POINTS OF VIEW ON ASYNCHRONOUS COMPUTABILITY

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WIP 2016

April 27, 2016

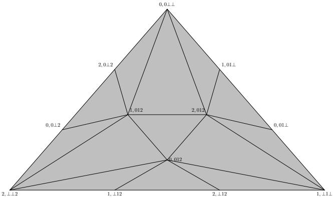
Asynchronous computability

In the 90s, Herlihy et al. have obtained major results on asynchronous computability.

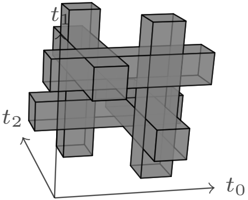
- What can a bunch of processes computing in parallel can compute in the presence of failures?
- For instance, they show that the consensus cannot be solved.
- Their proofs uses geometric arguments, they construct a geometric object corresponding to the possible states and
  - characterize those which can occur and their properties
  - obtain impossibility results from the fact that some maps should preserve (n-)connectivity
- The devil lies in the details.

Unifying points of view

Here, we unify different points of view on executions:



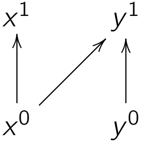
protocol complex
[Herlihy, . . .]



geometric semantics
[Goubault, . . .]

$$\left\langle u_i, s_i \;\middle|\; u_i u_j = u_j u_i, s_i s_j = s_j s_i \right\rangle$$

partially commutative traces



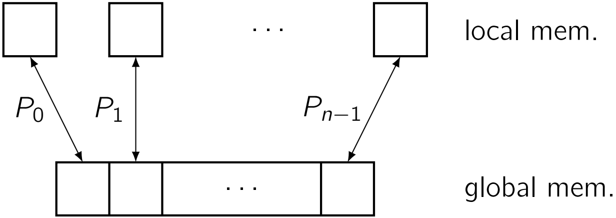
interval orders

ASYNCHRONOUS PROTOCOLS AND TASKS

Asynchronous protocols

We consider here a model with n processes P_i:

- each process has a local memory cell
- there is a global memory with n cells



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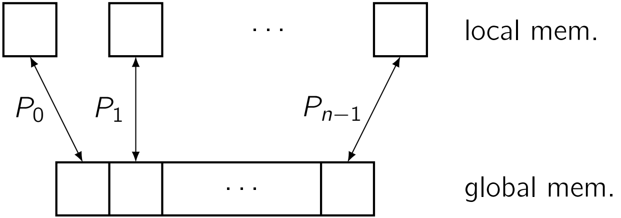

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  - `update`: write in its global memory cell
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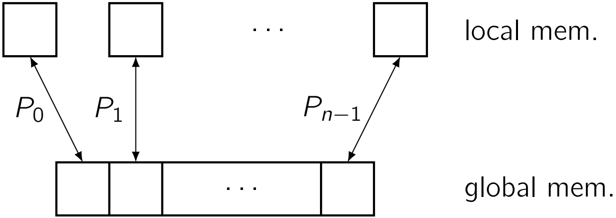

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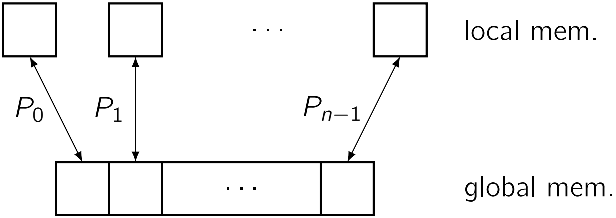

- each process alternatively does "rounds" made of
  - `update`: write in its global memory cell
  - `scan`: read the whole global memory and update its local cell (*immediate snapshot*)
- at any instant a process might die
- and the question is: what we can compute in such a model? (for this question we are only interested in local memories)

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- the local memory of each process is a partial information about the computation (called its *view*)
- we are mostly interested in local memory: it contains the input and output values
- the initial value for global memory is $\perp$ in every cell

Coherence between views

The main idea here is to introduce a semantics based on the same principles as in (hyper)coherence spaces.

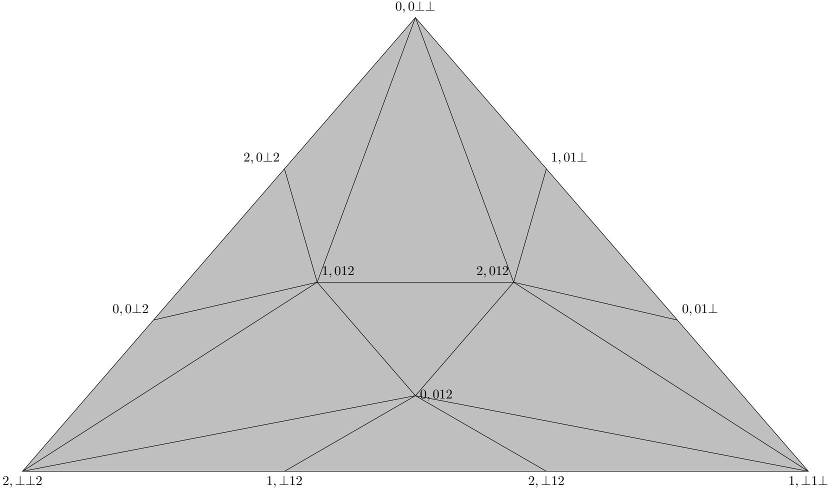
A set $X \subseteq \{(i, x) \mid i \in \mathbb{N}, x \in \mathcal{V}\}$ of local memories (= views) $(i, x) \in \mathbb{N} \times \mathcal{V}$ is **coherent** when

$$X \quad = \quad \{(i, l_i)\}$$

such that there is an execution leading to a local memory l.

