TIETZE EQUIVALENCES AS WEAK EQUIVALENCES

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Foreword

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Basic idea:

- algebraic geometry studies spaces up to deformation,
- we should study data structures up to implementation.

Presentations for monoids

A **presentation** $\langle G \mid R \rangle$ consists of

- a set G of *generators*
- a (multi)set R of *relations* of the form

$$u = v$$

  with $u, v \in G^*$.

Example

$$\langle a, b \mid ba = ab, bb = 1 \rangle$$

The presented monoid

Given a presentation $\langle G \mid R \rangle$, the **congruence** $\approx$ on G^* is the smallest relation such that

1. $u \approx v$ for $u = v \in R$
2. $u \approx u$
3. $u \approx v$ and $v \approx w$ implies $u \approx w$
4. $u \approx v$ implies $v \approx u$
5. $v \approx v'$ implies $uvw \approx uv'w$

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The **monoid presented** by $\langle G \mid R \rangle$ is the quotient monoid

$$M \quad = \quad G^*/{\approx} \quad = \quad G^*/R\,.$$

Some presentations

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$$\langle a \mid \rangle$$

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- $\mathbb{N} \times \mathbb{N}$ is presented by
$$\langle a, b \mid ba = ab \rangle$$
- S_3 is presented by
$$\langle a, b \mid aa = 1, bb = 1, bab = aba \rangle$$

Equivalence between presentations

Two presentations

$$\langle G \mid R \rangle \quad \text{and} \quad \langle G' \mid R' \rangle$$

are **equivalent** when they present the same monoid (up to isomorphism):

$$G^*/R \quad = \quad G'^*/R' \, .$$

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Question

When are two presentations equivalent?

Equivalent presentations

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The monoid $\mathbb{N}$ is presented by

- $\langle a \mid \rangle$

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Lemma

Tietze transformations preserve the presented monoid.

Tietze equivalence

Two presentations are **Tietze equivalent** when they are related by a finite zig-zag of Tietze transformations

$$\langle G_1 \,|\, R_1 \rangle \quad \rightsquigarrow \quad \langle G_2 \,|\, R_2 \rangle \quad \leftsquigarrow \quad \langle G_3 \,|\, R_3 \rangle \quad \leftsquigarrow \quad \ldots \quad \rightsquigarrow \quad \langle G_k \,|\, R_k \rangle$$

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Suppose $\langle a_i \mid u_i = v_i \rangle$ and $\langle a'_i \mid u'_i = v'_i \rangle$ present the same monoid.

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- The generators a_i can be expressed in terms of a_i' as w_i'.

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$$\begin{aligned} \langle a_i \mid u_i = v_i \rangle &\rightsquigarrow \langle a_i, a_i' \mid u_i = v_i, a_i' = w_i \rangle \\ &\rightsquigarrow \langle a_i, a_i' \mid u_i = v_i, a_i' = w_i, a_i = w_i' \rangle \end{aligned}$$

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&\rightsquigarrow \quad \langle a_i, a'_i \mid u_i = v_i, a'_i = w_i, a_i = w'_i \rangle \\
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$$\langle a_i, a'_i \mid u_i = v_i, a'_i = w_i, a_i = w'_i \rangle$$

$$\langle a_i \mid u_i = v_i \rangle \rightsquigarrow \quad \quad \leftsquigarrow \langle a'_i \mid u'_i = v'_i \rangle$$

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Tietze equivalence

Theorem
Two finite presentations are equivalent if and only if they are Tietze equivalent.

The following are presentations of $\mathbb{N}$:

- $\langle a \mid \rangle$
- $\langle a_x \mid a_x = a_0 \rangle_{x \in \mathbb{R}}$

Tietze equivalence

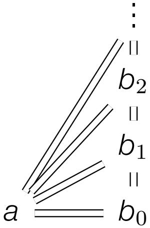
In order to encompass arbitrary presentations, one could be tempted to consider zig-zags of Tietze transformations of "arbitrary length":

$$\langle G_1 \mid R_1 \rangle \quad \rightsquigarrow \quad \langle G_2 \mid R_2 \rangle \quad \leftsquigarrow \quad \langle G_3 \mid R_3 \rangle \quad \leftsquigarrow \quad \dots \quad \rightsquigarrow \quad \langle G_k \mid R_k \rangle$$

Example

Consider the following presentation of $\mathbb{N}$:

$$\langle a, b_i \mid a = b_i, b_i = b_{i+1} \rangle_{i \in \mathbb{N}}$$



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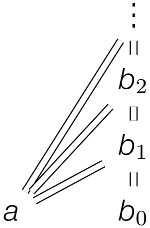

