Categorical coherence from term rewriting systems

Samuel Mimram
LIX, École polytechnique

Abstract

The celebrated Squier theorem allows to prove coherence properties of algebraic structures, such as MacLane's coherence theorem for monoidal categories, based on rewriting techniques. We are interested here in extending the theory and associated tools simultaneously in two directions. Firstly, we want to take in account situations where coherence is partial, in the sense that it only applies for a subset of structural morphisms (for instance, in the case of the coherence theorem for symmetric monoidal categories, we do not want to strictify the symmetry). Secondly, we are interested in structures where variables can be duplicated or erased. We develop theorems and rewriting techniques in order to achieve this, first in the setting of abstract rewriting systems, and then extend them to term rewriting systems, suitably generalized in order to take coherence in account. As an illustration of our results, we explain how to recover the coherence theorems for monoidal and symmetric monoidal categories.

2012 ACM Subject Classification Theory of computation → Rewrite systems

Keywords and phrases coherence, rewriting system, Lawvere theory

1 Introduction

A *monoidal category* consists of a category C equipped with a tensor bifunctor $\otimes : C \times C \to C$ and unit element $e : 1 \to C$ together with natural isomorphisms $\alpha_{x,y,z} : (x \otimes y) \otimes z \to x \otimes (y \otimes z)$, $\lambda_x : e \otimes x \to x$ and $\rho_x : x \otimes e \to x$, satisfying two well-known axioms. Thanks to these, the way tensor expressions are bracketed does not really matter: we can always rebracket expressions using the structural morphisms (α, λ and ρ), and any two ways of rebracketing an expression into the other are equal. In fact, there are various ways to formalize this [1]:

(C1) Every diagram in a free monoidal category made up of α, λ and ρ commutes [17, Corollary 1.6], [26, Theorem VI.2.1].

(C2) Every diagram in a monoidal category made up of α, λ and ρ commutes [27, Theorem 3.1], [26, Theorem XI.3.2].

(C3) Every monoidal category is monoidally equivalent to a strict monoidal category [17, Corollary 1.4], [26, Theorem XI.3.1].

(C4) The forgetful 2-functor from strict monoidal categories to monoidal categories has a left adjoint and the components of the unit are equivalences.

Condition (C1) implies (C2) as a particular case and the converse implication can also be shown. Condition (C4) implies (C3) as a particular case, and it can be shown that (C3) in turn implies (C2). Analogous statements hold for symmetric monoidal categories (monoidal categories equipped with a suitable symmetry $\gamma_{x,y} : x \otimes y \to y \otimes x$) although they are more subtle [2]: in (C2), we have to suppose that the diagrams are "generic enough", and in (C4) the notion of strict symmetric monoidal category does not impose that the symmetry should be an identity.

We first investigate here (in Section 2) an abstract version of this situation and formally compare the various coherence theorems: we show that quotienting a theory by a subtheory $\mathcal{W}$ gives rise to an equivalent theory if and only if $\mathcal{W}$ is coherent (or rigid), in the sense that all diagrams commute (Theorem 6). Moreover, this is the case if and only if they give rise to equivalent categories of algebras (Proposition 9), which can be thought of as a strengthened version of (C4). We also provide rewriting conditions which allow showing coherence in practice (Proposition 13). The idea of extending rewriting theory in order to take coherence in account dates back to pioneering work from people such as Power [30], Street [34] and Squier [32]. It has been generalized in higher dimensions in the context of polygraphs [33, 8], as well as homotopy type theory [19], and used to recover various coherence theorem [22, 13].

We then extend (in Section 3) our results to the 2-dimensional cartesian theories, which are able to axiomatize (symmetric) monoidal categories. Our work is based on the notion of Lawvere 2-theory [12, 35, 36], and unfortunately lead us to discover an important flaw in a main result about those [36]. The rewriting counterpart is based on a coherent extension of term rewriting systems, following [10, 5, 28]. One of the main novelties here consists in allowing for coherence with respect to a sub-theory (which is required to handle coherence for symmetric monoidal categories), building on recent works in order to work in structures modulo substructures [9, 29, 11].

2 Relative coherence and abstract rewriting systems

2.1 Quotient of categories

Suppose fixed a category $\mathcal{C}$ together with a set W of isomorphisms of $\mathcal{C}$. Although the situation is very generic, and the following explanation is only vague for now, it can be helpful to think of $\mathcal{C}$ as a theory describing a structure a category can possess and W as the morphisms we are interested in strictifying. For instance, if we are interested in the coherence theorem for symmetric monoidal categories, we can think of the objects of $\mathcal{C}$ as formal iterated tensor products, the morphisms of $\mathcal{C}$ as composites of α, λ, ρ and γ, and we would typically take W as consisting of all instances of α, λ and ρ (but not γ). This will be made formal in Section 3.

A functor $F : \mathcal{C} \to \mathcal{D}$ is W*-strict* when it sends every morphism of W to an identity. We write $\mathcal{C}/W$ for the *quotient* of $\mathcal{C}$ under W: this is the category equipped with a W-strict functor $\mathcal{C} \to \mathcal{C}/W$ such that any W-strict functor $F : \mathcal{C} \to \mathcal{D}$ extends uniquely as a functor $\tilde{F} : \mathcal{C}/W \to \mathcal{D}$. We write $\mathcal{W}$ for the subcategory of $\mathcal{C}$ generated by W (which we assimilate to a subset of the morphisms of $\mathcal{C}$). This is always a groupoid (a category in which every morphism is invertible) and it is easily shown that $\mathcal{C}/\mathcal{W} \cong \mathcal{C}/W$, so that we can always suppose that we quotient by a subgroupoid. Moreover, we can always suppose that this subgroupoid has the same objects as $\mathcal{C}$ (we can add all identities in it without changing the quotient).

We say that a groupoid $\mathcal{W}$ is *rigid* when any two morphisms $f, g : x \to y$ which are parallel (i.e. have the same source, and have the same target) are necessarily equal. Such a groupoid can be thought of as a “coherent” sub-theory of $\mathcal{C}$. It does not have non-trivial geometric structure in the following sense:

▶ **Lemma 1.** *A groupoid $\mathcal{W}$ is rigid if and only if either*
(i) *identities are the only automorphisms of $\mathcal{W}$,*
(ii) *$\mathcal{W}$ is a “set”, i.e. is equivalent to a coproduct of instances of the terminal category.*

The fact that $\mathcal{W} \subseteq \mathcal{C}$ is rigid is thought here as the fact that coherence condition (C1) holds for $\mathcal{C}$, relatively to $\mathcal{W}$.

General notions of quotients of categories are not trivial to construct (see for instance [4]), but in the case of rigid categories, we have the following simple description.

▶ **Proposition 2.** *When $\mathcal{W} \subseteq \mathcal{C}$ is rigid, the quotient category $\mathcal{C}/\mathcal{W}$ is isomorphic to the category where*

- *objects are equivalence classes $\mathcal{C}_0/\mathcal{W}$ of objects of $\mathcal{C}$ under the equivalence relation $\sim$ such that $x \sim y$ whenever there exists $w : x \to y$ in $\mathcal{W}$ (we write $[x]$ for the class of an object x),*

- *a morphism is of the form* $[f] : [x] \to [y]$ *for some morphism* $f : x \to y$ *of* $\mathcal{C}$*, under the equivalence relation such that* $f \sim f'$ *whenever there exists* v *and* w *in* $\mathcal{W}$ *such that* $w \circ f = f' \circ v$*,*
- *given* $f : x \to y$ *and* $g : y' \to z$ *with* $[y] = [y']$*, the composition is* $[g] \circ [f] = [g \circ w \circ f]$ *for the "mediating" morphism* $w : y \to y'$ *in* $\mathcal{W}$ *(uniquely determined by rigidity of* $\mathcal{W}$*),*
- *given* $x \in \mathcal{C}$*, the identity is* $\mathrm{id}_{[x]} = [\mathrm{id}_x]$*.*

When $\mathcal{W} \subseteq \mathcal{C}$ is not rigid, we can have a similar description, but we now have the choice between multiple mediating morphisms in the definition of the composition, and all the resulting composites in fact have to be identified in the quotient. This observation suggests that the construction of the quotient category $\mathcal{C}/\mathcal{W}$, when $\mathcal{W}$ is not rigid, is better described in two steps: we first formally make $\mathcal{W}$ rigid, and then apply Proposition 2. We say that a functor $F : \mathcal{C} \to \mathcal{D}$ is *$\mathcal{W}$-rigid* when any two parallel morphisms of $\mathcal{W}$ have the same image. The *$\mathcal{W}$-rigidification* of $\mathcal{C}$ is the category $\mathcal{C}/\!\!/\mathcal{W}$ equipped with a $\mathcal{W}$-rigid functor $\mathcal{C} \to \mathcal{C}/\!\!/\mathcal{W}$ such that any $\mathcal{W}$-rigid functor $F : \mathcal{C} \to \mathcal{D}$ extends uniquely as a functor $\mathcal{C}/\!\!/\mathcal{W} \to \mathcal{D}$.