Coherence between views

With 3 processes executing one round (update then scan), we typically obtain the following coherence space:



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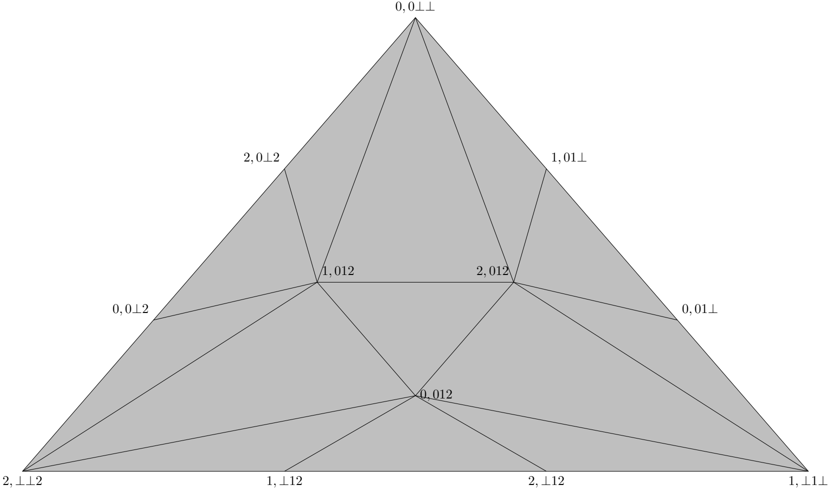

Notice that it is simply connected.

States

Formally, we suppose fixed a number $n \in \mathbb{N}$ of processes and a set $\mathcal{V}$ of **values** with

- $\mathcal{I} \subseteq \mathcal{V}$: *input values*
- $\mathcal{O} \subseteq \mathcal{V}$: *output values*
- $\bot \in \mathcal{I} \cap \mathcal{O}$: the *undefined value* / a *non-participating process*

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- $l \in \mathcal{V}^n$: the *local memories*
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The *standard* initial state has $l_i = i$ and $m_i = \bot$.

Protocols

A **protocol** π consists of, for $0 \leq i < n$,

- $\pi_{u_i} : \mathcal{V} \to \mathcal{V}$
  the values it will write in its global memory cell depending on its local memory
- $\pi_{s_i} : \mathcal{V} \times \mathcal{V}^n \to \mathcal{V}$
  the values it will write in its local memory depending on the values of its local memory and all the global memory cells

such that

- $\pi_{s_i}(x, m) = x$ for $x \in \mathcal{O}$
  once we decide an output we don't change our mind

Execution traces

The set of possible **actions** is

$$\mathcal{A} \quad = \quad \{u_i, s_i, d_i \mid 0 \leq i < n\}$$

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Given a protocol π, its **semantics**

$$[\![T]\!]_\pi : \mathcal{V}^n \times \mathcal{V}^n \to \mathcal{V}^n \times \mathcal{V}^n$$

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With two processes executing one round each there are "essentially" three traces:

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- $u_0 u_1 s_0 s_1$ / $u_0 u_1 s_1 s_0$ / $u_1 u_0 s_0 s_1$ / $u_1 u_0 s_1 s_0$:
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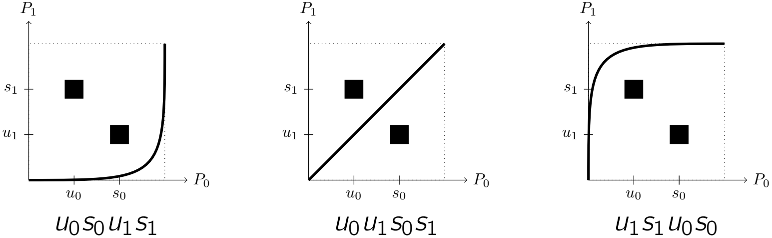
These execution traces can be represented geometrically by



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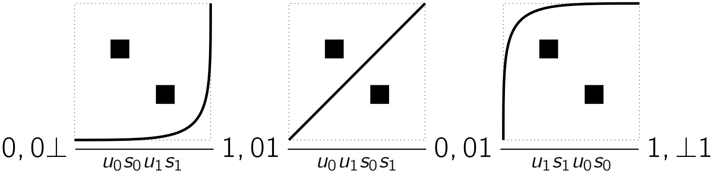

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Tasks

A **task** θ is a relation $\theta \subseteq \mathcal{I}^n \times \mathcal{O}^n$ such that for every $I, I' \in \Theta$

- $I_i = \bot$ if and only if $I'_i = \bot$,
- there exists $I'' \in \mathcal{O}^n$ such that $(I, I'') \in \Theta$ and $(I[i \leftarrow \bot], I''[i \leftarrow \bot]) \in \Theta$.

We write $\operatorname{dom} \Theta$ for the possible input values and $\operatorname{codom} \Theta$ for the possible output values.

The binary consensus

In the **binary consensus** problem each process

- starts with a value in $\{0, 1\}$
- end with the same value, among the initial values of the alive processes.

For instance, with $n = 2$, we have

$$\Theta \quad = \quad \Big\{(b\bot, b\bot), (\bot b, \bot b), (bb', bb), (b'b, bb) \;\Big|\; b, b' \in \{0, 1\}\Big\}$$

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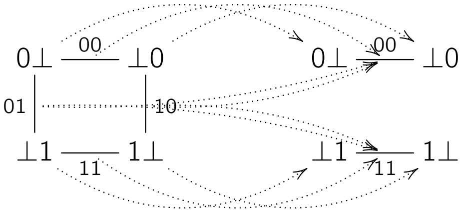
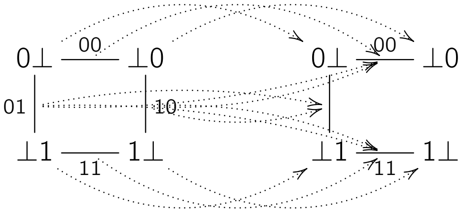