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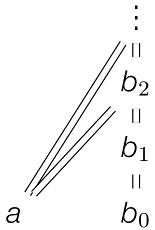

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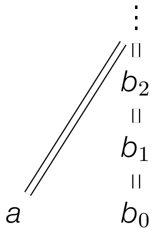

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$$\begin{array}{ccc} & \vdots & \\ & \shortparallel & \\ & b_2 & \\ & \shortparallel & \text{presents } \mathbb{N} * \mathbb{N}! \\ & b_1 & \\ & \shortparallel & \\ a & b_0 & \end{array}$$

Tietze equivalence

What I propose:

Definition

A **Tietze expansion** is a transfinite sequence of Tietze transformations

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and **Tietze equivalence** is the equivalence relation generated by Tietze expansions:

$$\langle G_1 \mid R_1 \rangle \quad \overset{*}{\rightsquigarrow} \quad \langle G_2 \mid R_2 \rangle \quad \overset{*}{\leftsquigarrow} \quad \langle G_3 \mid R_3 \rangle \quad \overset{*}{\leftsquigarrow} \quad \dots \quad \overset{*}{\rightsquigarrow} \quad \langle G_k \mid R_k \rangle$$

Tietze as weak equivalences

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More precisely, is there a model structure on presentations whose weak equivalences are Tietze equivalences?

Plus some intuitions:

- Tietze expansions look very much like trivial cofibrations
- the proof of the theorem looks very much like a mapping cylinder construction

The category of presentations

The category **Pres** has

- objects: the presentations
- morphisms: the “strict morphisms”, i.e. we send generators to generators, and relations to relations

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This category is complete and cocomplete. For instance,

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- the terminal object is $1 = \langle a \mid a^m = a^n \rangle_{m,n \in \mathbb{N}}$

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It is also locally presentable:

$$\vdots$$

$$G^m \xleftarrow{s_{m,n}} R_{m,n} \xrightarrow{t_{m,n}} G^n$$

$$\vdots$$

Model categories

A **model category** is a category C equipped with three classes of morphisms

- $\mathcal{W}$: **weak equivalences**
- $\mathcal{C}$: **cofibrations**
- $\mathcal{F}$: **fibrations**

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- a fibration
  $$A \twoheadrightarrow B$$
  means whenever a composition exists in A it must exist in B.

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Lifting properties

A morphism $f : X \to Y$ has the **right lifting property** wrt $i : A \to B$ when

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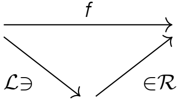
Given a class $\mathcal{I}$ of morphisms, we write

- $\mathcal{I}^{\boxslash}$ for the morphisms with the RLP wrt $\mathcal{I}$,
- ${}^{\boxslash}\mathcal{I}$ for the morphisms with the LLP wrt $\mathcal{I}$.

Weak factorization systems

A **weak factorization system** $(\mathcal{L}, \mathcal{R})$ on a category *C* is a pair of classes of morphisms such that

1. every morphism *f* of *C* factors as



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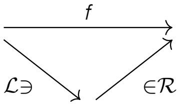

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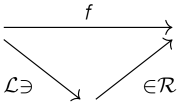

2. $\mathcal{L} = {}^{\boxslash}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\boxslash}$.

Remark

Note that $\mathcal{L}$ entirely determines $\mathcal{R} = \mathcal{L}^{\boxslash}$.

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A **model category** is a category C equipped with three classes of morphisms

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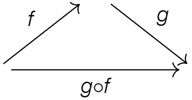
such that

1. $\mathcal{W}$ satisfies the 2-of-3 property:



2. $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ form a weak factorization system,
3. $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ form a weak factorization system.

Let's build a model structure on **Pres**!

Weak equivalences

We take as **weak equivalences** $\mathcal{W}$ the class of morphisms

$$f : \langle G \mid R \rangle \to \langle G' \mid R' \rangle$$

which induce an isomorphism on the presented monoids

$$G^*/R \quad = \quad G'^*/R' \,.$$

This satisfies 2-of-3.

