Tietze Equivalences as Weak Equivalences

Simon Henry Samuel Mimram

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Abstract

A given monoid usually admits many presentations by generators and relations and the notion of Tietze equivalence characterizes when two presentations describe the same monoid: it is the case when one can transform one presentation into the other using the two families of so-called Tietze transformations. The goal of this article is to provide an abstract and geometrical understanding of this well-known fact, by constructing a model structure on the category of presentations, in which two presentations are weakly equivalent when they present the same monoid. We show that Tietze transformations form a pseudo-generating family of trivial cofibrations and give a proof of the completeness of these transformations by an abstract argument in this setting.

In order to navigate between the various presentations of a monoid, a very convenient tool is provided by Tietze transformations, originally investigated for groups [12] (see also [9, chapter II]): these are two families of elementary transformations one can perform on a monoid while preserving the presented monoid. Typically, the Knuth-Bendix completion procedure for string rewriting systems uses such transformations in order to turn a presentation of a monoid into another presentation of the same monoid which has the property of being convergent [8, 6], and thus for which the word problem is easily decidable. The Tietze transformations moreover enjoy a completeness property: given any two presentations of a given monoid, there is a way of transforming the first into the second by performing a series of such transformations.

In this article, we provide a conceptual and geometrical point of view on Tietze transformations, by showing that they can be abstractly thought of as "continuously deforming" the presentations. In order to make this formal, we consider the category of presentations of monoids with suitably chosen morphisms (it turns out that we need to allow some sort of degeneracies) and construct a model structure on it, where weakly equivalent presentations are presentations of a same monoid. We then show that the Tietze transformations can then be interpreted in this setting as a pseudo-generating family of trivial cofibrations: they generate trivial cofibrations with fibrant codomain. Finally, the classical proof of completeness for Tietze transformations proceeds by constructing some kind of cospan of Tietze transformations between two presentations of the same monoid: we explain here how to reconstruct this proof by purely abstract arguments based on our model structure.

The main goal of this article is thus to shed new light on theses well-known concepts and proofs, and advocate the relevance of homotopical methods to people working with presentations of monoids, which is why we have done our best to have a self-contained exposition. We see this work as a first step in order to tackle generalizations of Tietze transformations to higher dimension (e.g. coherent presentations of categories [5, Section 2.1]) or more involved structures (Lawvere theories, operads, etc.).

We recall the notion of Tietze transformation between presentations of monoids in section 1, and of model category in section 2. We construct our model structure on the category of presentations in section 3, show that Tietze transformations form a pseudo-generating family of trivial cofibrations in section 4 and use this to abstractly study Tietze equivalences in section 5.

1 Tietze equivalences of presentations of monoids

1.1 Monoid. A *monoid* $(M,\cdot,1)$ consists of a set M equipped with a binary *multiplication* operation $\cdot$ and a *unit* element 1 such that multiplication is associative and the unit acts as a neutral element. A *morphism* $f : M \to N$ between two monoids is a function which preserves multiplication and unit. We write **Mon** for the resulting category.

1.2 Free and quotient monoids. Given a set X, we write X^* for the *free monoid* generated by X: its elements are words over X, multiplication uv of two words is their concatenation, and the unit is the empty word, noted 1.

Given a binary relation $\sim$ on a monoid M, we write $M/{\sim}$ for the *quotient monoid* whose elements and equivalence classes of elements of M by the congruence generated by $\sim$, and multiplication and unit are induced by those of M.

1.3 Presentation. A *presentation* $\mathsf{P} = \langle \mathsf{P}_1 \mid \mathsf{P}_2 \rangle$ consists of

- a set P_1 of *generators*,
- a set $\mathsf{P}_2 \subseteq \mathsf{P}_1^* \times \mathsf{P}_1^*$ of *relations*.

Such a presentation is *finite* when both the sets P_1 and P_2 are. A relation $(u,v) \in \mathsf{P}_1^*$ is generally denoted by "$u \Rightarrow v$" and we write $\overset{\mathsf{P}}{=}$ for the smallest congruence generated by P_2. A *morphism* $f : \mathsf{P} \to \mathsf{Q}$ between presentations is a function $f : \mathsf{P}_1 \to \mathsf{Q}_1$ such that, for every $u \Rightarrow v \in \mathsf{P}_2$, we have $f^*(u) \Rightarrow f^*(v) \in \mathsf{Q}_2$. A *subpresentation* P' of P is a presentation equipped with a morphism $\mathsf{P}' \to \mathsf{P}$ whose underlying function is an inclusion. We write **Pres** for the category of presentations and their morphisms. Note that, by definition, there is a forgetful functor **Pres** $\to$ **Set** sending a presentation P to its set P_1 of generators.

1.4 Presented monoid. The monoid $\overline{\mathsf{P}}$ *presented* by a presentation P is the quotient monoid $\overline{\mathsf{P}} = \mathsf{P}_1^*/\mathsf{P}_2$ i.e., the quotient of the free monoid P_1^* by the congruence $\overset{\mathsf{P}}{=}$ generated by P_2. We often write $q^{\mathsf{P}} : \mathsf{P}_1^* \to \overline{\mathsf{P}}$ for the quotient morphism and, given $u \in \mathsf{P}_1^*$, we write $\overline{u}$ for its equivalence class $q^{\mathsf{P}}(u)$. More generally, we say that a monoid M is *presented* by P when M is isomorphic to $\overline{\mathsf{P}}$, what we sometimes write $M \simeq \langle \mathsf{P}_1 \mid \mathsf{P}_2 \rangle$. This construction extends as a functor **Pres** $\to$ **Mon**.

Example 1. We have the following presentations:

$$\mathbb{N} \simeq \langle a \mid \rangle \qquad \mathbb{N}\times\mathbb{N} \simeq \langle a,b \mid ab \Rightarrow ba \rangle$$
$$\mathbb{N}/2\mathbb{N} \simeq \langle a \mid aa \Rightarrow 1 \rangle \qquad \mathbb{Z} \simeq \langle a,b \mid ab \Rightarrow 1, ba \Rightarrow 1 \rangle.$$

1.5 Standard presentation. To any monoid M, one can associate a presentation $\langle M \rangle$, called the *standard presentation* of M, defined by

$$\begin{aligned} \mathsf{P}_1 &= \{\underline{a} \mid a \in M\} \\ \mathsf{P}_2 &= \{\underline{a}_1 \ldots \underline{a}_n \Rightarrow \underline{b}_1 \ldots \underline{b}_m \mid a_1 \ldots a_n = b_1 \ldots b_m\}, \end{aligned}$$

i.e., it contains the elements of the monoids as generators and there is a relation between two words of generators when the product of their elements are equal. This construction extends as a functor $\mathbf{Mon} \to \mathbf{Pres}$. It can be used to show that any monoid admits at least one presentation:

Lemma 2. Given a monoid M, its standard presentation is a presentation of M: $\overline{\langle M \rangle} \simeq M$.