The binary quasi-consensus

In the case $n = 2$, we can also consider the **binary quasi-consensus**, which is similar but restricts the output so that it cannot happen that P_1 decides 0 and P_0 decide 1 at the same time:

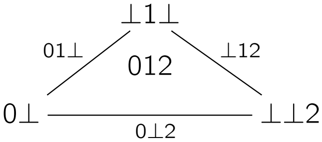


The way we draw tasks

Note that

- if $I \in \operatorname{dom} \Theta$ (the possible input values) then $I[i \leftarrow \bot]$ also belongs to $\operatorname{dom} \Theta$

$\operatorname{dom} \Theta$ can thus be pictured as a *simplicial complex* called the **input complex**:



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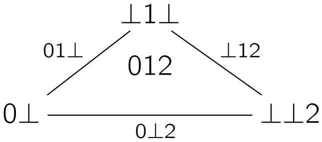

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Note also that the vertices are **colored** by $0 \leq i < n$:
the only active process

Tasks

A **task** θ is a relation $\theta \subseteq \mathcal{I}^n \times \mathcal{O}^n$ such that for every $I, I' \in \Theta$

1. $I_i = \perp$ if and only if $I'_i = \perp$,
2. there exists $I'' \in \mathcal{O}^n$ such that $(I, I'') \in \Theta$ and $(I[i \leftarrow \perp], I''[i \leftarrow \perp]) \in \Theta$.

which means

1. n-simplices are in relation with n-simplices
2. the relation is compatible with faces

Solving tasks

A protocol π **solves** a task Θ when

- for every initial local memory $l \in \operatorname{dom} \Theta$
- for every long enough and fair execution trace T

we have $l' \in \operatorname{codom} \Theta$, where

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$$(l', m') \quad = \quad [\![T]\!]_\pi(l, \bot\bot\ldots\bot)$$

For simplicity, we will suppose that $l_i = i$ initially (standard state) and thus write $[\![T]\!]_\pi$ instead of $[\![T]\!]_\pi(01\ldots(n-1), \bot\bot\ldots\bot)$.

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A more manageable setting

In order to study tasks which can be solved by protocols we should simplify as much as possible what we consider as

- protocols
- execution traces

Restricting executions

It can be shown that we can, without loss of generality, restrict to traces which are

- *well-bracketed*:

$$u_0\, u_1\, s_1\, u_2\, s_0\, s_2 \qquad \text{but not} \qquad u_0\, {\color{red}u_0\, s_1}\, s_0$$

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In particular, we have a notion of *round*.

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- *immediate snapshot*:

$$u_0 u_1 s_1 s_0 u_2 s_2 \qquad \text{but not} \qquad u_0 u_1 s_0 \, u_2 \, s_1 s_2$$

Full-information protocols

A protocol is **full-information** when

$$\pi_{u_i} \quad = \quad \mathrm{id}_{\mathcal{V}}$$

We can restrict to those without loss of generality (and we will).

A category of protocols

A **morphism** $\phi : \pi \to \pi'$ between protocols consists of functions

- $\phi_i : \mathcal{V} \to \mathcal{V}$ translating memory

such that

- $\phi_i(x) = x$ for $x \in \mathcal{I}$
- $\phi_i(x) \in \mathcal{O}$ for $x \in \mathcal{O}$
- and

$$\begin{array}{ccc} \mathcal{V} \times \mathcal{V}^n & \xrightarrow{\pi_{s_i}} & \mathcal{V} \\ {\scriptstyle \phi_i \times \prod_i \phi_i} \downarrow & & \downarrow {\scriptstyle \phi_i} \\ \mathcal{V} \times \mathcal{V}^n & \xrightarrow[\pi_{s_i}]{} & \mathcal{V} \end{array}$$

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We say that π' *simulates* π.

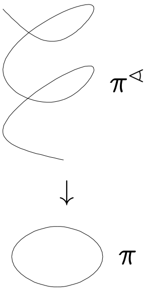
Actually, we only require ϕ_i and ϕ'_i to be defined on *reachable* values for a given task.

The view protocol

Theorem (GMT)

The category of protocols admits an initial object $\pi^{\lhd}$.

Morally, the space of executions of $\pi^{\lhd}$ is the "universal cover" of the space of executions of any process π: every execution of π corresponds to a unique execution of $\pi^{\lhd}$.



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The initial object $\pi^{\triangleleft}$ is called the **view protocol** and is defined by

- $\pi^{\triangleleft}_{u_i}(x) = x$ for $x \in \mathcal{V}$ (full-information),
- $\pi^{\triangleleft}_{s_i}(x, m) = \langle x, \langle m \rangle \rangle$ for $(x, m) \in \mathcal{V} \times \mathcal{V}^n$.

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Given a trace T, the local memory of i-th process after executing the trace T is called its **view**.

Theorem (GMT)

The category of protocols admits an initial object $\pi^{\triangleleft}$ *with* $\pi^{\triangleleft}_{s_i}(x, m) = \langle x, \langle m \rangle \rangle$.

Proof.

Suppose given a reachable memory

$$x = l_i \qquad \text{with} \qquad (l, m) = [\![T]\!]_{\pi^{\triangleleft}}$$

Because of the definition of morphisms, we are forced to define

$$\phi_i(x) = l'_i \qquad \text{with} \qquad (l', m') = [\![T]\!]_{\pi}$$

It only remains to check that this definition is well-defined, i.e. it does not depend on the chosen trace T... □

THE PROTOCOL COMPLEX

The protocol complex

Given a number r of rounds for each process, the **protocol complex** $\chi^r(\Theta)$ is the abstract simplicial complex whose

- vertices are $x \in \mathcal{V}$ such that x is the view (= local memory) of i-th process after executing a trace with $\pi^{\triangleleft}$
- simplices are sets of vertices occurring together after a same execution.