▶ **Lemma 3.** *The category* $\mathcal{C}/\!\!/\mathcal{W}$ *is the category obtained from* $\mathcal{C}$ *by quotienting morphisms under the smallest congruence (wrt composition) identifying any two parallel morphisms of* $\mathcal{W}$*.*

▶ **Proposition 4.** *The quotient* $\mathcal{C}/\mathcal{W}$ *is isomorphic to* $(\mathcal{C}/\!\!/\mathcal{W})/\tilde{\mathcal{W}}$ *where* $\tilde{\mathcal{W}}$ *is the set of equivalence classes of morphisms in* $\mathcal{W}$ *under the equivalence relation of Lemma 3.*

Proof. Follows directly from the universal properties of the quotient and the rigidification, and the fact that any $\mathcal{W}$-strict functor is $\mathcal{W}$-rigid. ◀

A consequence of the preceding explicit description of the quotient is the following:

▶ **Lemma 5.** *The quotient functor* $\mathcal{C} \to \mathcal{C}/\mathcal{W}$ *is surjective on objects and full.*

Proof. By Proposition 4, the quotient functor is the composite of the quotient functors $\mathcal{C} \to \mathcal{C}/\!\!/\mathcal{W} \to \mathcal{C}/\mathcal{W}$. The first one is surjective on objects and full by Lemma 3 and the second one is surjective on objects and full by Proposition 2. ◀

This entails the following theorem, which is the main result of the section. Its meaning can be explained by taking the point of view given above: thinking of $\mathcal{C}$ as describing a structure and of $\mathcal{W}$ as a part of the structure we want to strictify, the structure is equivalent to its strict variant if and only if the quotiented structure does not itself bear non-trivial geometry (in the sense of Lemma 1).

▶ **Theorem 6.** *Suppose that* $\mathcal{W}$ *is a subgroupoid of* $\mathcal{C}$*. The quotient functor* $[-] : \mathcal{C} \to \mathcal{C}/\mathcal{W}$ *is an equivalence of categories if and only if* $\mathcal{W}$ *is rigid.*

Proof. Since the quotient functor is always surjective and full by Lemma 5, it remains to show that it is faithful if and only if $\mathcal{W}$ is rigid. Suppose that the quotient functor is faithful. Given $w, w' : x \to y$ in $\mathcal{W}$, by Lemma 3 and Proposition 4 we have $[w] = [w']$ and thus $w = w'$ by faithfulness. Suppose that $\mathcal{W}$ is rigid. The category $\mathcal{C}/\mathcal{W}$ then admits the description given in Proposition 2. Given $f, g : x \to y$ in $\mathcal{C}$ such that $[f] = [g]$, there is $v : x \to x$ and $w : y \to y$ such that $w \circ f = g \circ v$. By rigidity, both v and w are identities and thus $f = g$. ◀

▶ **Example 7.** As a simple example, consider the groupoid $\mathcal{C}$ freely generated by the graph $x \overset{f}{\underset{g}{\rightrightarrows}} y$. The subgroupoid generated by $W = \{g\}$ is rigid, so that $\mathcal{C}$ is equivalent to the quotient category $\mathcal{C}/W$, which is the groupoid generated by $x \circlearrowleft f$. However, the groupoid generated by $W = \{f, g\}$ is not rigid (since we don't have $f = g$). And indeed, $\mathcal{C}$ is not equivalent to the quotient category $\mathcal{C}/W$, which is the terminal category.

2.2 Coherence for algebras

Given a category $\mathcal{C}$, we consider here a functor $\mathcal{C} \to \mathbf{Cat}$ as an *algebra* for $\mathcal{C}$. Namely, if we think of the category $\mathcal{C}$ as describing an algebraic structure (e.g. the one of monoidal categories), an algebra can be thought of as a category actually possessing this structure (an actual monoidal category).

We write $\mathrm{Alg}(\mathcal{C})$ for the category of algebras, with natural transformations as morphisms. Any functor $F : \mathcal{C} \to \mathcal{C}'$ induces, by precomposition, a functor $\mathrm{Alg}(F) : \mathrm{Alg}(\mathcal{C}') \to \mathrm{Alg}(\mathcal{C})$. We can characterize situations where two categories give rise to the same algebras:

▶ **Proposition 8.** *Suppose given a functor $F : \mathcal{C} \to \mathcal{C}'$ between categories. The functor F is an equivalence if and only if the induced functor $\mathrm{Alg}(F) : \mathrm{Alg}(\mathcal{C}') \to \mathrm{Alg}(\mathcal{C})$ is an equivalence.*

Proof. Given a 2-category $\mathcal{K}$, one can define a Yoneda functor $Y_\mathcal{K} : \mathcal{K}^{\mathrm{op}} \to [\mathcal{K}, \mathbf{Cat}]$, where $\mathbf{Cat}$ is the 2-category of categories, functors and natural transformations, and $[\mathcal{K}, \mathbf{Cat}]$ denotes the 2-category of 2-functors $\mathcal{K} \to \mathbf{Cat}$, transformations and modifications. In particular, given 0-cells $x \in \mathcal{K}^{\mathrm{op}}$ and $y \in \mathcal{K}$, we have $Y_\mathcal{K} xy = \mathcal{K}(x, y)$. The Yoneda lemma states that this functor is a local isomorphism (this is a particular case of the Yoneda lemma for bicategories detailed for instance in [16, chapter 8]). In particular, taking $\mathcal{K} = \mathbf{Cat}$ (and ignoring size issues), the Yoneda functor sends a category $\mathcal{C} \in \mathcal{K}^{\mathrm{op}}$ to $Y_\mathcal{K}\mathcal{C} = \mathrm{Alg}(\mathcal{C})$, and the result follows from the Yoneda lemma. ◀

As a particular application, given a category $\mathcal{C}$ and a subgroupoid $\mathcal{W}$, we have by Theorem 6 that $\mathcal{W}$ is rigid if and only if the quotient functor $\mathcal{C} \to \mathcal{C}/\mathcal{W}$ is an equivalence. By Proposition 8, we thus have the following property which can be interpreted as the equivalence of coherence conditions (C1) and a strengthened variant of (C4).

▶ **Proposition 9.** *Given a category $\mathcal{C}$ and a subgroupoid $\mathcal{W}$, the morphism $\mathrm{Alg}(\mathcal{C}/\mathcal{W}) \to \mathrm{Alg}(\mathcal{C})$ induced by the quotient functor is an equivalence of categories if and only if $\mathcal{W}$ is rigid.*

2.3 Coherent abstract rewriting systems

We now explain how the theory rewriting can be used to show the rigidity of a groupoid in practice. In the same way the theory of rewriting can be studied abstractly [15, 3, 6], i.e. without taking in consideration the structure of the objects getting rewritten, we first develop the coherence theorems of interest in this article in an abstract setting. Although the terminology is different, the formalization given here is based on the notion of polygraph [33, 8].

Extended abstract rewriting systems. An *abstract rewriting system*, or ARS, $\mathsf{P} = (\mathsf{P}_0, s_0, t_0, \mathsf{P}_1)$ consists of a set P_0, a set P_1 and two functions $s_0, t_0 : \mathsf{P}_1 \to \mathsf{P}_0$. The elements of P_0 are generally thought as the *objects* of interest, the elements of P_1 as *rewriting rules*, and the function s_0 (resp. t_0) associating to a rewriting rule its *source* (resp. *target*). We write $a : x \to y$ for a rewriting rule a with $s_0(a) = x$ and $t_0(a) = y$. We write P_1^* for the set of *rewriting paths* in the ARS: its elements are (possibly empty) finite sequences $a_1, \ldots, a_n$ of rewriting steps, which are composable in the sense that $t_0(a_i) = s_0(a_{i+1})$ for $1 \leq i < n$. The source (resp. target) of such a rewriting path is $s_0(a_1)$ (resp. $t_0(a_n)$); we sometimes write $p : x \xrightarrow{*} y$ to indicate that p is a rewriting path with x as source and y as target. Given two composable paths $p : x \xrightarrow{*} y$ and $q : y \xrightarrow{*} z$, we write $p \cdot q$ for their concatenation.

A morphism $f : \mathsf{P} \to \mathsf{Q}$ of ARS is a pair of functions $f_0 : \mathsf{P}_0 \to \mathsf{Q}_0$ and $f_1 : \mathsf{P}_1 \to \mathsf{Q}_1$ such that $s_0 \circ f_1 = f_0 \circ s_0$ and $t_0 \circ f_1 = f_0 \circ t_0$, and we write $\mathbf{Pol}_1$ for the resulting category. There is a forgetful functor $\mathbf{Cat} \to \mathbf{Pol}_1$, sending a category C to the ARS whose objects are those of C and whose rewriting steps are the morphisms of C. This functor admits a left adjoint

$-^* : \mathbf{Pol}_1 \to \mathbf{Cat}$ sending an ARS to the category with P_0 as objects and P_1^* as morphisms (composition is given by concatenation of paths and identities are the empty paths).