Lemma 3. The presentation functor is left adjoint to the standard presentation functor

$$\begin{tikzcd} \mathbf{Pres} \arrow[r, bend left, "\overline{-}"] \arrow[r, phantom, "\perp"] & \mathbf{Mon} \arrow[l, bend left, "\langle - \rangle"] \end{tikzcd}$$

the counit of the adjunction being an isomorphism.

1.6 Reflexive presentations. A presentation P is *reflexive* when for every word $u \in \mathsf{P}_1^*$ there is a relation $u \Rightarrow u \in \mathsf{P}_2$. We write $\mathbf{rPres}$ for the full subcategory of $\mathbf{Pres}$ on reflexive presentations.

Lemma 4. The expected forgetful functor admits a left adjoint

$$\begin{tikzcd} \mathbf{Pres} \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathbf{rPres} \arrow[l, bend left] \end{tikzcd}$$

sending a presentation P to the presentation Q with

$$\mathsf{Q}_1 = \mathsf{P}_1 \qquad\qquad \mathsf{Q}_2 = \mathsf{P}_2 \cup \{u \Rightarrow u \mid u \in \mathsf{P}_1^*\}$$

and $\mathbf{rPres}$ is equivalent to the Kleisli category of the monad on $\mathbf{Pres}$ induced by the adjunction.

Lemma 5. The category $\mathbf{rPres}$ is equivalent to the category whose objects are presentations (not necessarily reflexive) and a morphism $f : \mathsf{P} \to \mathsf{Q}$ is a function $f : \mathsf{P}_1 \to \mathsf{Q}_1$ such that for every relation $u \Rightarrow v \in \mathsf{P}_2$ we have either $f(u) \Rightarrow f(v) \in \mathsf{Q}_2$ or $f(u) = f(v)$.

In the following, when describing concrete examples of reflexive presentations, we generally omit mentioning reflexivity relations (or, alternatively, the description of morphisms given by previous lemma could be considered).

Remark 6. The standard presentation is clearly reflexive and thus the adjunction of lemma 3 restricts to an adjunction between reflexive presentations and monoids.

1.7 Equivalence between presentations. There is a very natural notion of equivalence of presentations: two presentations can be considered as *equivalent* when they present isomorphic monoids. In order to provide a concrete and amenable description of this relation, Tietze has introduced a family of transformations on presentations which characterize the equivalence. Those were originally formulated in the context of presentations of groups [12].

We begin with a simpler but useful characterization of the equivalence:

Lemma 7. Two presentations P and Q are such that $\overline{\mathsf{P}} \simeq \overline{\mathsf{Q}}$ if and only if there is a cospan of presentations

$$\mathsf{P} \xrightarrow{f} \mathsf{R} \xleftarrow{g} \mathsf{Q}$$

such that the induced monoid morphisms $\overline{f} : \overline{\mathsf{P}} \to \overline{\mathsf{R}}$ and $\overline{g} : \overline{\mathsf{Q}} \to \overline{\mathsf{R}}$ are isomorphisms.

Proof. If there is a cospan as above then we have $\overline{\mathsf{P}} \simeq \overline{\mathsf{R}} \simeq \overline{\mathsf{Q}}$ and P and Q are thus equivalent. Conversely, suppose that P presents the monoid M, i.e., there is an isomorphism $\overline{\mathsf{P}} \to M$. Under the adjunction of lemma 3, this induces a map $f : \mathsf{P} \to \langle M \rangle$ such that $\overline{f} : \overline{\mathsf{P}} \to \overline{\langle M \rangle} = M$. Similarly, we can construct a map $g : \mathsf{P} \to \langle M \rangle$. $\square$

1.8 Tietze transformation. The *elementary Tietze transformations* are the following transformations producing a new presentation Q from a presentation P:

(T1) *adding a derivable generator*: given a new generator $a \notin \mathsf{P}_1$ and word $u \in \mathsf{P}_1^*$, we define the presentation Q by

$$\mathsf{Q}_1 = \mathsf{P}_1 \sqcup \{a\} \qquad\qquad \mathsf{Q}_2 = \mathsf{P}_2 \cup \{u \Rightarrow a\}\,,$$

(T2) *adding a derivable relation*: given two words $u, v \in \mathsf{P}_1^*$ such that $u \overset{\mathsf{P}}{=} v$, we define the presentation Q by

$$\mathsf{Q}_1 = \mathsf{P}_1 \qquad\qquad \mathsf{Q}_2 = \mathsf{P}_2 \cup \{u \Rightarrow v\}\,.$$

It is easy to see that those transformations preserve the presented monoids:

Lemma 8. Given an elementary Tietze transformation from P to Q, we have an isomorphism $\overline{\mathsf{P}} \simeq \overline{\mathsf{Q}}$.

A *Tietze transformation* from P to Q consists in a finite sequence of presentations

$$\mathsf{P} = \mathsf{P}^0, \mathsf{P}^1, \mathsf{P}^2, \ldots, \mathsf{P}^n = \mathsf{Q}$$

such that for every i with $0 \leq i < n$ there is an elementary Tietze transformation from P^i to P^{i+1}. In this situation, we sometimes write

$$\mathsf{P} \rightsquigarrow \mathsf{Q}$$

Note that contrarily to the usual convention, we do not allow here removing generators or relations.