The protocol complex

Suppose that we have 2 processes and the input is the standard one:

0 ——————————— 1

The protocol complex $\chi^1(\Theta)$ for 1 round is as follows:

The protocol complex

Suppose that we have 2 processes and the input is the standard one:

$$0 \longrightarrow 1$$

The protocol complex $\chi^1(\Theta)$ for 1 round is as follows:

$$0, 0\bot \text{ —— } 1, 01$$

After executing 1 round for each process, we have the executions

- $u_0 s_0 u_1 s_1$:

0	1
$\bot$	$\bot$

$\xrightarrow{u_0}$

0	1
0	$\bot$

$\xrightarrow{s_0}$

$0, 0\bot$	1
0	$\bot$

$\xrightarrow{u_1}$

$0, 0\bot$	1
0	1

$\xrightarrow{s_1}$

$0, 0\bot$	$1, 01$
0	1

The protocol complex

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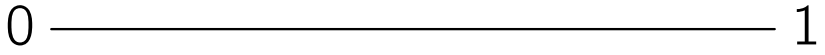

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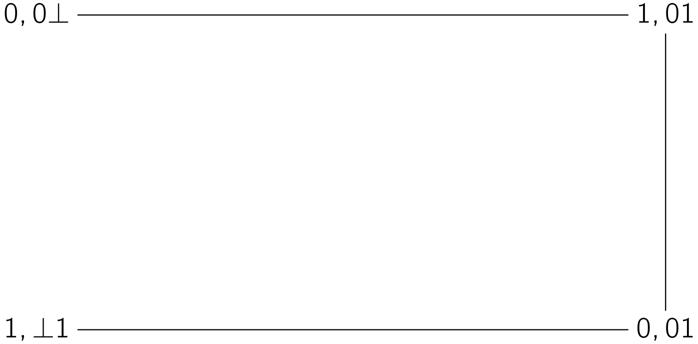

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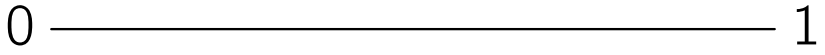

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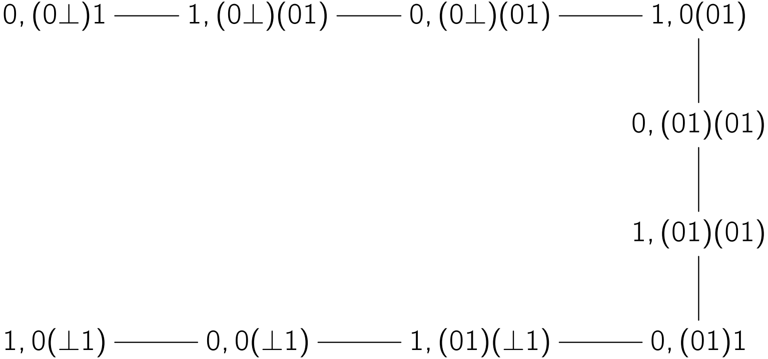

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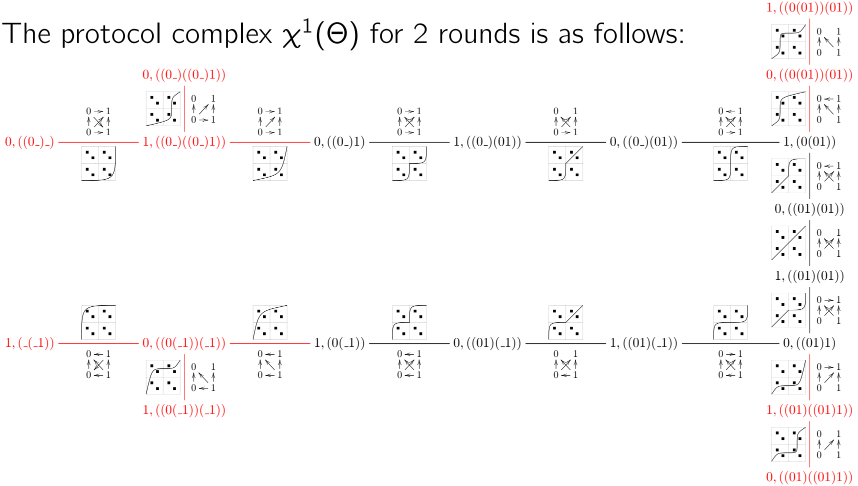
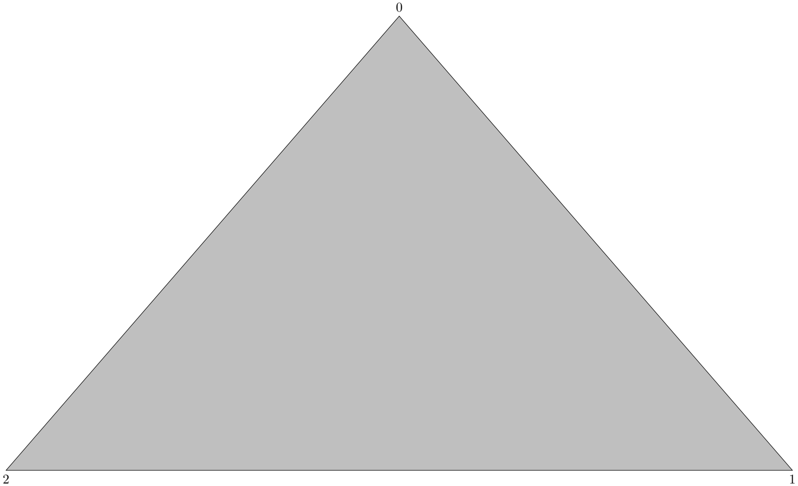