As a variant of the preceding situation, we can consider the forgetful functor $\mathbf{Gpd} \to \mathbf{Pol}_1$, from the category of groupoids. It also admits a left adjoint $-^\sim : \mathbf{Pol}_1 \to \mathbf{Gpd}$, and we write $\mathsf{P}_1^\sim$ for the set of morphisms of the groupoid generated by an ARS. The elements of P_1 are *rewriting zig-zags* in the ARS: they consist in finite sequences $a_1^{\epsilon_1}, \ldots, a_n^{\epsilon_n}$ with $a_i \in \mathsf{P}_1$ and $\epsilon_i \in \{-,+\}$ for $1 \leq i \leq n$, which are

- composable: $t_0(a_i^{\epsilon_i}) = s_0(a_{i+1}^{\epsilon_{i+1}})$ for $1 \leq i < n$,
  by convention $s_0(a_i^+) = s_0(a_i)$, $t_0(a_i^+) = t_0(a_i)$, $s_0(a_i^-) = t_0(a_i)$, $t_0(a_i^-) = s_0(a_i)$, and
- reduced: if $a_i = a_{i+1}$ then $\epsilon_i = \epsilon_{i+1}$ for $1 \leq i < n$.

The intuition is that a zig-zag is a "non-directed" rewriting path, consisting of rewriting steps, some of which are taken backward (i.e. formally inverted: those for which the exponent is "$-$"). The source (resp. target) of a zig-zag as above is $s_0(a_1^{\epsilon_1})$ (resp. $t_0(a_n^{\epsilon_n})$) and we write $p : x \overset{\sim}{\to} y$ to indicate that p is a zig-zag from x to y. Composition $p \cdot q$ of composable zig-zags $p : x \overset{\sim}{\to} y$ and $q : y \overset{\sim}{\to} y$ is given by taking their concatenation and iteratively removing the subpaths of the form $a^- \cdot a^+$ or $a^+ \cdot a^-$ at the interface, which ensures that the composite is reduced. Given a zig-zag p, we write p^- for its inverse, obtained by inverting the polarity of the exponents in p (we exchange "+" and "−"): it satisfies $p \cdot p^- = \mathrm{id}$ and $p^- \cdot p = \mathrm{id}$, where id denotes an empty zig-zag. Note that there is a canonical inclusion $\mathsf{P}_1^* \to \mathsf{P}_1^\sim$, which adds a "+" exponent to every step of a rewriting path, witnessing for the fact that rewriting paths are particular zig-zags.

An *extended abstract rewriting system*, or 2-ARS, P consists of an ARS as above, together with a set P_2 and two functions $s_1, t_1 : \mathsf{P}_2 \to \mathsf{P}_1^\sim$, such that $s_0 \circ s_1 = s_0 \circ t_1$ and $t_0 \circ s_1 = t_0 \circ t_1$. The elements of P_2 are *coherence relations* and the functions respectively describe their source and target (which are rewriting paths). We sometimes write $A : p \Rightarrow q$ to indicate that $A \in \mathsf{P}_2$ admits p (resp. q) as source (resp. target), which can be thought of as a 2-cell $x \underset{q}{\overset{p}{\Downarrow A}} y$ where x (resp. y) is the common source (resp. target) of p and q. The notion of 2-ARS is a groupoidal variant of the one of 2-*computad* [33] aka 2-*polygraph* [8], which generalizes in arbitrary dimension. The *groupoid presented* by a 2-ARS P, denoted by $\overline{\mathsf{P}}$, is the groupoid obtained from the free groupoid generated by the underlying ARS by quotienting morphisms under the smallest congruence identifying the source and the target of any element of P_2. The groupoid $\overline{\mathsf{P}}$ thus has P_0 as set of objects, the set $\mathsf{P}_1^\sim$ of rewriting zig-zags as morphisms, quotiented by the smallest equivalence relation $\equiv$ such that $p \cdot q \cdot r \equiv p \cdot q' \cdot r$ for every rewriting zig-zags p and r and coherence relation $A : q \Rightarrow q'$, which are suitably composable. Given a rewriting zig-zag $p \in \mathsf{P}_1^\sim$, we write $\overline{p}$ for the corresponding morphism in $\overline{\mathsf{P}}$ (i.e. its equivalence class under $\equiv$).

Rewriting properties. Now, suppose fixed a 2-ARS P together with a set $W \subseteq \mathsf{P}_1$. We can think of W as inducing a rewriting subsystem W of P, with P_0 as objects, W as rewriting steps and $\mathsf{W}_2 = \{A \in \mathsf{P}_2 \mid s_1(A) \in W^* \text{ and } t_1(A) \in W^*\}$ as coherence relations, and formulate the various traditional rewriting concepts with respect to it. We are considering here the quotient of a category $\mathcal{C} = \overline{\mathsf{P}}$ presented by a 2-ARS P by the subgroupoid $\mathcal{W}$ generated by W, with the aim of showing coherence results wrt the strictification of $\mathcal{W}$ as previously.

We say that P is *W-terminating* if there is no infinite sequence $a_1, a_2, \ldots$ of elements of W such that every finite prefix is a rewriting path (i.e. belongs to W^*). An element $x \in \mathsf{P}_0$ is a *W-normal form* when there is no rewriting step in W with x as source. We say that P is *weakly W-normalizing* when for every $x \in \mathsf{P}_0$ there exists a normal form $\hat{x}$ and a rewriting path $n_x : x \overset{*}{\to} \hat{x}$. We necessarily have $\hat{\hat{x}} = \hat{x}$ and we always suppose that $n_{\hat{x}} = \mathrm{id}_{\hat{x}}$.

▶ **Lemma 10.** *If* P *is* W*-terminating then it is weakly* W*-normalizing.*

Proof. Traditional rewriting argument: a maximal path (wrt prefix order) starting from x exists (because W is terminating) and its target is necessarily a normal form. ◀

A W*-branching* is a pair of rewriting steps $a_1 : x \to y_1$ and $a_2 : x \to y_2$ in W which are coinitial, i.e. have the same source. Such a branching is *confluent* when there is a pair of cofinal (with the same target) rewriting paths $p_1 : y_1 \to z$ and $p_2 : y_2 \to z$ in W^* such that $\overline{a_1 \cdot p_1} = \overline{a_2 \cdot p_2}$ (as morphisms of $\overline{\mathsf{P}}$, or, equivalently, of $\overline{\mathcal{W}}$). We say that P is *locally* W*-confluent* when W-branching is confluent. This condition is in particular satisfied when there exists a coherence relation $A : a_1 \cdot p_1 \Rightarrow a_2 \cdot p_2$, or $A : a_2 \cdot p_2 \Rightarrow a_1 \cdot p_1$ in P_2. Note that, here, not only we require that we can close a span of rewriting steps by a cospan of rewriting paths (as in the traditional definition of confluence), but also that the confluence square can be filled coherence relations. Similarly, P is W*-confluent* when for every $p_1 : x \overset{*}{\to} y_1$ and $p_2 : x \overset{*}{\to} y_2$ in W^*, there exist $q_1 : y_1 \overset{*}{\to} z$ and $q_2 : y_2 \overset{*}{\to} z$ in W^* such that $\overline{p_1 \cdot q_1} = \overline{p_2 \cdot q_2}$. We say that P is W*-convergent* when it is both W-terminating and W-confluent.

The celebrated Newman's lemma (also sometimes called the diamond lemma) along with its traditional proof [6, Theorem 1.2.1 (ii)] easily generalizes to our setting:

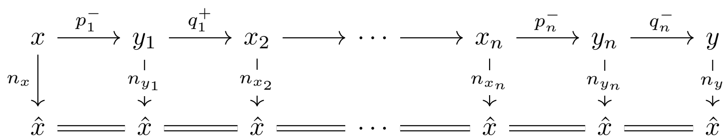
▶ **Proposition 11.** *If* P *is* W*-terminating and locally* W*-confluent then it is* W*-confluent.*

Proof. Classical argument, by well-founded induction on x, using local W-confluence. ◀

We say that P is W*-coherent* if for any parallel paths $p, q : x \overset{\sim}{\to} y$ in $W^\sim$, we have $\overline{p} = \overline{q}$. In other words, P is W-coherent precisely when $\overline{\mathsf{W}}$ is a rigid subgroupoid of $\overline{\mathsf{P}}$. The traditional Church-Rosser property [6, Theorem 1.2.2] generalizes as follows in our setting:

▶ **Proposition 12.** *If* P *is weakly* W*-normalizing and* W*-confluent then for any zig-zag* $p : x \overset{\sim}{\to} y$ *in* $W^\sim$*, we have* $\overline{p \cdot n_y} = \overline{n_x}$*.*

Proof. By confluence, given a rewriting path $p : x \overset{*}{\to} y$ in W^*, we have $\hat{x} = \hat{y}$ and $\overline{p \cdot n_y} = \overline{n_x}$ (where n_x and n_y are paths to a normal form given by the weak normalization property), and thus $\overline{p^+ \cdot n_y} = \overline{n_x}$ and $\overline{n_x \cdot p^-} = \overline{n_y}$. Any zig-zag $p : x \overset{\sim}{\to} y$ in $W^\sim$ decomposes as $p = p_1^- q_1^+ p_2^- p_2^+ \ldots p_n^- p_n^+$ for some $n \in \mathbb{N}$ and paths p_i and q_i in W^*. We thus have $\overline{p \cdot n_y} = \overline{n_x}$, since all the squares of the following diagram commute in $W^\sim$ by the preceding remark:

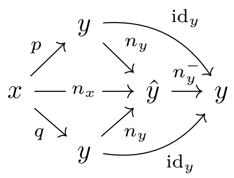


which allows us to conclude. ◀

This implies the following "abstract" variant of Squier's homotopical theorem [32, 21, 14]:

▶ **Proposition 13.** *If* P *is weakly* W*-normalizing and is* W*-confluent then it is* W*-coherent.*

Proof. Given two parallel zig-zags $p, q : x \overset{\sim}{\to} y$ in $W^\sim$, we have $\overline{p} = \overline{q}$, since the following diagram commutes in $\overline{\mathsf{P}}$:



Namely, we have $\hat{x} = \hat{y}$ by confluence, the two triangles above commute by Proposition 12, and the two triangles below do because n_y^- is an inverse for n_y. ◀

▶ **Example 14.** As a variant of Example 7, consider the 2-ARS P with $\mathsf{P}_0 = \{x, y\}$, $\mathsf{P}_1 = \{a, b : x \to y\}$ and $\mathsf{P}_2 = \emptyset$, i.e. $x \underset{b}{\overset{a}{\rightrightarrows}} y$. With $W = \{a\}$, we have that P is W-terminating and locally W-confluent, thus W-confluent by Proposition 11, and thus W-coherent by Lemma 10 and Proposition 13. With $W = \{a, b\}$, we have seen in Example 7 that the groupoid $\overline{W}$ is not rigid and, indeed, P is not W-confluent because $\overline{a} \neq \overline{b}$ (because $\mathsf{P}_2 = \emptyset$).

In a situation as above, we write $\mathrm{N}(\overline{\mathsf{P}})$ for the full subcategory of $\overline{\mathsf{P}}$ whose objects are W-normal forms. When P is weakly W-normalizing, we have that every object x of $\overline{\mathsf{P}}$ is isomorphic to one in the image by $\overline{n_x}$, and thus the inclusion functor $\mathrm{N}(\overline{\mathsf{P}}) \to \overline{\mathsf{P}}$ is an equivalence of categories. This equivalence is precisely the one with the quotient category when P is W-convergent:

▶ **Proposition 15.** *If P is W-convergent, the quotient category is isomorphic to the category of normal forms: $\overline{\mathsf{P}}/W \cong \mathrm{N}(\overline{\mathsf{P}})$.*

Proof. Since P is W-convergent, by Proposition 13, the groupoid generated by W is rigid and we thus have the description of the quotient $\overline{\mathsf{P}}/W$ given by Proposition 2. We have a canonical functor $\mathrm{N}(\overline{\mathsf{P}}) \to \overline{\mathsf{P}}/W$, obtained as the composite of the inclusion functor $\mathrm{N}(\overline{\mathsf{P}}) \to \overline{\mathsf{P}}$ with the quotient functor $\overline{\mathsf{P}} \to \overline{\mathsf{P}}/W$. By convergence, an equivalence class $[x]$ of objects contains a unique normal form (which is $\hat{x}$), and the functor is bijective on objects. By weak normalization (Lemma 10), any morphism $f : x \to y$ is equivalent to one with both normal source and target, namely $n_y \circ f \circ n_x^- : \hat{x} \to \hat{y}$, hence the functor is full. Suppose given two morphisms $f, g : \hat{x} \to \hat{y}$ in $\hat{\mathsf{P}}$ with the same image $[f] = [g]$: there exist morphisms $v : \hat{x} \to \hat{x}$ and $w : \hat{y} \to \hat{y}$ in $W^\sim$ such that $w \circ f = g \circ v$. By the Church-Rosser property (Proposition 13), we have $v = n_{\hat{x}} \circ n_{\hat{x}}^-$ and thus $v = \mathrm{id}_x$ (since $n_{\hat{x}} = \mathrm{id}_{\hat{x}}$ by hypothesis), and similarly $w = \mathrm{id}_y$. Hence $f = g$ and the functor is faithful. ◀

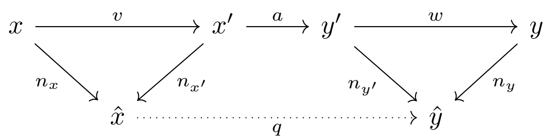
We would now like to provide an explicit description of $\mathrm{N}(\overline{\mathsf{P}})$. Since the rules in W are quotiented out, we can expect that they can simply be removed from P. This is not the case in general, but we provide here conditions which ensure that it holds, see also [9, 29] for alternative conditions. We write $\mathsf{P} \setminus W$ for the 2-ARS where $(\mathsf{P} \setminus W)_0 = \mathsf{P}_0$, $(\mathsf{P} \setminus W)_1 = \mathsf{P}_1 \setminus W$ and $(\mathsf{P} \setminus W)_2$ is P_2 restricted to rewriting rules whose source and target both belong to $(\mathsf{P}_1 \setminus W)^\sim$.

▶ **Proposition 16.** *Writing $\mathsf{P}' = \mathsf{P} \setminus W$, suppose that*

1. *P is W-convergent,*
2. *every rule $a : x \to y$ in P'_1 has a normal target $\hat{y} = y$,*
3. *for every coinitial rules $a : x \to y$ in P'_1 and $w : x \to x'$ in W, there is a path $p : x' \overset{*}{\to} y$ in P'^*_1 such that $\overline{a} = \overline{w \cdot p}$,*
4. *for every $A : p \Rightarrow q : x \to y$ in P_2 and rule $w : x \to x'$ in W, there is a relation $A' : p' \Rightarrow q' : x' \to y$ in P'_2 (or, more generally, $\overline{p'} = \overline{q'}$ in $\overline{\mathsf{P}'}$) where $p', q' : x' \to y$ are paths such that $\overline{p} = \overline{w \cdot p'}$ and $\overline{q} = \overline{w \cdot q'}$.*

Then $\mathrm{N}(\overline{\mathsf{P}})$ is isomorphic to $\mathrm{N}(\overline{\mathsf{P}'})$.

Proof. We claim that for every zig-zag $p : x \overset{\sim}{\to} y$ in $\mathsf{P}_1^\sim$ there is zig-zag $q \in \mathsf{P}'^\sim_1$ such that $\overline{p} = \overline{n_x \cdot q \cdot n_y^-}$. We have that p is of the form $p = w_0 \cdot a_1 \cdot w_1 \cdot a_2 \cdot w_2 \cdot \ldots \cdot a_n \cdot w_n$ where the a_i are rules in P'_1 (possibly taken backward) and the w_i are in $W^\sim$. For instance, consider the case $n = 1$ and a path p of the form $p = v \cdot a \cdot w$ with $a \in \mathsf{P}'_1$ and $v, w \in W^\sim$ (the case where a is reversed is similar, and the general case follows by induction):



By hypothesis 1 and Proposition 12, we have $\overline{v} = \overline{n_x \cdot n_{x'}^-}$ and $\overline{w} = \overline{n_{y'} \cdot n_y^-}$. By hypothesis 2, $n_{y'}$ is the empty path. By iterated use of hypothesis 3, there is $q \in \mathsf{P}_1'^*$ such that $\overline{a} = \overline{n_{x'} \cdot q}$.

The canonical functor $F : \mathrm{N}(\overline{\mathsf{P}}') \to \mathrm{N}(\overline{\mathsf{P}})$ is the identity on objects. The above reasoning shows that it is full. It can also be shown to be faithful by iterated use of hypothesis 4. ◀

We can finally summarize the results obtained in this section as follows. Given a 2-ARS P and a set $W \subseteq \mathsf{P}_1$, we have the following possible reasonable definitions of the fact that P is *coherent* wrt W:

(1) Every parallel zig-zags with edges in W are equal
(i.e. the subgroupoid of $\overline{P}$ generated by W is rigid).

(2) The quotient map $\overline{\mathsf{P}} \to \overline{\mathsf{P}}/W$ is an equivalence of categories.

(3) The canonical morphism $\mathrm{N}(\mathsf{P}) \to \overline{\mathsf{P}}$ is an equivalence.

(4) The inclusion $\mathrm{Alg}(\overline{\mathsf{P}}/W) \to \mathrm{Alg}(\overline{\mathsf{P}})$ is an equivalence of categories.

▶ **Theorem 17.** *If* P *is* W*-convergent then all the above coherence properties hold.*

Proof. (1) is given by Proposition 13, (2) is given by (1) and Theorem 6, (3) is given by Proposition 15, and (4) is given by (1) and Proposition 9. ◀

3 Relative coherence and term rewriting systems

In order to use the previous developments in concrete situations, such as (symmetric) monoidal categories, we need to consider a more structured notion of theory. For this reason, we consider here Lawvere 2-theories, as well as the adapted notion of rewriting, which is a coherent extension of the traditional notion of string rewriting systems.

3.1 Lawvere 2-theories

A *Lawvere theory* $\mathcal{T}$ is a cartesian category, with $\mathbb{N}$ as set of objects, and cartesian product given on objects by addition [25] (for simplicity, we restrict here to unsorted theories). In such a theory, we usually restrict our attention to morphisms with 1 as codomain, since $\mathcal{T}(n, m) \cong \mathcal{T}(n, 1)^m$ by cartesianness. A morphism between Lawvere theories is a product-preserving functor and we write $\mathbf{Law}_1$ for the category of Lawvere theories.