The transformation (T2) can be replaced by the following four transformations:

(T2r) *reflexivity*: given $u \in \mathsf{P}_1^*$, we define Q by

$$\mathsf{Q}_1 = \mathsf{P}_1 \qquad\qquad \mathsf{Q}_2 = \mathsf{P}_2 \cup \{u \Rightarrow u\}\,,$$

(T2s) *symmetry*: given $u, v \in \mathsf{P}_1^*$ such that $u \Rightarrow v \in \mathsf{P}_2$, we define Q by

$$\mathsf{Q}_1 = \mathsf{P}_1 \qquad\qquad \mathsf{Q}_2 = \mathsf{P}_2 \cup \{v \Rightarrow u\}\,,$$

(T2t) *transitivity*: given $u, v, w \in \mathsf{P}_1^*$ such that $u \Rightarrow v, v \Rightarrow w \in \mathsf{P}_2$, we define Q by

$$\mathsf{Q}_1 = \mathsf{P}_1 \qquad\qquad \mathsf{Q}_2 = \mathsf{P}_2 \cup \{u \Rightarrow w\}\,.$$

(T2c) *context*: given $u, v, v', w \in \mathsf{P}_1^*$ such that $v \Rightarrow v' \in \mathsf{P}_2$, we define Q by

$$\mathsf{Q}_1 = \mathsf{P}_1 \qquad\qquad \mathsf{Q}_2 = \mathsf{P}_2 \cup \{uvw \Rightarrow uv'w\}\,,$$

The resulting systems are the same in the following sense:

Lemma 9. The following assertions are equivalent: there is a Tietze transformation from P to Q

(i) using (T1) and (T2),

(ii) using (T1), (T2r), (T2s), (T2t) and (T2c).

In the following, unless otherwise mentioned, we use the second set of Tietze transformations which are easier to work with because they are more “atomic”.

1.9 Tietze equivalence. A *Tietze equivalence* from P to Q is a finite sequence of presentations $\mathsf{P} = \mathsf{P}^0, \mathsf{P}^1, \mathsf{P}^2, \ldots, \mathsf{P}^n = \mathsf{Q}$ such that for every i with $0 \leq i < n$ there is a Tietze transformation from P^i to P^{i+1} or from P^{i+1} to P^i. Two presentations are *Tietze equivalent* when there is a Tietze equivalence between them. Otherwise said, the Tietze equivalence is the smallest equivalence relation relating any two presentations between which there is an (elementary) Tietze transformation. By lemma 8 above, Tietze equivalences preserve the presented monoids. It well known that, for finite presentations, the converse holds [9, chapter II]:

Theorem 10. Given two finite presentations P and Q, we have $\overline{\mathsf{P}} \simeq \overline{\mathsf{Q}}$ if and only if P and Q are Tietze equivalent.

Proof. The right-to-left implication follows from lemma 8. For the left-to-right implication, suppose given an isomorphism $\overline{\mathsf{P}} \simeq \overline{\mathsf{Q}}$. For the sake of simplicity we suppose that we actually have $\mathsf{P} = \mathsf{Q}$ and more generally that Tietze equivalent presentation give rise to identical presented monoids (the proof without this assumption can be constructed from the one below by inserting isomorphisms at required places). Given a generator $a \in \mathsf{P}_1$, there exists an element $u \in \mathsf{Q}_1^*$ such that $q^{\mathsf{P}}(a) = q^{\mathsf{Q}}(u)$. We write a^{Q} for a choice of such an element. Dually, given $b \in \mathsf{Q}_1$, we write $b^{\mathsf{P}} \in \mathsf{P}_1^*$ for a word such that $q^{\mathsf{P}}(b^{\mathsf{P}}) = q^{\mathsf{Q}}(b)$. We generalize this notation to words $u = a_1 \ldots a_n \in \mathsf{P}_1^*$, by setting $u^{\mathsf{Q}} = a_1^{\mathsf{Q}} \ldots b_n^{\mathsf{Q}}$ (and we define v^{Q} for $v \in \mathsf{Q}_1^*$ similarly). Note that, for $u \in \mathsf{P}_1^*$, we have

$$(u^{\mathsf{Q}})^{\mathsf{P}} \overset{\mathsf{P}}{=} u \tag{1}$$

(and dually). We construct a presentation R by

$$\mathsf{R}_1 = \mathsf{P}_1 \sqcup \mathsf{Q}_1 \qquad\qquad \mathsf{R}_2 = \mathsf{P}_2 \sqcup \mathsf{Q}_2 \sqcup \mathsf{R}_2^{\mathsf{P}} \sqcup \mathsf{R}_2^{\mathsf{Q}}$$

where

$$\mathsf{R}_2^{\mathsf{P}} = \left\{ a^{\mathsf{Q}} \Rightarrow a \,\middle|\, a \in \mathsf{P}_1 \right\} \qquad\qquad \mathsf{R}_2^{\mathsf{Q}} = \left\{ b^{\mathsf{P}} \Rightarrow b \,\middle|\, b \in \mathsf{Q}_1 \right\}$$

We now construct a Tietze transformation from P to R. Dually, we will be able to construct a Transformation from Q to R and we will be able to conclude that P and Q are Tietze equivalent:

$$\mathsf{P} \rightsquigarrow \mathsf{R} \leftsquigarrow \mathsf{Q}.$$

By using Tietze transformations (T1), starting from P, we can add each generator $b \in \mathsf{Q}_1$ along with the relation $b^{\mathsf{P}} \Rightarrow b$, thus obtaining a transformation

$$\mathsf{P} = \langle \mathsf{P}_1 \,|\, \mathsf{P}_2 \rangle \rightsquigarrow \mathsf{P}' = \langle \mathsf{P}_1, \mathsf{Q}_1 \,|\, \mathsf{P}_2, \mathsf{R}_2^{\mathsf{Q}} \rangle.$$

Note that, given a word $u \in \mathsf{Q}_1^*$, we have $u^{\mathsf{P}} \overset{\mathsf{P}'}{=} u$. Therefore, given $a \in \mathsf{P}_1$, we have $a^{\mathsf{Q}} \overset{\mathsf{P}'}{=} (a^{\mathsf{Q}})^{\mathsf{P}} \overset{\mathsf{P}'}{=} a$ by (1). By using Tietze transformations (T2) we can add each derivable relation $a^{\mathsf{Q}} \Rightarrow a$ to P' thus reaching the presentation R:

$$\mathsf{P} \rightsquigarrow \mathsf{P}' \rightsquigarrow \mathsf{R}.$$

□

Remark 11. The proof above uses Tietze transformations (T1) and (T2). The proof can be performed by using the other set of transformations given by lemma 9, at the cost of having to take a slightly bigger R.

The proof of theorem 10 constructs a "cospan" of Tietze transformations. We will see that it can be constructed by using tools coming from model categories.