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At least, we need the following class $\mathcal{I}$ of morphisms

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$$\langle a_1, \dots, a_m, b_1, \dots, b_n \mid \rangle \to \langle a_i, b_i \mid a_1 \dots a_m = b_1 \dots b_m \rangle$$

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called **generating cofibrations** and we define

$$\mathcal{C} \quad = \quad {}^{\boxslash}(\mathcal{I}^{\boxslash}) \, .$$

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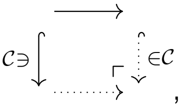
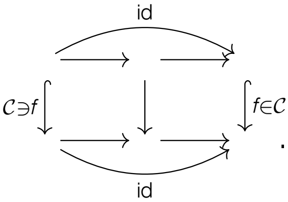
$$\mathcal{C} \quad = \quad {}^{\boxslash}(\mathcal{I}^{\boxslash})$$

is the smallest class of morphisms of **Pres** which

- contains $\mathcal{I}$,
- is closed under pushouts: 

,
- is closed under transfinite compositions,
- is closed under retracts: 

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Cofibrations

We deduce that

- every object $P = \langle G \mid R \rangle$ is cofibrant:

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Trivial cofibrations

The **trivial cofibrations** in $\mathcal{C} \cap \mathcal{W}$ should add structure while being weak equivalences.

At least, we need the following class $\mathcal{J}$ of morphisms, corresponding to atomic *Tietze transformations*:

- $\langle a_1, \ldots, a_n \mid \rangle \to \langle a_1, \ldots, a_n, b \mid a_1 \ldots a_n = b \rangle$
- $\langle a_1, \ldots, a_n \mid \rangle \to \langle a_1, \ldots, a_n \mid a_1 \ldots a_n = a_1 \ldots a_n \rangle$
- [transitivity, symmetry, congruence]

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Thus trivial cofibrations are retracts of transfinite compositions of Tietze transformations.

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i.e. f is surjective on generators.

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i.e. if $f(u) = f(v)$ then $u = v$, both in one relation step.

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Given $f : \langle G \mid R \rangle \to \langle G' \mid R' \rangle$ a trivial fibration (= surjective + lifting relations), do we have $f \in \mathcal{W}$?

$$
\begin{array}{ccccccccc}
\langle G \mid R \rangle & & u & & & & & & u' \\
\downarrow{\scriptstyle f} & & \downarrow & & & & & & \downarrow \\
\langle G' \mid R' \rangle & & v_1 & = & v_2 & = & \dots & = & v_n
\end{array}
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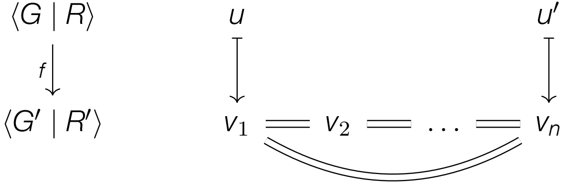

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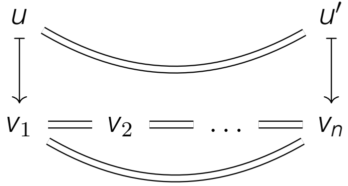

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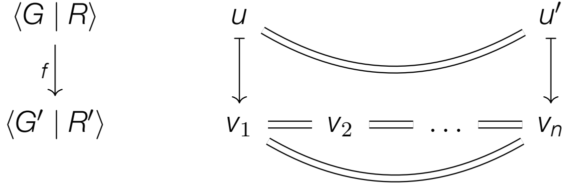

In general, $\mathcal{F} \cap \mathcal{W} \neq \mathcal{I}^{\boxslash}$!

Fibrant objects

The **fibrant objects** are presentations such that

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...or we can change the generating cofibrations and say that we can add a chain of relations between two words (instead of only one relation).

Weak equivalences

We said:

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A Tietze equivalence is a zig-zag of Tietze expansions (i.e. transfinite sequences of Tietze transformations).

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Proposition

A weak equivalence can be expressed as a zig-zag of trivial cofibrations.

⚠ trivial cofibrations are *retracts* of Tietze expansions.

Factorizations

We have two factorizations of morphisms as required:

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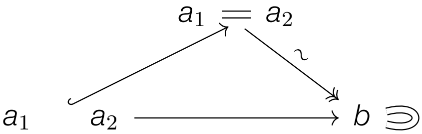

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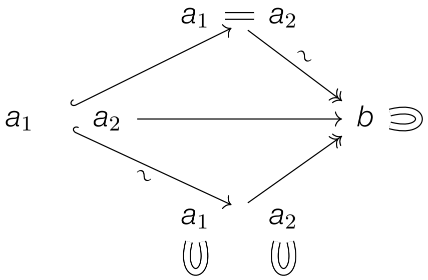

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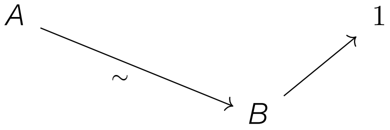