The protocol complex

With 3 processes and 1 one round, starting from the input complex



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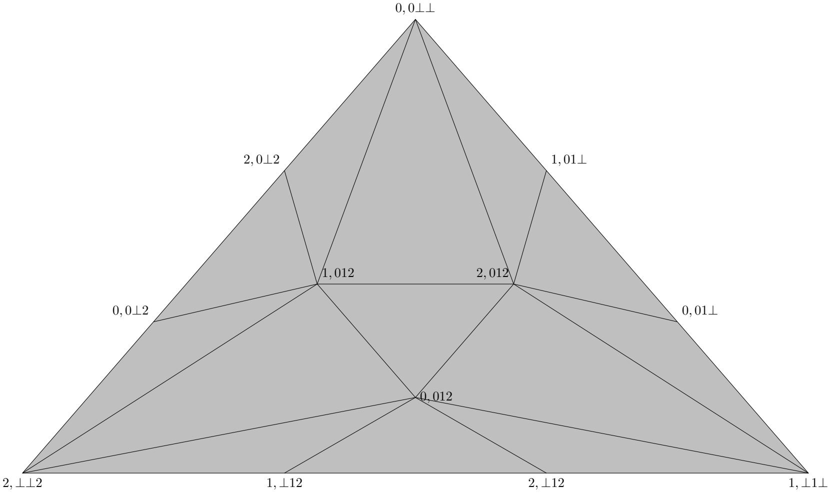

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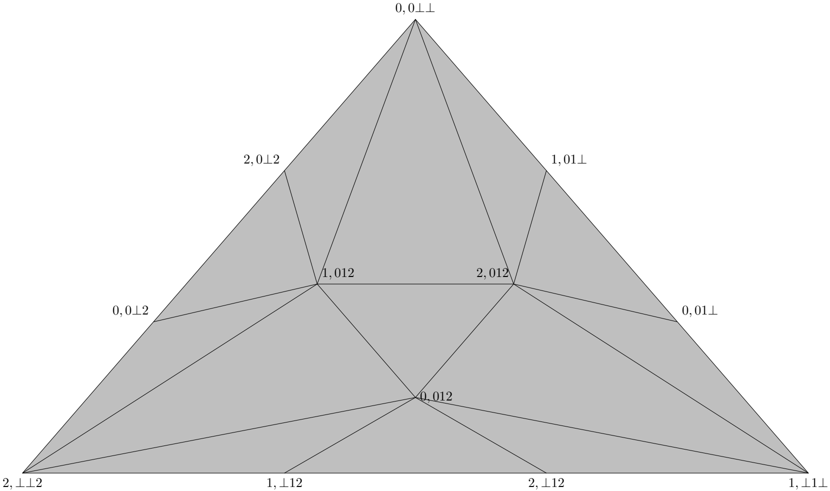

Notice that this is a particular subdivision of the original complex.

The chromatic subdivision

In general, the protocol complex on r rounds is obtained by

- starting from the input complex
- performing a **chromatic subdivision** of it r times

and this subdivision can be defined and studied independently.

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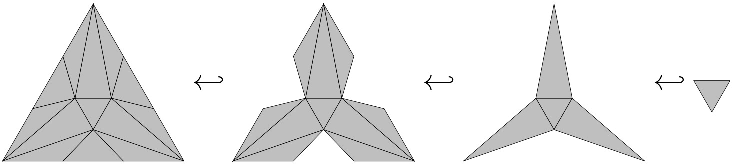
Theorem (Herliy-Shavit, GMT, Koszlov)

If the input complex is contractible then the protocol complex is (in fact, collapsible).



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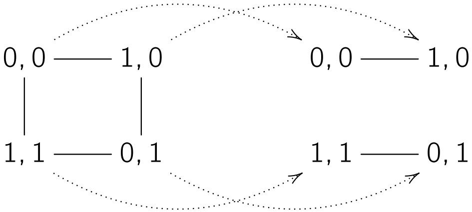
NB: simplicial maps preserve contractibility!

The binary consensus

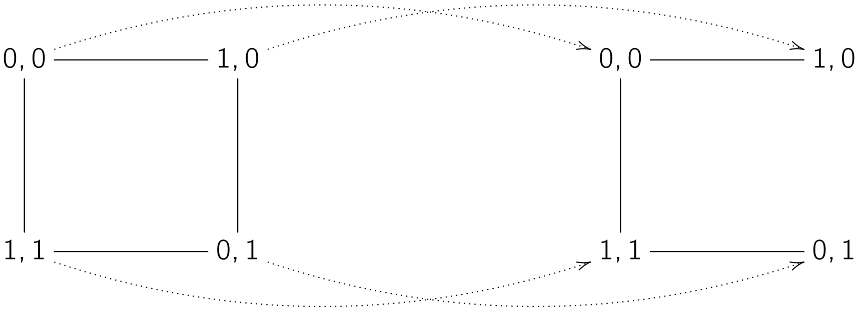
Consider again the **binary consensus** task:



There can be no protocol solving it (even after some rounds).