1.10 An example. Consider the presentations

$$\langle a \mid \rangle \qquad \text{and} \qquad \langle a, b \mid b \Rightarrow bb, 1 \Rightarrow bb \rangle.$$

Both present the additive monoid $\mathbb{N}$, and indeed there is a Tietze equivalence between them:

$$\begin{aligned}
\langle a \mid \rangle &\to \langle a, b \mid 1 \Rightarrow b \rangle && \text{(T1)}\\
&\to \langle a, b \mid 1 \Rightarrow b, b \Rightarrow bb \rangle && \text{(T2c)}\\
&\to \langle a, b \mid 1 \Rightarrow b, b \Rightarrow bb, 1 \Rightarrow bb \rangle && \text{(T2t)}\\
&\to \langle a, b \mid 1 \Rightarrow b, b \Rightarrow bb, 1 \Rightarrow bb, bb \Rightarrow b \rangle && \text{(T2s)}\\
&\leftarrow \langle a, b \mid b \Rightarrow bb, 1 \Rightarrow bb, bb \Rightarrow b \rangle && \text{(T2t)}\\
&\leftarrow \langle a, b \mid b \Rightarrow bb, 1 \Rightarrow bb \rangle && \text{(T2s)}
\end{aligned}$$

Also note that both presentations are "minimal": there is no way to remove a derivable generator or a relation without changing the presented monoid. In particular, starting from the second presentation, we have to add relations first in order to be able remove the generator b and all the relations.

1.11 Generalization to infinite presentations. The above theorem 10 holds only for finite presentations, which is the way it is usually stated. It can easily be generalized to presentations of arbitrary cardinality by allowing the Tietze transformations to add *sets* of derivable generators and *sets* of derivable relations (instead of only one), what we call *generalized Tietze transformations*. The right way to think of those is as being obtained as cellular extensions of elementary Tietze transformations and we will prove in theorem 58 the following generalization of theorem 10, which was already known, see for instance [10, section 1.5]:

Theorem 12. Given two presentation P and Q, we have $\overline{\mathsf{P}} \simeq \overline{\mathsf{Q}}$ if and only if they are related by a zig-zag of generalized Tietze transformations, i.e., there exists a finite sequence of presentations

$$\mathsf{P} = \mathsf{P}^0, \mathsf{P}^1, \ldots, \mathsf{P}^{2n} = \mathsf{Q}$$

such that for every index i, there is a generalized Tietze transformation from P^{2i} to P^{2i+1} and from P^{2i+2} to P^{2i+1}.

Remark 13. The naive generalization of theorem 10, which states that two presentations have the same presented monoid if and only if they are related by a "possibly infinite zig-zag" of elementary Tietze transformations, is plain wrong (and this is not what the above theorem states). For instance, consider the following presentation of the monoid $\mathbb{N}$:

$$\mathsf{P} = \langle a, b_i \mid a \Rightarrow b_i, b_i \Rightarrow b_{i+1} \rangle_{i \in \mathbb{N}}$$

and write P^i for P with the relations $a \Rightarrow b_i$ removed for $i < k$. We have $\mathsf{P}^0 = \mathsf{P}$ and the relation $a \Rightarrow b_k$ is derivable in P^k, so that there is an elementary Tietze transformation from P^{k+1} to P^k. However, writing

$$\mathsf{P}^\infty = \langle a, b_i \mid b_i \Rightarrow b_{i+1} \rangle_{i \in \mathbb{N}}$$

we have that P^0 does not present the same monoid as P^∞ even though there is an "infinite sequence of elementary Tietze transformations" between them. Namely, P presents $\mathbb{N}$ whereas P^∞ presents $\mathbb{N} * \mathbb{N}$, the free product of two copies of $\mathbb{N}$, and two are not isomorphic (the former is commutative whereas the later is not).

2 Model categories

In this section, we recall elementary definitions and facts about model categories which we will use in the following and refer the reader to classical textbooks for details [7].

2.1 Lifting properties. Suppose fixed a category. A morphism $p : X \to Y$ has the *right lifting property*, or *rlp*, with respect to a morphism $i : A \to B$ when for every morphisms $f : A \to X$ and $g : B \to Y$ such that $p \circ f = g \circ i$ there exists a morphism $h : B \to X$ making the following diagram commute:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dotted, "h"] & Y \end{tikzcd}$$

In this situation, we also say that i has the *left lifting property*, or *llp*, with respect to f, and write $i \boxslash p$. Given two classes $\mathcal{L}$ and $\mathcal{R}$, we write $\mathcal{L} \boxslash \mathcal{R}$ whenever $i \boxslash p$ for every $i \in \mathcal{L}$ and $p \in \mathcal{R}$. We also write $\mathcal{L}^{\boxslash}$ (resp. ${}^{\boxslash}\mathcal{R}$) for the class of morphism with the rlp (resp. llp) with respect to $\mathcal{L}$ (resp. $\mathcal{R}$).

Lemma 14. Given classes $\mathcal{L}$, $\mathcal{L}'$, $\mathcal{R}$ and $\mathcal{R}'$ of morphisms,

$$\begin{array}{lll} \mathcal{L} \subseteq {}^{\boxslash}(\mathcal{L}^{\boxslash}) & ({}^{\boxslash}(\mathcal{L}^{\boxslash}))^{\boxslash} = \mathcal{L}^{\boxslash} & \mathcal{L} \subseteq \mathcal{L}' \text{ implies } \mathcal{L}^{\boxslash} \supseteq \mathcal{L}'^{\boxslash}, \\ \mathcal{R} \subseteq ({}^{\boxslash}\mathcal{R})^{\boxslash} & {}^{\boxslash}(({}^{\boxslash}\mathcal{R})^{\boxslash}) = {}^{\boxslash}\mathcal{R} & \mathcal{R} \subseteq \mathcal{R}' \text{ implies } {}^{\boxslash}\mathcal{R} \supseteq {}^{\boxslash}\mathcal{R}'. \end{array}$$

Lemma 15. We suppose the ambient category cocomplete. A class of the form $\mathcal{L} = {}^{\boxslash}\mathcal{R}$ contains isomorphisms and is closed under