A $(2, 1)$*-category* is a 2-category in which every 2-cell is invertible (i.e. a category enriched in groupoids). A *Lawvere 2-theory* $\mathcal{T}$, as introduced in [12, 35, 36] (as well as [31] for the enriched point of view), is a cartesian $(2, 1)$-category with $\mathbb{N}$ as objects, and cartesian product given on objects by addition. A *morphism* $F : \mathcal{T} \to \mathcal{U}$ between 2-theories is a functor which preserves products. We write $\mathbf{Law}_2$ for the resulting category (which can be extended to a 3-category by respectively taking natural transformations and modifications as 2- and 3-cells).

3.2 Coherence for 2-theories

We can reuse the properties developed in Section 2 by working "hom-wise" as follows. Suppose fixed a 2-theory $\mathcal{T}$ together with a subset W of the 2-cells. We write $\mathcal{W}$ for the sub-2-theory of $\mathcal{T}$, with the same 0- and 1-cells, and whose 2-cells contain W (we often assimilate this 2-theory to its set of 2-cells). The *quotient* 2*-theory* $\mathcal{T}/W$ is the one representing the morphisms from $\mathcal{T}$ sending 2-cells in W to identities; it comes equipped with a quotient 2-functor $\mathcal{T} \to \mathcal{T}/W$. We have $\mathcal{T}/W \cong \mathcal{T}/\mathcal{W}$, so that we can always assume that we are quotienting by a sub-2-theory. On hom-categories, the quotient corresponds to the one introduced in Section 2.1: for every $m, n \in \mathbb{N}$, we have $(\mathcal{T}/\mathcal{W})(m, n) = \mathcal{T}(m, n)/\mathcal{W}(m, n)$.

We say that a morphism $F : \mathcal{T} \to \mathcal{U}$ is a *local equivalence* when for every objects $m, n \in \mathcal{T}$, the induced functor $F_{m,n} : \mathcal{T}(m, n) \to \mathcal{U}(m, n)$ between hom-categories is an equivalence. We say that $\mathcal{W}$ is 2*-rigid* when any two parallel 2-cells are equal, i.e. the category $\mathcal{W}(m, n)$ is rigid for every 0-cells m and n. By direct application of Theorem 6, we have

▶ **Theorem 18.** *The quotient* 2*-functor* $\mathcal{T} \to \mathcal{T}/\mathcal{W}$ *is a local equivalence iff* $\mathcal{W}$ *is* 2*-rigid.*

3.3 Extended rewriting systems

We briefly recall here the categorical setting for term rewriting systems. A more detailed presentation can be found in [10, 5, 28].

A *signature* consists of a set S_1 of *symbols* together with a function $s_0 : \mathsf{S}_1 \to \mathbb{N}$ associating to each symbol an *arity* and we write $a : n \to 1$ for a symbol a of arity n. A morphism of signatures is a function between the corresponding sets of symbols which preserves arity, and we write $\mathbf{Pol}_1^\times$ for the corresponding category. There is a forgetful functor $\mathbf{Law}_1 \to \mathbf{Pol}_1^\times$, sending a theory $\mathcal{T}$ on the set $\bigsqcup_{n\in\mathbb{N}} \mathcal{T}(n, 1)$ with first projection as arity. This functor admits a left adjoint $-^* : \mathbf{Pol}_1^\times \to \mathbf{Law}_1$. Given a signature S_1, and $n \in \mathbb{N}$, $\mathsf{S}_1^*(n, 1)$ is the set of *terms* formed using operations, with variables in $\{x_1^n, x_2^n, \ldots, x_n^n\}$ (the superscript is necessary to unambiguously recover the type of a variable, i.e. $x_i^n : n \to 1$, but for simplicity we will often omit it in the following). More generally, a morphism in $\mathsf{S}_1^*(n, m)$ is an m-uple $\langle t_1, \ldots, t_m\rangle$ of terms with variables in $\{x_1^n, \ldots, x_n^n\}$, which can be thought of as a formal *substitution*, and composition is given by parallel substitution:

$$\langle u_1, \ldots, u_k\rangle \circ \langle t_1, \ldots, t_m\rangle = \langle u_1[t_1/x_1, \ldots, t_n/x_n], \ldots, u_k[t_1/x_1, \ldots, t_m/x_m]\rangle$$

(the term on the right is obtained by substituting each occurrence of a variable x_i in a term u_j by t_i). By abuse of notation, we write S_1^* for the set of substitutions and $s_0, t_0 : \mathsf{S}_1^* \to \mathbb{N}$ for the source and target maps.

A *term rewriting system*, or TRS, consists of a signature S_1 together with a set S_2 of *rewriting rules* and functions $s_1, t_1 : \mathsf{S}_2 \to \mathsf{S}_1^*$ which indicate the source and target of each rewriting rule, and are supposed to satisfy $s_0 \circ s_1 = s_0 \circ t_1$ and $t_0 \circ s_1 = t_0 \circ t_1 = 1$ (i.e. the source and target of a rewriting rule are parallel terms). We sometimes write $\rho : t \Rightarrow u$ for a rule ρ with t as source and u as target. A *context* C of arity n is a term with variables in $\{x_1, \ldots, x_n, \square\}$ where the variable $\square$ is a particular variable, the *hole*, occurring exactly once. Given a term t of arity n, we write $C[t]$ for the term obtained from C by replacing $\square$ by t. The composition of contexts C and D is given by substitution $D \circ C = D[C]$. A *bicontext* is a pair (C, f) consisting of a context C and a substitution f. A *rewriting step* $C[\rho \circ f]$ of arity n is a triple consisting of a rewriting rule $\rho : t \Rightarrow u$, with t and u of arity k, together with a hole C of arity n, as well as a substitution $f : n \to k$ in S_1^*: a rewriting step can thus be thought of as a rewriting rule in a bicontext. Its source is the term $C[t \circ f]$ and its target is the term $C[u \circ f]$. We write $\mathsf{S}_2^{[]}$ for the set of rewriting steps. As in Section 2.3, we write S_2^* for the set of rewriting paths, which consist of composable sequence of rewriting steps, and $\mathsf{S}_2^{\sim}$ for the set of rewriting zig-zags in a TRS, and use associated notations. Every term rewriting system S freely generates a 2-Lawvere theory, with S_1^* as 1-cells and $\mathsf{S}_2^{\sim}$ as 2-cells. Given a rewriting step $C[\rho \circ f]$, a context D and a substitution g of suitable types, we have $D[C[\rho \circ f] \circ g] = (D \circ C)[\rho \circ (f \circ g)]$ so that bicontexts act on rewriting steps, and this action extends to rewriting paths and zig-zags by functoriality, i.e. $C[(p \cdot q) \circ f] = C[p \circ f] \cdot C[q \circ f]$.

An *extended term rewriting system*, or 2-TRS, consists of a term rewriting system as above, together with a set S_3 of *coherence relations* and functions $s_2, t_2 : \mathsf{S}_3 \to \mathsf{S}_2^{\sim}$, indicating their source and target, such that $s_1 \circ s_2 = s_1 \circ t_2$ and $t_1 \circ s_2 = t_1 \circ t_2$. The Lawvere 2-theory *presented* by a 2-TRS S is the $(2,1)$-category noted $\overline{\mathsf{S}}$, with $\mathbb{N}$ as 0-cells, S_1^* as 1-cells and, as 2-cells the quotient of $\mathsf{S}_2^{\sim}$ under the smallest congruence identifying the source and target of any elements of S_3.

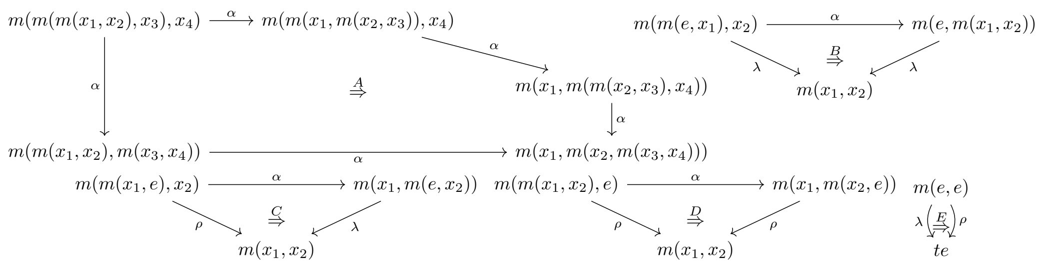
▶ **Example 19.** The extended rewriting system Mon for monoids has symbols and rules

$$\mathsf{Mon}_1 = \{m : 2 \to 1, e : 0 \to 1\}$$
$$\mathsf{Mon}_2 = \{\alpha : m(m(x_1, x_2), x_3) \Rightarrow m(x_1, m(x_2, x_3)), \lambda : m(e, x_1) \Rightarrow x_1, \rho : m(x_1, e) \Rightarrow x_1\}$$

There are coherence relations A, B, C, D and E, respectively corresponding to a confluence for the five critical branchings of the rewriting system, whose 0-sources are

$$m(m(m(x_1, x_2), x_3), x_4) \quad m(m(e, x_1), x_2) \quad m(m(x_1, e), x_2) \quad m(m(x_1, x_2), e) \quad m(e, e)$$

Those coherence relations can be pictured as follows:



For concision, for each arrow, we did not indicate the proper rewriting step, but only the rewriting rule of the rewriting step (hopefully, the reader will easily be able to reconstruct it). For instance, the coherence relation C has target $\alpha(x_1, e, x_2) \cdot m(x_1, \lambda(x_2))$.