The binary quasi-consensus

Consider the **binary quasi-consensus**:



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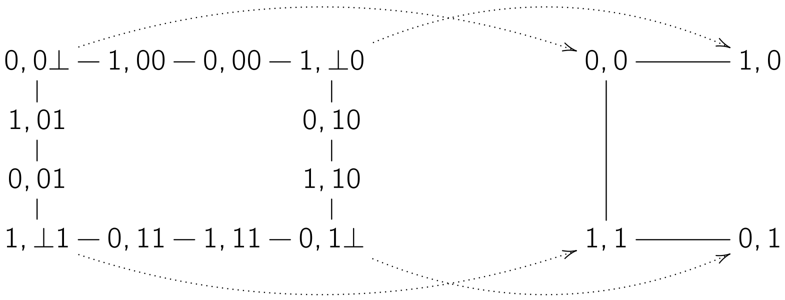

EQUIVALENCE BETWEEN TRACES

Execution traces

The (well-bracketed) execution traces in $\{u_i, s_i\}^*$ are semantically invariant under the congruence $\approx$ generated by

$$u_j u_i \approx u_i u_j \qquad\qquad s_j s_i \approx s_i s_j$$

which means that

$$T \approx T' \qquad \text{implies} \qquad [\![T]\!]_\pi = [\![T']\!]_\pi$$

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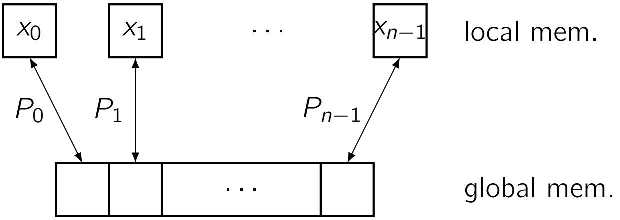

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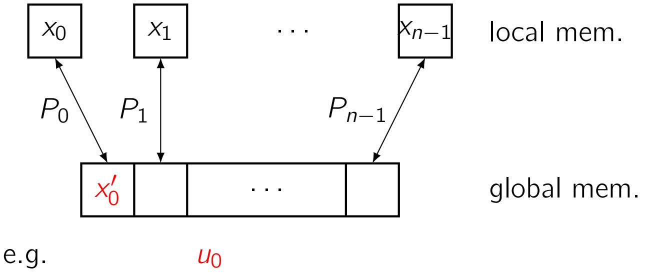

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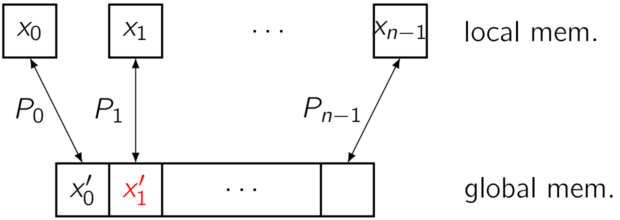

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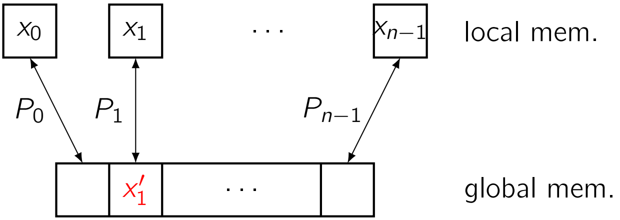

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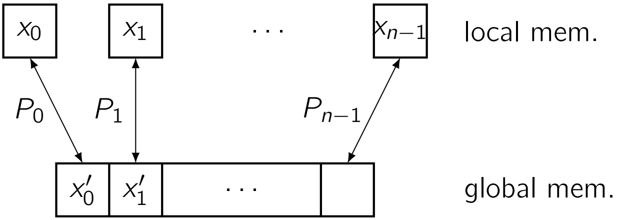
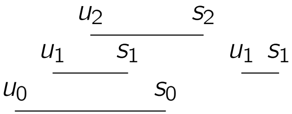


e.g. $\quad u_0 u_1 \approx u_1 u_0$

Interval orders

In a well-bracketed trace, the u_i and s_i form intervals:

$$u_0 u_1 u_2 s_1 s_0 s_2 u_1 s_1 \quad \rightsquigarrow$$



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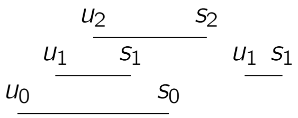

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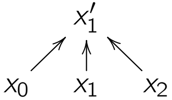

An **interval order** $(X, \preceq)$ is a poset such that there exists a function $I : X \to \wp(\mathbb{R})$ associating an interval I_x to each x in such a way that

$$x \prec y \qquad \text{if and only if} \qquad \forall s \in I_x, \forall t \in I_y,\ s < t$$

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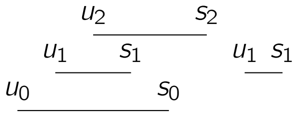

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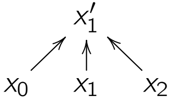

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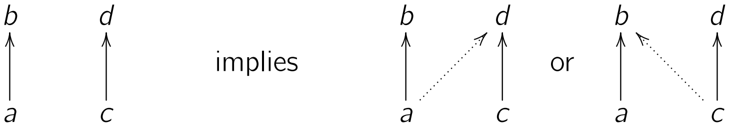
There is a *colored variant* with $\ell : X \to \mathbb{N}$ such that $\ell(x) = \ell(y)$ implies that x and y are comparable.

Remark (Fishburn)

A poset is an interval order if it is "$(2+2)$-free":

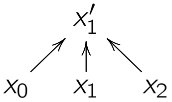


Interval orders

Theorem

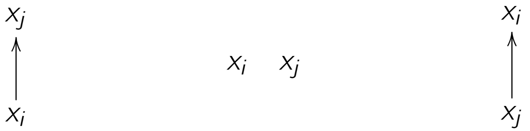
Well-bracketed traces up to equivalence are in bijection with colored interval orders.

$$u_0 u_1 u_2 s_1 s_0 s_2 u_1 s_1 \quad \rightsquigarrow$$



Views of interval orders

Suppose given two elements x_i and x_j of an interval order. We have the following possible situations:



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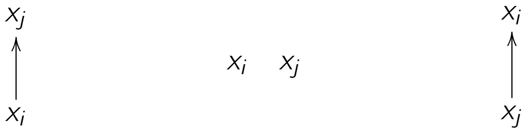

which correspond to the following traces:

$u_i s_i u_j s_j$ $\qquad$ $u_i u_j s_i s_j$ $\qquad$ $u_j s_j u_i s_i$

In the two first cases, s_j sees u_i.