- coproducts: for any family $(i_k : A_k \to B_k)_{k \in K}$ of morphisms in $\mathcal{L}$, the morphism

  $$\coprod_{k \in K} i_k : \coprod_{k \in K} A_k \to \coprod_{k \in K} B_k$$

  is also in the class,

- pushouts: for any morphism $i : A \to B$ in $\mathcal{L}$ and morphism $f : A \to A'$, for any pushout diagram

  $$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & A' \arrow[d, "j"] \\ B \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & B' \end{tikzcd}$$

  the morphism j also belongs to $\mathcal{L}$,

- countable compositions: for any diagram

  $$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} \cdots \tag{2}$$

  consisting of morphisms $f_k : A_k \to A_{k+1}$ in $\mathcal{L}$ for $k \in \mathbb{N}$, the canonical morphism

  $$A_0 \to \operatorname{colim}_k A_k$$

  also belongs to $\mathcal{L}$,

– retracts: given a morphism $i : A \to B$ and two retracts $r \circ s = \mathrm{id}_{A'}$ and $r' \circ s' = \mathrm{id}_{B'}$, any morphism $j : A' \to B'$ for which there is a commutative diagram

$$
\begin{tikzcd}
A' \arrow[r, "s"'] \arrow[d, "j"'] \arrow[rr, bend left, "\mathrm{id}_{A'}"] & A \arrow[r, "r"'] \arrow[d, dashed, "i"] & A' \arrow[d, "j"] \\
B' \arrow[r, "s'"] \arrow[rr, bend right, "\mathrm{id}_{B'}"'] & B \arrow[r, "r'"] & B'
\end{tikzcd}
\tag{3}
$$

also belongs to $\mathcal{L}$.

Dually, any class for the form $\mathcal{L}^{\boxslash}$ contains isomorphisms and is closed under products, pullbacks, countable compositions and retracts.

Given a class $\mathcal{I}$ of morphisms, the class $\mathcal{I}$-cell of *$\mathcal{I}$-cellular extensions* is defined as the smallest class of morphisms closed under sums, pushouts and countable compositions (note that we do not require closure under retracts).

Lemma 16. A morphism is an $\mathcal{I}$-cellular extension if and only if it is a composite of pushouts of sums of elements of $\mathcal{I}$.

Lemma 17. Given a class $\mathcal{I}$ of morphisms, the class of $\mathcal{I}$-cellular extensions is included in ${}^{\boxslash}(\mathcal{I}^{\boxslash})$.

Proof. By lemma 14, we have $\mathcal{I}$ included in ${}^{\boxslash}(\mathcal{I}^{\boxslash})$ and, by lemma 15, this class is closed under sums, pushouts and countable compositions. □

Lemma 18 (Retract lemma). Given a factorization $f = p \circ i$ such that $f \boxslash p$, f is a retract of i. Dually, given a factorization $f = p \circ i$ such that $i \boxslash f$, f is a retract of p.

Proof. Since $f \boxslash p$, we have a map h such that

$$
\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "f"'] & Y \arrow[d, "p"] \\
Z \arrow[r, equal] \arrow[ur, "h"] & Z
\end{tikzcd}
$$

and the map f is thus a retract of i:

$$
\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "f"'] & X \arrow[r, equal] \arrow[d, dashed, "i"] & X \arrow[d, "f"] \\
Z \arrow[r, "h"'] & Y \arrow[r, "p"'] & Z
\end{tikzcd}
$$

as claimed. □

2.2 Weak factorization system. A *weak factorization system* on a category is a pair $(\mathcal{L}, \mathcal{R})$ of classes of morphisms such that

– every morphism f factors as $f = p \circ i$ with $i \in \mathcal{L}$ and $p \in \mathcal{R}$,

– $\mathcal{L} = {}^{\boxslash}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\boxslash}$.

Remark 19. From lemma 15 and lemma 18, the second condition can be equivalently be replaced by the two following conditions

- $\mathcal{L} \boxslash \mathcal{R}$,
- the classes $\mathcal{L}$ and $\mathcal{R}$ are closed under retracts.

One of the main technique in order to construct weak factorization systems is due to the following proposition [7, Section 2.1.2]. The notion of locally finitely presentable category is recalled in section 3.5.

Proposition 20 (Small object argument). Suppose that the category is cocomplete and locally finitely presentable. For any class $\mathcal{I}$ of morphisms, $({}^{\boxslash}(\mathcal{I}^{\boxslash}), \mathcal{I}^{\boxslash})$ is a weak factorization system. Moreover, every morphism f factors as $f = p \circ i$ where $i \in {}^{\boxslash}(\mathcal{I}^{\boxslash})$ is an $\mathcal{I}$-cellular extension and $p \in \mathcal{I}^{\boxslash}$. Moreover, every element of ${}^{\boxslash}(\mathcal{I}^{\boxslash})$ is a retract of an $\mathcal{I}$-cellular extension.

2.3 Model category. A *model category* is a category equipped with three classes of morphisms

- $\mathcal{C}$: cofibrations,
- $\mathcal{W}$: weak equivalences,
- $\mathcal{F}$: fibrations

such that

- the category is complete and cocomplete,
- weak equivalences satisfy the 2-out-of-3 property: given a diagram

$$\begin{tikzcd} & {} \arrow[dr, "g"] & \\ {} \arrow[ur, "f"] \arrow[rr, "g\circ f"'] & & {} \end{tikzcd}$$

  if two morphisms belong to $\mathcal{W}$ then so does the third,
- $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ forms a weak factorization system,
- $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ forms a weak factorization system.

An object X is *cofibrant* when the initial morphism $\varnothing \to X$ is a cofibration, and *fibrant* when the terminal morphism $X \to 1$ is a fibration.

From previous section, we can expect that the weak factorization system can be generated as lifting completions of some classes. Indeed, many model categories are *cofibrantly generated* (also sometimes called *combinatorial* since we work here with locally presentable categories) [7, Theorem 2.1.19]:

Proposition 21. In a locally presentable complete and cocomplete category, suppose given a class $\mathcal{W}$ of morphisms satisfying the 2-out-of-3 property and two sets $\mathcal{I}$ and $\mathcal{J}$ of morphisms such that the inclusions

$$\mathcal{I}^{\boxslash} \subseteq \mathcal{J}^{\boxslash} \cap \mathcal{W} \qquad\qquad {}^{\boxslash}(\mathcal{J}^{\boxslash}) \subseteq {}^{\boxslash}(\mathcal{I}^{\boxslash}) \cap \mathcal{W}$$

hold, one of them being an equality. Then we have a model category with $\mathcal{W}$ as weak equivalences, ${}^{\boxslash}(\mathcal{I}^{\boxslash})$ as cofibrations and $\mathcal{J}^{\boxslash}$ as fibrations. In this case, the elements of $\mathcal{I}$ as $\mathcal{J}$ are respectively called *generating cofibrations* and *generating trivial cofibrations*.