3.4 Rewriting properties

Suppose fixed a 2-TRS S together with $W \subseteq \mathsf{S}_2$. The 2-TRS S induces an 2-ARS in each hom-set: this point of view will allow reusing the work done on 2-ARS on Section 2.

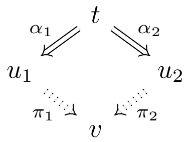
▶ **Definition 20.** *Given a 2-TRS S and $m, n \in \mathbb{N}$, we write $\mathsf{S}(m, n)$ for the 2-ARS with $\mathsf{S}(m, n)_0 = \mathsf{S}_1^*(n, m)$, $\mathsf{S}(m, n)_1 = \mathsf{S}_2^{[]}$ and $\mathsf{S}(m, n)_2 = \mathsf{S}_3$.*

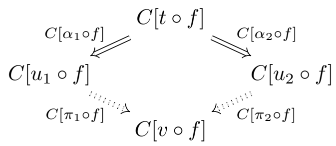
Similarly, a set W induces a set $W(m, n) \subseteq \mathsf{S}(m, n)_1 = \mathsf{S}_2^{[]}$, where $W(m, n)$ is the set of rewriting steps whose rewriting rule belongs to W (they are of the form $C[\alpha \circ f]$ with $\alpha \in W$). We say that a 2-TRS S is *W-terminating* / *locally W-confluent* / *W-confluent* / *W-coherent* when each $\mathsf{S}(m, n)$ is with respect to $W(m, n)$. We say that S is *confluent* when it is W-confluent for $W = \mathsf{S}_2$ (and similarly for other properties).

A *W-branching* (α_1, α_2) is a pair of rewriting steps $\alpha_1 : t \Rightarrow u_1$ and $\alpha_2 : t \Rightarrow u_2$ in $\mathcal{W}^{\square}$ with the same source:

$$u_1 \overset{\alpha_1}{\Longleftarrow} t \overset{\alpha_2}{\Longrightarrow} u_2$$

It is *W-confluent* when there are cofinal rewriting paths $\pi_1 : u_1 \Rightarrow v$ and $\pi_2 : u_2 \Rightarrow v$ in W^* such that $\overline{\alpha_1 \cdot \pi_1} = \overline{\alpha_2 \cdot \pi_2}$, which is depicted on the left





By extension of Proposition 11, we have

▶ **Proposition 21.** *If* S *is W-terminating and locally W-confluent then it is W-confluent.*

In practice, termination can be shown as follows [3, Section 5.2]. A *reduction order* $>$ is a well-founded partial order on terms in S_1^* which is compatible with context extension: given terms $t, u \in \mathsf{S}_1^*$, $t > u$ implies $C[t \circ f] > C[u \circ f]$ for every context C and substitution $f \in \mathsf{S}_1^*$ (whose types are such that the expressions make sense).

▶ **Proposition 22.** *A 2-TRS* S *equipped with a reduction order such that* $t > u$ *for any rule* $\alpha : t \Rightarrow u$ *in* W *is W-terminating.*

In order to construct a reduction order one can use the following "interpretation method" [3, Section 5.3]. Suppose given a well-founded poset $(X, <)$ and an interpretation $[\![a]\!] : X^n \to X$ of each symbol $a \in \mathsf{S}_1$ of arity n as a function which is strictly decreasing in each argument. This induces, by composition and product, an interpretation $[\![t]\!]$ of every term. We define an order on functions $f, g : X^n \to X$ by $f \succ g$ whenever $f(x_1, \ldots, x_n) \succ g(x_1, \ldots, x_n)$ for every $x_i \in X$; this order is well-founded because the order on X is. By extension, we define an order on terms $t, u \in \mathsf{S}_1^*(n, 1)$ by $t \succ u$ whenever $[\![t]\!] \succ [\![u]\!]$: this order is always a reduction order. By Proposition 22, if we have $t \succ u$ for every rule $\alpha : t \Rightarrow u$ the 2-TRS is thus W-terminating.

▶ **Example 23.** Consider the 2-TRS Mon of Example 19. We interpret the symbols as $[\![m(x_1, x_2)]\!] = 2x_1 + x_2$ and $[\![e]\!] = 1$ on $X = \mathbb{N} \setminus 0$. All the rules are decreasing since we have

$$[\![m(m(x_1, x_2), x_3)]\!] = 4x_1 + 2x_2 + x_3 > 2x_1 + 2x_2 + x_3 = [\![m(x_1, m(x_2, x_3))]\!]$$
$$[\![m(e, x_1)]\!] = 2 + x_1 > x_1 = [\![x_1]\!] \qquad [\![m(x_1, e)]\!] = 2x_1 + 1 > x_1 = [\![x_1]\!]$$

and the rewriting system is terminating.

We now briefly recall the notion of *critical branching*, see [28] for a more detailed presentation. We say that a branching (α_1, α_2) is *smaller* than a branching (β_1, β_2) when the second can be obtained from the first by "extending the context", i.e. when there exists a context C and a morphism f of suitable types such that $\beta_i = C[\alpha_i \circ f]$ for $i = 1, 2$. In this case, the confluence of the first branching implies the confluence of the second one (see the diagram on the right above). The notion of context can be generalized to define the notion of a binary context C, with two holes, each of which occurs exactly once: we write $C[t, u]$ for the context where the holes have respectively been substituted with terms t and u. A branching is *non-overlapping* when it consists of two rewriting steps at disjoint positions, i.e. when it is of the form

$$C[u_1 \circ f_1, t_2 \circ f_2] \overset{C[\alpha_1 \circ f_1, t_2 \circ f_2]}{\Longleftarrow} C[t_1 \circ f_1, t_2 \circ f_2] \overset{C[t_1 \circ f_1, \alpha_2 \circ f_2]}{\Longrightarrow} C[t_1 \circ f_1, u_2 \circ f_2]$$

for some binary context C, rewriting rules $\alpha_i : t_i \Rightarrow u_i$ in S_2 and morphisms f_i in S_1^* of suitable types. A branching is *critical* when it is not non-overlapping and minimal (wrt the above order). A TRS with a finite number of rewriting rules always have a finite number of critical branchings and those can be computed efficiently [3].

▶ **Lemma 24.** *A 2-*TRS S *is locally W-confluent when all its critical W-branchings are W-confluent.*

Proof. Suppose that all critical W-branchings are confluent. A non-overlapping W-branching is easily shown to be W-confluent. A non-minimal W-branching is greater than a minimal one, which is W-confluent by hypothesis, and is thus itself also W-confluent. ◀

We write $W_3 \subseteq \mathsf{S}_3$ for the set of coherence relations $A : \pi \Rightarrow \rho$ such that both π and ρ belong to $W^{\sim}$. As a useful particular case, we have the following variant of the Squier theorem:

▶ **Lemma 25.** *If 2-*TRS S *has a coherence relation in W_3 corresponding to a choice of confluence for every critical W-branching then it is locally W-confluent.*

▶ **Example 26.** The 2-TRS Mon of Example 19. By definition, every critical branching is confluent and Mon is thus locally confluent. From Example 23 and Proposition 21, we deduce that it is confluent.

As a direct adaptation of Proposition 13, we have

▶ **Lemma 27.** *If* S *is W-terminating and locally W-confluent then it is W-coherent.*

From Examples 23 and 26, we deduce that the 2-TRS Mon is coherent, thus showing the coherence property (C1) for monoidal categories.

Suppose given a W-convergent 2-TRS S. By Lemma 27, $\overline{\mathsf{S}}$ is W-coherent, by Theorem 18, the quotient functor $\overline{\mathsf{S}} \to \overline{\mathsf{S}}/W$ is a local equivalence, and by Proposition 15, $\overline{\mathsf{S}}/W$ is obtained from $\overline{\mathsf{P}}$ by restricting to 1-cells in normal form. Moreover, in good situations, we can provide a description of the quotient category $\overline{\mathsf{S}}/W$ by applying Proposition 16 hom-wise.

3.5 Algebras for Lawvere 2-theories

The notion of algebra for 2-theories was extensively studied by Yanofsky [35, 36], we refer to his work for details.

