \multimap \mathbb{B} \quad = \quad \mathbb{B}^* \parr \mathbb{B}^* \parr \mathbb{B}$$

The scheduling criterion

Two kinds of tensors: $\otimes$ and $\parr$.



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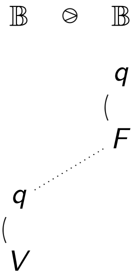

The scheduling criterion

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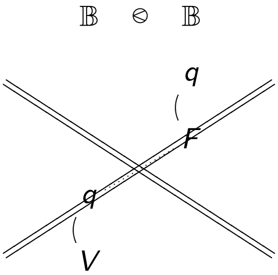


Functoriality

Definition

A strategy $\sigma : A$ is **receptive** when for every path $s : * \longrightarrow\!\!\!\!\!\rightarrow x$ in σ and for every Opponent move $m : x \longrightarrow y$ the path $s \cdot m : * \longrightarrow\!\!\!\!\!\rightarrow y$ is also in σ.

Functoriality

Definition

A strategy $\sigma : A$ is **receptive** when for every path $s : * \longrightarrow\!\!\!\!\!\rightarrow x$ in σ and for every Opponent move $m : x \longrightarrow y$ the path $s \cdot m : * \longrightarrow\!\!\!\!\!\rightarrow y$ is also in σ.

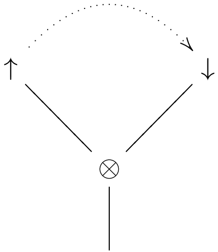
Theorem

Ingenuous strategies which satisfy the scheduling criterion and are receptive compose and satisfy

$$(\sigma ; \tau)^\circ \quad = \quad \sigma^\circ ; \tau^\circ$$

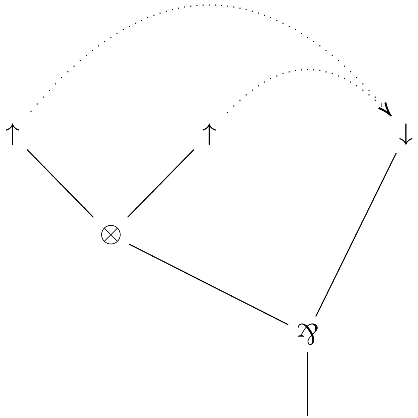
Towards innocence

The scheduling criterion detects directed cycles.



Towards innocence

The scheduling criterion does not detect non-directed cycles.



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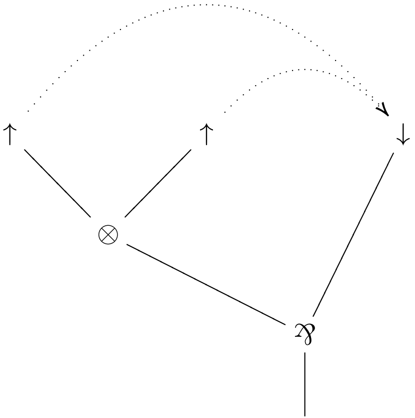

We need a more elaborate scheduling criterion.

Thanks for your attention

Any question?