3 A model structure on reflexive presentations

Our aim is to construct a model structure on the category of reflexive presentations where weak equivalences correspond to presenting isomorphic categories and trivial cofibrations are Tietze transformations. The general strategy here is to use proposition 21 and thus to satisfy all the required hypothesis: in particular, we want to show the equality $\mathcal{I}^{\boxslash} = \mathcal{J}^{\boxslash} \cap \mathcal{W}$. Unless otherwise mentioned, all the presentations considered in this section are supposed to be reflexive; the reason for this shall be discussed in section 6.1. We first study some of the properties of the category of reflexive presentations.

3.1 Colimits. The category **rPres** has coproducts. Namely, given two presentations P and Q, their coproduct $\mathsf{P} \sqcup \mathsf{Q}$ is given by

$$(\mathsf{P} \sqcup \mathsf{Q})_1 = \mathsf{P}_1 \sqcup \mathsf{Q}_1 \qquad\qquad (\mathsf{P} \sqcup \mathsf{Q})_2 = \mathsf{P}_2 \sqcup \mathsf{Q}_2$$

and the argument generalizes to show that the category has small coproducts. In particular, the initial object $\varnothing$ is the empty presentation, with $\varnothing_1 = \emptyset$ and $\varnothing_2 = \emptyset$. Suppose given two morphisms of presentations

$$\mathsf{P} \underset{g}{\overset{f}{\rightrightarrows}} \mathsf{Q}$$

Their coequalizer is the presentation R whose set of generators is the coequalizer

$$\mathsf{P}_1 \underset{g}{\overset{f}{\rightrightarrows}} \mathsf{Q}_1 \xrightarrow{h} \mathsf{R}_1$$

i.e., the quotient set $\mathsf{R}_1 = \mathsf{Q}_1/{\sim}$ under the smallest equivalence relation such that $f(a) \sim f(b)$ for $a \in \mathsf{P}_1$, the function h being the quotient map, and the set of relations is

$$\mathsf{R}_2 = \{h^*(u) \Rightarrow h^*(v) \,|\, u \Rightarrow v \in \mathsf{Q}_1\}\,.$$

The category is thus cocomplete. In particular, the pushout of a diagram

$$\mathsf{Q}^1 \xleftarrow{f^1} \mathsf{P} \xrightarrow{f^2} \mathsf{Q}^2$$

is the presentation R whose set R_1 of generators is the pushout of the underlying sets of generators, with cocoone maps $h^1 : \mathsf{Q}^1 \to \mathsf{R}$ and $h^2 : \mathsf{Q}^2 \to \mathsf{R}$, and relations

$$R_2 = \left\{h^1(u) \Rightarrow h^1(v) \,\middle|\, u \Rightarrow v \in \mathsf{Q}^1_2\right\} \cup \left\{h^2(u) \Rightarrow h^2(v) \,\middle|\, u \Rightarrow v \in \mathsf{Q}^2_2\right\}.$$

Note that the forgetful functor $\mathbf{rPres} \to \mathbf{Set}$, sending a presentation to its underlying set of generators, preserves colimits.

3.2 Limits. The product $\mathsf{P}\times\mathsf{Q}$ of two reflexive presentations P and Q has generators $(\mathsf{P}\times\mathsf{Q})_1 = \mathsf{P}_1\times\mathsf{Q}_1$ and the set $(\mathsf{P} \times \mathsf{Q})_2$ of relations is

$$\left\{(a_1, a'_1) \ldots (a_m, a'_m) \Rightarrow (b_1, b'_1) \ldots (b_n, b'_n) \,\middle|\, \begin{array}{l} a_1 \ldots a_m \Rightarrow b_1 \ldots b_n \in \mathsf{P}_2 \\ a'_1 \ldots a'_m \Rightarrow b'_1 \ldots b'_n \in \mathsf{Q}_2 \end{array}\right\}.$$

This generalizes to small products. In particular, the terminal presentation 1 has one generator a and all relations of the form $a^m \Rightarrow a^n$ for $m, n \in \mathbb{N}$. Given two morphisms of polygraphs

$$\mathsf{P} \underset{g}{\overset{f}{\rightrightarrows}} \mathsf{Q}$$

their equalizer R is given by

$$\mathsf{R}_1 = \{a \in \mathsf{P}_1 \mid f(a) = g(a)\}$$

i.e., this is the equalizer of the underlying sets, and relations are

$$\mathsf{R}_2 = \{u \Rightarrow v \in \mathsf{P}_2 \mid f^*(u) = g^*(u) \text{ and } f^*(v) = g^*(u)\}\,.$$

The category is thus complete and the forgetful functor **rPres** $\to$ **Set** preserves limits.

3.3 Monomorphisms. A monomorphism $f : \mathsf{P} \to \mathsf{Q}$ is a morphism whose underlying function $f : \mathsf{P}_1 \to \mathsf{Q}_1$ is injective, i.e., the forgetful functor **rPres** $\to$ **Set** reflects monomorphisms. In this sense, the monomorphisms of presentations inherit the properties of those of the categories of sets. For instance,

Lemma 22. In **rPres**, monomorphisms are stable under coproducts, pushouts and countable compositions.

Proof. The forgetful functor to sets preserves coproducts, pushouts and countable compositions, and reflects monomorphisms. ☐

Remark 23. These stability conditions are not generally true in a category. As a counter-example, in the category of commutative rings, the inclusion $i : \mathbb{Z} \to \mathbb{Q}$ is a mono, but the sum (which is here the tensor product, and corresponds to the usual tensor product of $\mathbb{Z}$-modules)

$$\mathrm{id}_{\mathbb{Z}/2} \otimes i : \mathbb{Z}/2 = \mathbb{Z}/2 \otimes \mathbb{Z} \to \mathbb{Z}/2 \otimes \mathbb{Q} = 1$$

is not a mono. It is however the case that monomorphisms are stable under pushout in a topos (and, more generally, an adhesive category).

3.4 Epimorphisms. Similarly, an epimorphism $f : \mathsf{P} \to \mathsf{Q}$ is a morphism whose underlying function $\mathsf{P}_1 \to \mathsf{Q}_1$ is a surjection.

3.5 Local presentability. We refer to [1] for a detailed presentations of the notions introduced here. An object X of a category $\mathcal{C}$ is *finitely presentable* when the representable functor

$$\mathrm{Hom}(X, -) : \mathcal{C} \to \mathbf{Set}$$

preserves filtered limits: this means that for a diagram $(Y_i)_{i \in I}$ indexed by a filtered category I, the canonical morphism

$$\mathrm{colim}_i \, \mathrm{Hom}(X, Y_i) \to \mathrm{Hom}(X, \mathrm{colim}_i \, Y_i)$$

is an isomorphism. In particular, finitely presentable presentations objects are precisely the finite presentations.