An *algebra* for a Lawvere 2-theory $\mathcal{T}$ is a 2-functor $C : \mathcal{T} \to \mathbf{Cat}$ which preserves products. By abuse of notation, we often write C instead of $C1$ and suppose that products are strictly preserved, so that $Cn = C^n$. A *pseudo-natural transformation* $F : C \Rightarrow D$ between algebras C and D consists in a functor $F : C \to D$ together with a family $\phi_f : Df \circ F^n \Rightarrow F \circ Cf$ of natural transformations indexed by 1-cells $f : n \to 1$ in $\mathcal{T}$, which is compatible with products, composition and 2-cells of $\mathcal{T}$. A *modification* $\mu : F \Rrightarrow G : C \Rightarrow D$ between two pseudo-natural transformations is a natural transformation $\mu : F \Rrightarrow G$ which is compatible with 2-cells of $\mathcal{T}$. We write $\mathrm{Alg}(\mathcal{T})$ for the 2-category of algebras of a 2-theory $\mathcal{T}$, pseudo-natural transformations and modifications.

▶ **Example 28.** Consider the 2-TRS Mon of Example 19. The 2-category $\mathrm{Alg}(\overline{\mathsf{Mon}})$ of algebras of the presented 2-theory is isomorphic to the category $\mathbf{MonCat}$ of monoidal categories, strong monoidal functors and monoidal natural transformations. It might be surprising that Mon has five coherence relations whereas the traditional definition of monoidal categories only features two axioms (which correspond to the coherence relations A and C). There is no contradiction here: the commutation of the two axioms can be shown to imply the one of the three other [18, 13].

A morphism $F : \mathcal{T} \to \mathcal{U}$ of 2-theories is a *biequivalence* when there is a morphism $G : \mathcal{U} \to \mathcal{T}$ and natural transformations $\eta : \mathrm{Id}_\mathcal{T} \Rightarrow G \circ f$ and $\varepsilon : F \circ G \Rightarrow \mathrm{Id}_\mathcal{U}$ whose components are equivalences. A generalization of Proposition 8 is shown in [36, Proposition 7]:

▶ **Proposition 29.** *A morphism $F : \mathcal{T} \to \mathcal{T}'$ between theories is a biequivalence if and only if the functor $\mathrm{Alg}(F) : \mathrm{Alg}(\mathcal{T}') \to \mathrm{Alg}(\mathcal{T})$ induced by precomposition is a biequivalence (in a suitable sense).*

In particular, in the case where $\mathcal{W}$ is 2-rigid, it seems that we can deduce from Theorem 18 that the projection functor $\mathcal{T} \to \mathcal{T}/\mathcal{W}$ is a local equivalence, and thus a biequivalence, and thus that the categories $\mathrm{Alg}(\mathcal{T})$ and $\mathrm{Alg}(\mathcal{T}/\mathcal{W})$ are biequivalent (for instance, it is claimed that the categories of monoidal and strict monoidal categories are equivalent). However, the claim that any local equivalence is a biequivalence [36, Proposition 6] is wrong: given a local equivalence $F : \mathcal{C} \to \mathcal{D}$ between 2-categories, one can in general construct a pseudo-functor $G : \mathcal{D} \to \mathcal{C}$ satisfying suitable properties, but not a strict one, see [20, Example 3.1] for a counter-example. Intuitively, in the case where $\mathcal{C} = \mathcal{T}$ and $\mathcal{D} = \mathcal{T}/\mathcal{W}$ with rewriting properties as in Section 3.4, G will send a 1-cell to a normal form in its equivalence class, but the composite of two normal forms is not itself a normal form in general, we can only expect that it is isomorphic to a normal form.

We however conjecture that one can generalize the classical proof that any monoidal category is monoidally equivalent to a strict one [26, Theorem XI.3.1] to show the following general (C3) coherence theorem, as well as its (C4) generalization:

▶ **Conjecture 30.** *When $\mathcal{W}$ is 2-rigid, every $\mathcal{T}$-algebra is equivalent to a $\mathcal{T}/\mathcal{W}$ algebra.*

▶ **Conjecture 31.** *When $\mathcal{W}$ is 2-rigid, the 2-functor $\mathrm{Alg}(\mathcal{T}/\mathcal{W}) \to \mathrm{Alg}(\mathcal{T})$ induced by precomposition with the quotient functor $\mathcal{T} \to \mathcal{T}/W$ has a left adjoint such that the components of the unit are equivalences.*

This is left for future works. Note that, apart from informal explanations, we could not find a proof of Conjectures 30 and 31 for symmetric or braided monoidal categories in the literature, e.g. in [27, 17, 26] (in [17, Theorem 2.5] the result is only shown for free braided monoidal categories).

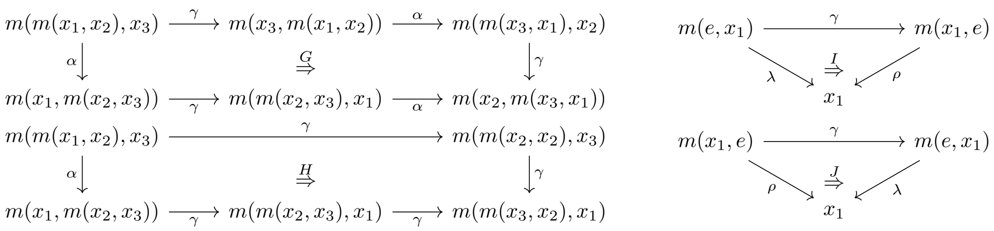
3.6 Symmetric monoidal categories

A *symmetric monoidal category* is a monoidal category equipped with a natural isomorphism $\gamma_{x,y} : x \otimes y \to y \otimes x$, called *symmetry*, satisfying three classical axioms. A symmetric monoidal category is *strict* when the structural isomorphisms α, λ and ρ are identities (but we do not require γ to be an identity). We write **SMonCat** (resp. $\mathbf{SMonCat}_{\mathrm{str}}$) for the category of symmetric monoidal categories (resp. strict ones). Using the same method as above, we can show the coherence theorems for symmetric monoidal categories [17]. This example illustrates more the interest of the previous developments since we are quotienting by a $(2,1)$-category $\mathcal{W}$ which is not the whole category.

We write SMon for the 2-TRS obtained from Mon (see Example 19) by adding a rewriting rule $\gamma : m(x_1, x_2) \Rightarrow m(x_2, x_1)$ (corresponding to symmetry), together with a coherence relation

$$
\begin{array}{ccc}
m(x_1,x_2) & \xrightarrow{\gamma} & m(x_2,x_1) \\
\| & \overset{F}{\Rrightarrow} & \downarrow{\scriptstyle\gamma} \\
m(x_1,x_2) & = & m(x_1,x_2)
\end{array}
$$

as well as four relations corresponding to the critical branchings between the rule γ and the rules α, λ or ρ:



The category $\mathrm{Alg}(\overline{\mathsf{SMon}})$ is isomorphic to the category **SMonCat**. The traditional definition of symmetric monoidal categories only features axioms corresponding to F, G and I, but it can be shown that they implies the commutation of the axiom corresponding to H (by using G twice) and J (by using F and I). We write $W = \{\alpha,\lambda,\rho\}$. The category $\mathrm{Alg}(\mathsf{SMon}/W)$ is isomorphic to $\mathbf{SMonCat}_{\mathrm{str}}$. We have that the 2-TRS is W-terminating by Example 23 and W-locally confluent by definition (Example 19), it is thus W-coherent by Lemma 27. From Conjecture 30, we would deduce that any symmetric monoidal category is monoidally equivalent to a strict one.

Note that the above reasoning only depends on the convergence of the subsystem induced by W, i.e. on the fact that every diagram made of α, λ and ρ commutes, but it does not require anything on diagrams containing γ's. In particular, if we removed the compatibility relations G, H, I and J, the strictification theorem would still hold. The resulting notion of strict symmetric monoidal category would however be worrying since, for instance, in absence of I, the morphism $\gamma_{e,x_1} : m(e,x_1) \to m(x_1,e)$ would induce, in the quotient, a non-trivial automorphism $\gamma_{e,x_1} : x_1 \to x_1$ of each object x_1. The following variant of the coherence theorem is "stronger" in the sense that it requires these axioms to hold.

We have seen that for the theory of monoidal categories "every diagram commutes", in the sense that $\overline{\mathsf{Mon}}$ is a 2-rigid $(2,1)$-category. For symmetric monoidal categories, we do not expect this to hold since γ_{x_1,x_1} and $\mathrm{id}_{m(x_1,x_1)}$ both are rewriting paths from $m(x_1,x_1)$ to itself, and are not equal in general (one can easily find an example of a symmetric monoidal category in which the symmetry is not the identity, this is in fact the case for most usual examples). It can however be shown that it holds for a subclass of 2-cells whose source and target are affine terms: a term is *affine* if no variable occurs twice. We now explain this, thus recovering a well-known property [27, Theorem 4.1] using rewriting techniques. Note that the property of being affine, as well as the variables occurring in terms, are preserved by rewriting steps. By inspection of critical branchings (Lemma 24), we have

▶ **Lemma 32.** *The 2-TRS* SMon *is locally confluent.*

It is not terminating, even when restricted to affine terms, since we for instance have the loop

$$m(x_1,x_2) \xrightarrow{\gamma(x_1,x_2)} m(x_2,x_1) \xrightarrow{\gamma(x_2,x_1)} m(x_1,x_2)$$

In order to circumvent this problem, we are going to formally "remove" the second morphism above and only keep instances of γ which tend to make variables in decreasing order. Note that the coherence relation F ensures that $\gamma(x_2,x_1) = \gamma(x_1,x_2)^-$ so that this removal does not change the presented $(2,1)$-category.