Views of interval orders

This suggests defining the *i*-**view** of a colored interval order $(X, \preceq)$ by

1. restricting to elements which are below or independent from the maximum element x_i^k labeled by i
2. remove dependencies from x_i^k

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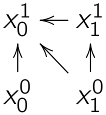
Theorem

- *an interval order can be reconstructed from all the i-views*
- *the execution of the i-th process in the view protocol* $\pi^{\lhd}$ *is uniquely determined by the i-view*

Views of interval orders

For instance, with two processes, consider $u_0 u_1 s_1 u_1 s_0 s_1 u_0 s_0$:

- it corresponds to the colored interval order



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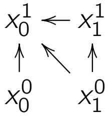
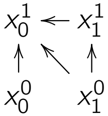
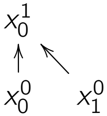


- the views are





Views of interval orders

For instance, with two processes, consider $u_0 u_1 s_1 u_1 s_0 s_1 u_0 s_0$:

0	1
$\bot$	$\bot$

$\xrightarrow{u_0}$

0	1
0	$\bot$

$\xrightarrow{u_1}$

0	1
0	1

$\xrightarrow{s_1}$

0	$\langle 1, 01 \rangle$
0	1

$\xrightarrow{u_1}$

0	$\langle 1, 01 \rangle$
0	$\langle 1, 01 \rangle$

$\xrightarrow{s_0}$

$\langle 0, 0\langle 1, 01 \rangle \rangle$	$\langle 1, 01 \rangle$
0	$\langle 1, 01 \rangle$

$\xrightarrow{s_1}$

$\langle 0, 0\langle 1, 01 \rangle \rangle$	$\langle \langle 1, 01 \rangle , 0\langle 1, 01 \rangle \rangle$
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$\xrightarrow{s_0}$

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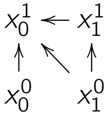
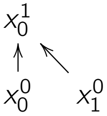
- we have a correspondence:



$$\langle\langle 0, 0\langle 1, 01\rangle\rangle\,, \langle 0, 0\langle 1, 01\rangle\rangle\,\langle 1, 01\rangle\rangle$$



$$\langle\langle 1, 01\rangle\,, 0\langle 1, 01\rangle\rangle$$

Completeness results

From this we deduce:

Theorem

The equivalence is complete: given two traces t and t'

$$t \approx t' \qquad \textit{iff} \qquad [\![t]\!]_{\pi^{\lhd}} = [\![t']\!]_{\pi^{\lhd}}$$

Theorem

$\pi^{\lhd}$ *is actually initial in the category of protocols.*

The interval order complex

Definition

The **interval order complex** is the simplicial complex whose

- *vertices* are (i, V_i) where V_i is an i-view
- *maximal simplices* are $\{(0, V_0), \dots, (n, V_n)\}$ such that there is an interval order $(X, \prec)$ (with given number of rounds) whose i-view is V_i.

Theorem

The interval order complex is isomorphic to the protocol complex.

DIRECTED GEOMETRIC SEMANTICS

Directed geometric semantics

The idea of geometric semantics is to formalize the dictionary:

program	$\Leftrightarrow$	**topological space**
state	$\Leftrightarrow$	point of the space
execution trace	$\Leftrightarrow$	path
equivalent traces	$\Leftrightarrow$	homotopic paths

so that we can import tools from (algebraic) topology in order to study concurrent programs.

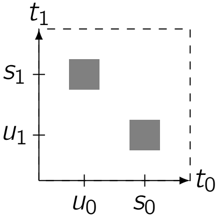
We actually need to use spaces equipped with a notion of direction in order to take in account irreversible time.

An example

Consider two processes executing one round of update/scan, i.e.

$$u_0.s_0 \quad \| \quad u_1.s_1$$

The geometric semantics of this program will be



i.e. a square $[0, 1] \times [0, 1]$ minus two holes, which is directed componentwise.

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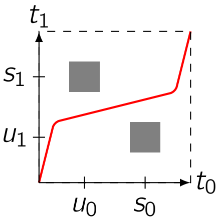

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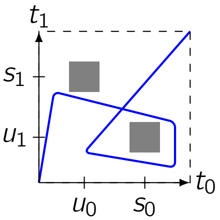
directed path : $u_1 u_0 s_0 s_1$

An example

Consider two processes executing one round of update/scan, i.e.

$$u_0.s_0 \quad \| \quad u_1.s_1$$

The geometric semantics of this program will be



i.e. a square $[0,1] \times [0,1]$ minus two holes, which is directed componentwise.

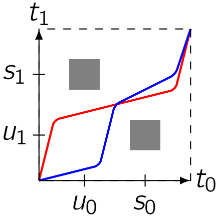
non directed path : ???

An example

Consider two processes executing one round of update/scan, i.e.

$$u_0.s_0 \quad \| \quad u_1.s_1$$

The geometric semantics of this program will be



i.e. a square $[0,1] \times [0,1]$ minus two holes, which is directed componentwise.

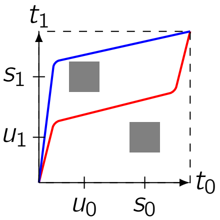
homotopy between paths : $u_1 u_0 s_0 s_1 \approx u_0 u_1 s_0 s_1$

An example

Consider two processes executing one round of update/scan, i.e.

$$u_0.s_0 \quad \| \quad u_1.s_1$$

The geometric semantics of this program will be



i.e. a square $[0,1] \times [0,1]$ minus two holes, which is directed componentwise.