A locally small category $\mathcal{C}$ is *locally finitely presentable* when it is cocomplete and there is a set of finitely presentable objects such that every object of $\mathcal{C}$ is a filtered colimit of objects in this set. In the case of the category of presentations, every presentation is the filtered colimit of its finite subpresentations, and the category **rPres** is thus locally finitely presentable (and **Mon** as well).

3.6 Weak equivalences. We write $\mathcal{W}$ for the class of morphisms $f : \mathsf{P} \to \mathsf{Q}$ such that the induced morphism $\overline{f} : \overline{\mathsf{P}} \to \overline{\mathsf{Q}}$ between presented monoids is an isomorphism. Many of the properties of isomorphisms are thus reflected on weak equivalences:

Lemma 24. The class $\mathcal{W}$ satisfies the 2-out-of-3 property and is closed under coproducts, pushouts, countable compositions and retracts.

Proof. The class of isomorphisms in any category satisfies the 2-out-of-3 property. Isormorphisms are closed under sums

$$
\begin{tikzcd}
A \arrow[rr, "i"] \arrow[dr] \arrow[rrrr, bend left, "\mathrm{id}_A"] & & B \arrow[rr, "i^{-1}"] \arrow[dr] & & A \arrow[dr] & \\
& A + A' \arrow[rr, "i+i'"] & & B + B' \arrow[rr, "i^{-1}+i'^{-1}"] & & A + A' \\
A' \arrow[rr, "i'"'] \arrow[ur] \arrow[rrrr, bend right, "\mathrm{id}_B"'] & & B' \arrow[rr, "i'^{-1}"'] \arrow[ur] & & A' \arrow[ur] &
\end{tikzcd}
$$

(this shows that the sum of two isomorphisms i and i' is still an isomorphism since the diagram also commutes if we replace $(i^{-1} + i'^{-1}) \circ (i + i')$ by $\mathrm{id}_{A+A'}$) and pushouts

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"'] \arrow[dr, phantom, "\ulcorner"] & A' \arrow[d, "j"] \arrow[ddr, bend left, "\mathrm{id}_{A'}"] \arrow[dddrr, bend left, "j"] & & \\
B \arrow[r, "g"'] \arrow[drr, bend right, "f \circ i^{-1}"'] \arrow[ddrrr, bend right, "g"'] & B' \arrow[dr, dotted, "j^{-1}"] & & \\
& & A' \arrow[dr, "j"] & \\
& & & B'
\end{tikzcd}
$$

(this shows that the pushout j of an isomorphism i along an arbitrary morphism f is an isomorphism since the diagram also commutes if we replace $j \circ j^{-1}$ by $\mathrm{id}_{B'}$). Consider a countable composition of isomorphisms $f_i : A_i \to A_{i+1}$ as in (2). There is a cocone on A_0 consisting of the morphisms

$$f_0^{-1} \circ f_1^{-1} \circ \ldots \circ f_{i-1}^{-1} : A_i \to A_0$$

which is easily seen to be universal and the composite is thus (isomorphic to) id_{A_0}. Consider a retract j of an isomorphism i as in (3). We claim that the morphism $j' = s \circ i^{-1} \circ r'$ is the inverse of j. Namely, one has

$$
\begin{aligned}
j' \circ j &= s \circ i^{-1} \circ r' \circ j \\
&= s \circ i^{-1} \circ i \circ r \\
&= s \circ r \\
&= \mathrm{id}_{A'}
\end{aligned}
\qquad\qquad
\begin{aligned}
j \circ j' &= j \circ s \circ i^{-1} \circ r' \\
&= s' \circ i \circ i^{-1} \circ r' \\
&= s' \circ r' \\
&= \mathrm{id}_{B'}
\end{aligned}
$$

□

3.7 Generating cofibrations. Consider the presentation with one generator and no relation:

$$\mathsf{G} = \langle a \mid \rangle.$$

Given $m, n \in \mathbb{N}$, we introduce notations for the following presentations, respectively with n generators and no relation, and with one relation between words of respective lengths m and n:

$$\mathsf{G}^n = \langle a_1, \ldots, a_n \mid \rangle \qquad \mathsf{R}^{m,n} = \langle a_1, \ldots, a_{m+n} \mid a_1 \ldots a_m \Rightarrow a_{m+1} \ldots a_{m+n} \rangle.$$

We write $\mathcal{I}$ for the class of morphisms, called *generating cofibrations*, consisting of the obvious inclusions of presentations

$$g : \varnothing \hookrightarrow \mathsf{G} \qquad r^{m,n} : \mathsf{G}^{m+n} \hookrightarrow \mathsf{R}^{m,n}$$

for some $m, n \in \mathbb{N}$.

3.8 Cofibrations. We write $\mathcal{C} = {}^{\boxslash}(\mathcal{I}^{\boxslash})$ for the class of morphisms whose elements are called *cofibrations*. Note that, given a presentation P, the pushouts

$$\begin{tikzcd} \varnothing \arrow[r, "g"] \arrow[d] & \mathsf{G} \arrow[d, dotted] \\ \mathsf{P} \arrow[r, dotted] & \mathsf{Q} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{G}^{m+n} \arrow[r, "r^{m,n}"] \arrow[d, "f"'] & \mathsf{R}^{m,n} \arrow[d, dotted] \\ \mathsf{P} \arrow[r, dotted] & \mathsf{Q} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

are respectively the polygraph obtained from P by adding a generator and the polygraph obtained from P by adding a relation (between the two words of P_1^* specified by f).

Lemma 25. Every presentation P is cofibrant, in the sense that the initial morphism $\varnothing \hookrightarrow \mathsf{P}$ is a cofibration.