Given a term t, we write $\|t\|$ the list of variables occurring in it, from left to right, e.g. $\|m(m(x_2,e),x_1)\| = x_2x_1$. We order variables by $x_i \succeq x_j$ whenever $i \leq j$ and extend it to lists of variables by lexicographic ordering (which is well-founded since we compare only words of the same length). Given a rewriting step $C[\gamma(t_1, t_2)]$ involving the rule γ is *decreasing* when $\|t_1\|\|t_2\| \succ \|t_2\|\|t_1\|$. Fix $n \in \mathbb{N}$, consider the 2-ARS $\mathsf{P}' = \mathsf{SMon}(n, 1)$, and write P for the 2-ARS obtained from P' by

- removing from P'_1 all non-decreasing rewriting steps involving γ,
- replacing in the source or target of a relation in P'_2 all non-decreasing steps $C[\gamma(t_1, t_2)]$ by $C[\gamma(t_2, t_1)^-]$.

▶ **Lemma 33.** *The 2-ARS P is locally confluent.*

Proof. The above reasoning shows that we have $\overline{\mathsf{P}}' = \overline{\mathsf{P}}$. Since SMon is locally confluent (Lemma 32), P' is locally confluent and thus also P. ◀

Since we restricted ourselves to decreasing symmetries, P is "almost" terminating since rewriting rules tend to put variables in decreasing order. However, it sill does not prevent loops when there is no variable: for instance, $\gamma(e, e)$ is a rewriting step from $m(e, e)$ to itself. Fortunately, we can always remove units by restricting to terms which are in normal form wrt λ and ρ. Namely, P satisfies the hypothesis of Proposition 16 with W consisting of all rewriting steps generated by λ and ρ (condition 3. uses the compatibility relations G, H, I and J). We can thus restrict to terms in which the unit e does not occur and for those the system is terminating. Any two rewriting paths between affine terms are equals and the algebras thus satisfy:

▶ **Proposition 34.** *In a symmetric monoidal category, every diagram whose* 0*-source is a tensor product of distinct objects commutes.*

4 Future works

We believe that the developed framework applies to a wide variety of algebraic structures, which will be explored in subsequent work. In fact, the full generality of the framework was not needed for (symmetric) monoidal categories, since the rules of the corresponding theory never need to duplicate or erase variables (and, in fact, those can be handled by traditional polygraphs [22, 13]). This is however, needed for the case of rig categories [24], which feature two monoidal structures $\oplus$ and $\otimes$, and natural isomorphisms such as $\delta_{x,y,z} : x \otimes (y \oplus z) \to (x \otimes y) \oplus (x \otimes z)$ (note that x occurs twice in the target), generalizing the laws for rings. Those were a motivating example for this work, and we will develop elsewhere a proof of coherence of those structures based on our rewriting framework, as well as related approaches on the subject [7, Appendix G].

A notion of Tietze transformation for term rewriting system, which are transformations allowing one to navigate between the various presentations of a given Lawvere theory, were given in [28]. It would be interesting to develop an analogous notion for 2-TRS, presenting a given Lawvere 2-theory: this would allow us to formalize reasoning about superfluous generators or relations (such as in Example 28).

Finally, the importance of the notion of polygraph can be explained by the fact that they are the cofibrant objects in a model structure on ω-categories [23]. It would be interesting to develop a similar point of view for higher term rewriting systems: a first step in this direction is the model structure developed in [36].

References

1 Coherence theorem for monoidal categories. URL: `https://ncatlab.org/nlab/show/coherence+theorem+for+monoidal+categories`.

2 Coherence theorem for symmetric monoidal categories. URL: `https://ncatlab.org/nlab/show/coherence+theorem+for+symmetric+monoidal+categories`.

3 Franz Baader and Tobias Nipkow. *Term rewriting and all that*. Cambridge university press, 1999.

4 Marek A Bednarczyk, Andrzej M Borzyszkowski, and Wieslaw Pawlowski. Generalized congruences – Epimorphisms in Cat. *Theory and Applications of Categories*, 5(11):266–280, 1999.

5 Tibor Beke. Categorification, term rewriting and the Knuth-Bendix procedure. *Journal of Pure and Applied Algebra*, 215(5):728–740, 2011.

6 Marc Bezem, Jan Willem Klop, and Roel de Vrijer. *Term rewriting systems*. Cambridge University Press, 2003.

7 Filippo Bonchi, Alessandro Di Giorgio, and Alessio Santamaria. Tape diagrams for rig categories with finite biproducts. Technical report, Università di Pisa, 2022.

8 Albert Burroni. Higher-dimensional word problems with applications to equational logic. *Theoretical computer science*, 115(1):43–62, 1993.

9 Florence Clerc and Samuel Mimram. Presenting a category modulo a rewriting system. In *26th International Conference on Rewriting Techniques and Applications (RTA 2015)*. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2015.

10 Jonathan Asher Cohen. *Coherence for rewriting 2-theories*. PhD thesis, Australian National University, 2008. `arXiv:0904.0125`.

11 Benjamin Dupont and Philippe Malbos. Coherent confluence modulo relations and double groupoids. *Journal of Pure and Applied Algebra*, 226(10):107037, 2022. `arXiv:1810.08184`.

12 John W Gray. 2-algebraic theories and triples. *Cahiers de Topologie et Géométrie Différentielle*, 14:178–180, 1973.

13 Yves Guiraud and Philippe Malbos. Coherence in monoidal track categories. *Mathematical Structures in Computer Science*, 22(6):931–969, 2012.

14 Yves Guiraud and Philippe Malbos. Polygraphs of finite derivation type. *Mathematical Structures in Computer Science*, 28(2):155–201, 2018.

15 Gérard Huet. Confluent reductions: Abstract properties and applications to term rewriting systems. *Journal of the ACM (JACM)*, 27(4):797–821, 1980.

16 Niles Johnson and Donald Yau. *2-dimensional categories*. Oxford University Press, USA, 2021.

17 André Joyal and Ross Street. Braided tensor categories. *Advances in Math*, 102:20–78, 1993.

18 Gregory Maxwell Kelly. On MacLane's conditions for coherence of natural associativities, commutativities, etc. *Journal of Algebra*, 1(4):397–402, 1964.

19 Nicolai Kraus and Jakob von Raumer. A rewriting coherence theorem with applications in homotopy type theory. *Mathematical Structures in Computer Science*, 32(7):982–1014, 2022. `arXiv:2107.01594`.

20 Stephen Lack. A Quillen model structure for 2-categories. *K-theory*, 26(2):171–205, 2002.

21 Yves Lafont. A new finiteness condition for monoids presented by complete rewriting systems (after Craig C. Squier). *Journal of Pure and Applied Algebra*, 98(3):229–244, 1995.

22 Yves Lafont. Towards an algebraic theory of boolean circuits. *Journal of Pure and Applied Algebra*, 184(2-3):257–310, 2003.

23 Yves Lafont, François Métayer, and Krzysztof Worytkiewicz. A folk model structure on omega-cat. *Advances in Mathematics*, 224(3):1183–1231, 2010.

24 Miguel L Laplaza. Coherence for distributivity. In *Coherence in categories*, pages 29–65. Springer, 1972.

25 F William Lawvere. Functorial semantics of algebraic theories. *Proceedings of the National Academy of Sciences*, 50(5):869–872, 1963.

26 Saunders Mac Lane. *Categories for the Working Mathematician*, volume 5. Springer Science & Business Media, 1998.

27 Saunders MacLane. Natural associativity and commutativity. *Rice Institute Pamphlet-Rice University Studies*, 49(4), 1963.

28 Philippe Malbos and Samuel Mimram. Homological computations for term rewriting systems. In *1st International Conference on Formal Structures for Computation and Deduction (FSCD 2016)*, volume 52, pages 27–1, 2016.

29 Samuel Mimram and Pierre-Louis Curien. Coherent Presentations of Monoidal Categories. *Logical Methods in Computer Science*, 13, 2017.

30 John Power. An abstract formulation for rewrite systems. In *Category Theory and Computer Science*, pages 300–312, Berlin, Heidelberg, 1989. Springer Berlin Heidelberg. `doi:10.1007/BFb0018358`.

31 John Power. Enriched Lawvere theories. *Theory and Applications of Categories*, 6(7):83–93, 1999.

32 Craig C Squier, Friedrich Otto, and Yuji Kobayashi. A finiteness condition for rewriting systems. *Theoretical Computer Science*, 131(2):271–294, 1994.

33 Ross Street. Limits indexed by category-valued 2-functors. *Journal of Pure and Applied Algebra*, 8(2):149–181, 1976.

34 Ross Street. Categorical structures. *Handbook of algebra*, 1:529–577, 1996.

35 Noson S Yanofsky. The syntax of coherence. *Cahiers de topologie et géométrie différentielle catégoriques*, 41(4):255–304, 2000.

36 Noson S Yanofsky. Coherence, Homotopy and 2-Theories. *K-Theory*, 23(3):203–235, 2001.