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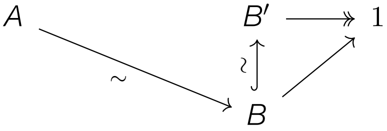

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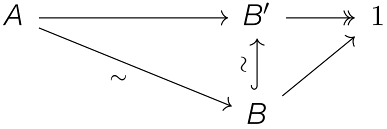

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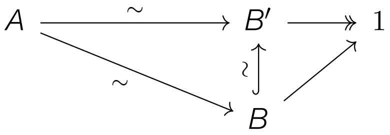

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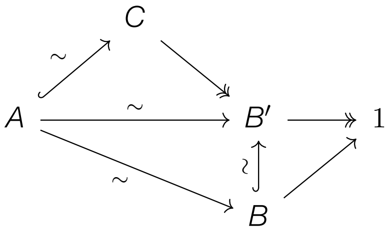

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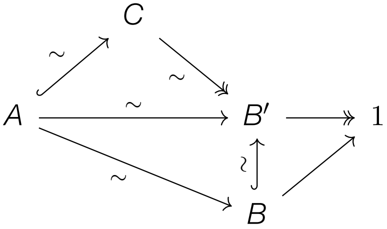

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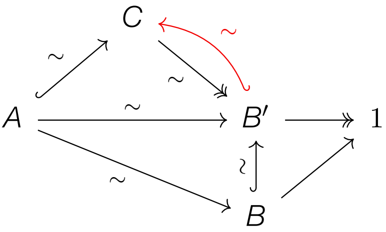

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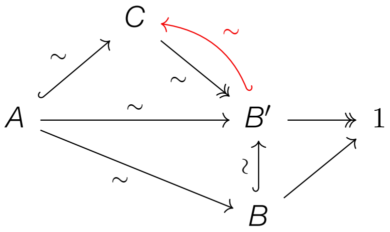

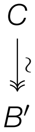

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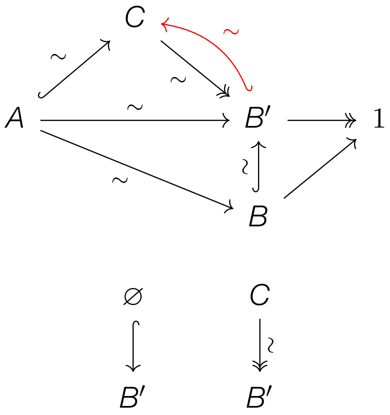

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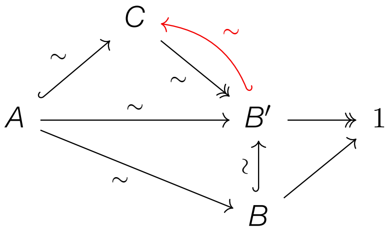

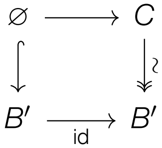

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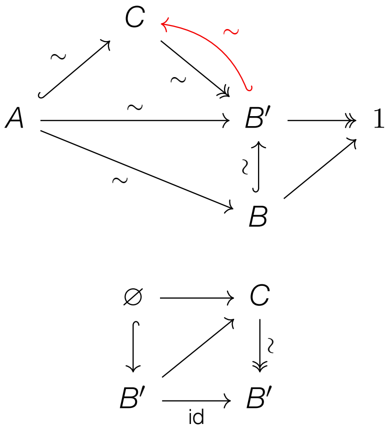

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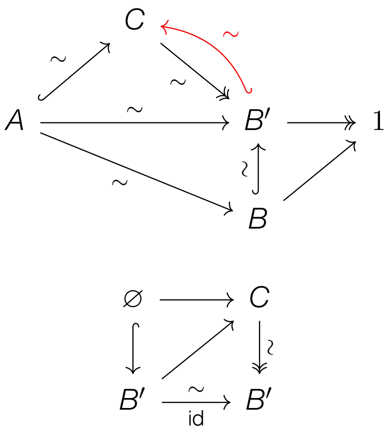

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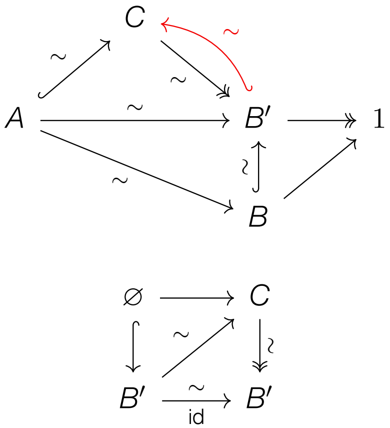

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