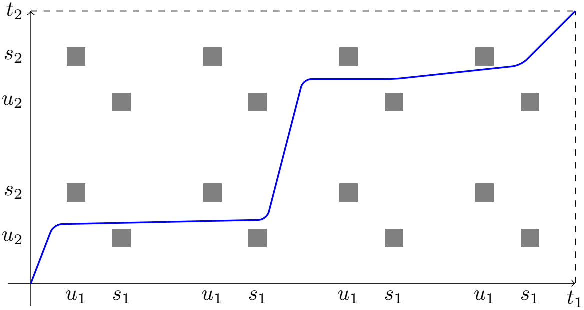
some paths are not homotopic

More examples

This generalizes to *more rounds*:
consider two processes executing 2 and 4 rounds of update/scan,

$$u_0.s_0.u_0.s_0 \quad \| \quad u_1.s_1.u_1.s_1.u_1.s_1.u_1.s_1$$

The geometric semantics of this program will be

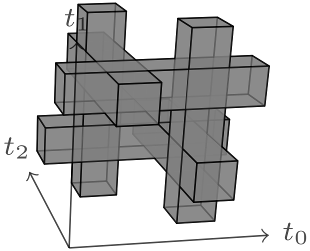


More examples

This generalizes to *more processes*:
consider three processes executing one round of update/scan,

$$u_0.s_0 \quad \| \quad u_1.s_1 \quad \| \quad u_2.s_2$$

The geometric semantics of this program will be



NB: we will illustrate in dimension 2, where things are simpler

Directed spaces

Formally,

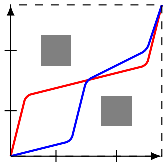
Definition

A **pospace** $(X, \leq)$ consists of a topological space X equipped with a partial order $\leq \subseteq X \times X$, which is closed.

A **dipath** p is a continuous non-decreasing map $p : [0,1] \to X$.

A **dihomotopy** H from a path p to a path q is a continuous map $H : [0,1] \times [0,1] \to X$ such that

- $H(0,t) = p(t)$ for every t
- $H(1,t) = q(t)$ for every t
- $t \mapsto H(s,t)$ is a dipath for every s
- $s \mapsto H(s,0)$ and $s \mapsto H(s,1)$ are constant



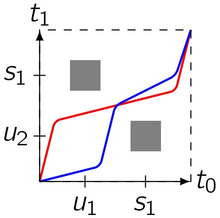
Directed paths vs traces

Theorem

Fixing a number of rounds for each process, there is a bijection between

(i) *directed paths up to directed homotopy in the geometric semantics*

(iii) *execution traces up to* $\approx$



$\Leftrightarrow$ $\quad u_1\, u_0\, s_0\, s_1 \approx u_0\, u_1\, s_0\, s_1$

Directed paths vs traces

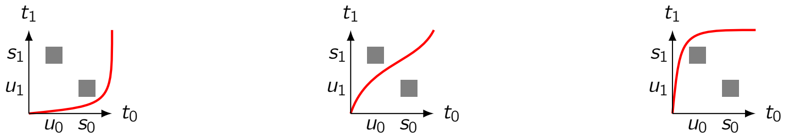
Theorem

Fixing a number of rounds for each process, there is a bijection between

(i) *directed paths up to directed homotopy in the geometric semantics*

(ii) *colored interval orders*

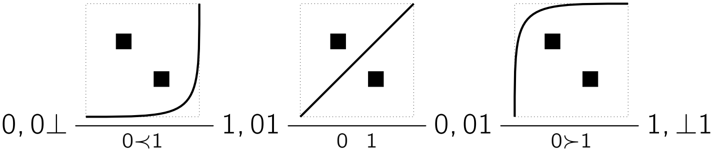
(iii) *execution traces up to* $\approx$



$[u_0, s_0] \prec [u_1, s_1]$ $\qquad$ $[u_0, s_0] \parallel [u_1, s_1]$ $\qquad$ $[u_0, s_0] \succ [u_1, s_1]$

From geometry to the complex

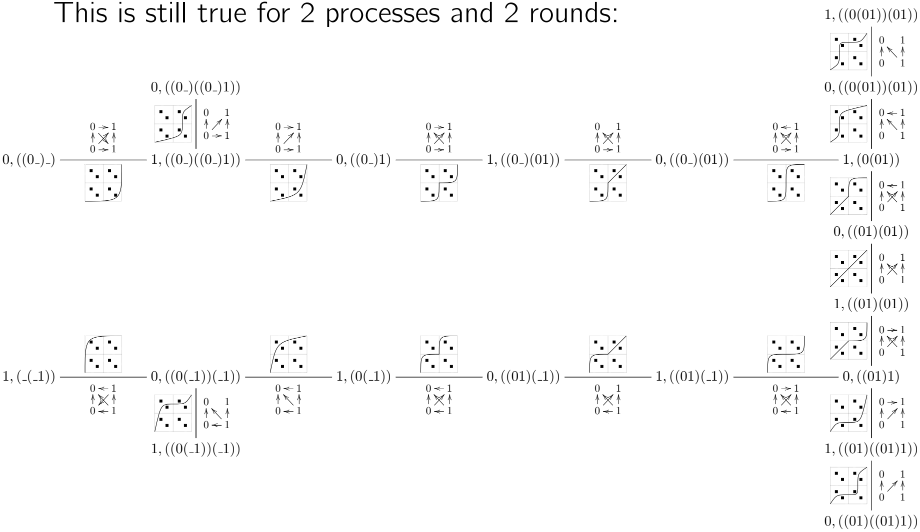
One can notice in the last example that edges are in bijection with directed paths up to homotopy (and with interval orders):



(more generally maximal simplices are in bijection with maximal directed paths up to homotopy).

From geometry to the complex

This is still true for 2 processes and 2 rounds:



CONCLUSION

Perspectives

- Links with game stuff?

async. comp.	game semantics
update	question
scan	answer
view	view :)
interval order	event structure
trace up to equivalence	asynchronous transition system

- Links with quantum stuff?

async. comp.	quantum
...	...

- Any question?