Proof. By proposition 20, it is enough to show that the initial morphism $\varnothing \hookrightarrow \mathsf{P}$ can be obtained as a composite of pushouts of generating cofibrations, which amounts to show that every presentation can be obtained from the empty one by adding generators and relations, which we will do in this order (generators first, and then relations). Given a relation $u \Rightarrow v \in \mathsf{P}_2$, we have a canonical inclusion

$$\mathsf{G}^{|u|+|v|} \xrightarrow{r^{|u|+|v|}} \mathsf{R}^{|u|,|v|}$$

and a canonical inclusion

$$\mathsf{G}^{|u|+|v|} \longrightarrow \coprod_{a \in \mathsf{P}_1} \mathsf{G}.$$

By summing those morphisms over relations $(u, v) \in \mathsf{P}_2$, and post-composing with the codiagonal

$$\coprod_{(u,v) \in \mathsf{P}_2} \coprod_{a \in \mathsf{P}_1} \mathsf{G} \longrightarrow \coprod_{a \in \mathsf{P}_1} \mathsf{G},$$

we obtain a diagram

$$\begin{tikzcd} \coprod_{(u,v) \in \mathsf{P}_2} \mathsf{G}^{|u|+|v|} \arrow[r] \arrow[d] & \coprod_{(u,v) \in \mathsf{P}_2} \mathsf{R}^{|u|,|v|} \\ \coprod_{a \in \mathsf{P}_1} \mathsf{G} & \end{tikzcd}$$

whose pushout is precisely P. Finally, we consider the composite of morphism

$$\varnothing \longrightarrow \coprod_{a \in \mathsf{P}_1} \mathsf{G} \longrightarrow \mathsf{P}$$

where the second morphism is constructed in the cocone of the pushout. Again, this composite expresses the fact that any presentation can be constructed from the empty one by first adding all the generators, and then adding all the relations. $\square$

The construction given in the above proof easily generalizes to show:

Lemma 26. Any monomorphism $f : \mathsf{P} \to \mathsf{Q}$ is a cofibration (and, in fact, an $\mathcal{I}$-cellular extension).

Conversely, one has:

Lemma 27. Cofibrations are monomorphisms.

Proof. The generating cofibrations are monomorphisms. Moreover, monomorphisms are closed under coproducts, under pushouts and countable compositions by lemma 22. By proposition 20, cofibrations are thus retracts of monomorphisms. We conclude using the fact that monomorphisms are closed under retracts. Namely, suppose given a retract j of a monomorphism i, as in (3), and two morphisms h_1, h_2 such that $j \circ h_1 = j \circ h_2$, we have

$$\begin{aligned} j \circ h_1 &= j \circ h_2 \\ s' \circ j \circ h_1 &= s' \circ j \circ h_2 \\ i \circ s \circ h_1 &= i \circ s \circ h_2 \\ s \circ h_1 &= s \circ h_2 \\ r \circ s \circ h_1 &= r \circ s \circ h_2 \\ h_1 &= h_2 \end{aligned}$$

and we conclude. □

Corollary 28. The class $\mathcal{C}$ of cofibrations is the class of monomorphisms in **rPres**.

3.9 Trivial fibrations. The morphisms in the class $\mathcal{I}^{\boxslash}$ are called *trivial fibrations.* From the lifting property with respect to the generators we immediately deduce,

Lemma 29. The morphisms $f : \mathsf{P} \to \mathsf{Q}$ in $\mathcal{I}^{\boxslash}$ are those

- whose underlying function $f : \mathsf{P}_1 \to \mathsf{Q}_1$ is surjective, and
- such that for every $u, v \in \mathsf{P}_1^*$, $f^*(u) \Rightarrow f^*(v) \in \mathsf{Q}_2$ implies $u \Rightarrow v \in \mathsf{P}_2$.

Lemma 30. Trivial fibrations are weak equivalences: $\mathcal{I}^{\boxslash} \subseteq \mathcal{W}$.

Proof. Since $f : \mathsf{P}_1 \to \mathsf{Q}_1$ is surjective, we have that $\overline{f} : \overline{\mathsf{P}} \to \overline{\mathsf{Q}}$ is also surjective. We have to show that it is also injective in order to conclude. Suppose given $u, v \in \mathsf{P}_1^*$ such that $f^*(u) \overset{\mathsf{Q}}{=} f^*(v)$: we have a sequence

$$f^*(u) = w_0 \Leftrightarrow w_1 \Leftrightarrow \ldots \Leftrightarrow w_n = f^*(v)$$

where the arrows "$\Leftrightarrow$" mean that, for $0 \le i < n$, there is a decomposition of w_i and w_{i+1} as

$$w_i = t_i u_i v_i \quad \text{and} \quad w_{i+1} = t_i' u_i' v_i' \qquad \text{with} \qquad u_i \Rightarrow u_i' \in \mathsf{Q}_2 \quad \text{or} \quad u_i \Leftarrow u_i' \in \mathsf{Q}_2.$$

Moreover, since Q is reflexive, we can always suppose that this sequence is non-empty, i.e., $n > 0$: we can replace the empty sequence by the reflexivity relation $f^*(u) \Rightarrow f^*(u)$. By surjectivity, for $0 \le i \le n$, there are words t_i^{P}, u_i^{P}, v_i^{P}, $t_i'^{\mathsf{P}}$, $u_i'^{\mathsf{P}}$, $v_i'^{\mathsf{P}}$ in P_1^* whose image under f is respectively t_i, u_i, v_i, t_i', u_i', v_i', and we may moreover assume $t_0^{\mathsf{P}} u_0^{\mathsf{P}} v_0^{\mathsf{P}} = u$ and $t_{n-1}'^{\mathsf{P}} u_{n-1}'^{\mathsf{P}} v_{n-1}'^{\mathsf{P}} = v$. Finally, since f is a trivial fibration, we have $u_i \Rightarrow u_i'$ or $u_i \Leftarrow u_i'$ and we conclude that $u \overset{\mathsf{P}}{=} v$. □

From the results of section 3.4, one has:

Lemma 31. Every trivial fibration is an epimorphism.

3.10 Trivial cofibrations. The class of *trivial cofibrations* is $\mathcal{C} \cap \mathcal{W}$ and consists of monomorphisms $f : \mathsf{P} \to \mathsf{Q}$ such that the induced morphism of monoids $\overline{f} : \overline{\mathsf{P}} \to \overline{\mathsf{Q}}$ is an isomorphism.

Lemma 32. A morphism $f : \mathsf{P} \to \mathsf{Q}$ is a trivial cofibration when

– f is a monomorphism,

– for every $a \in \mathsf{Q}_1$, there exists $u \in \mathsf{P}_1^*$ such that $f(u) \overset{\mathsf{Q}}{=} a$,

– for $u, v \in \mathsf{P}_1^*$ such that $f(u) \overset{\mathsf{Q}}{=} f(v)$, we have $u \overset{\mathsf{P}}{=} v$.

Proof. Suppose that f is a trivial cofibration. Since f is a cofibration, it is a monomorphism by lemma 27. Given $a \in \mathsf{Q}_1$, since $\overline{f}$ is surjective there is $u \in \mathsf{P}_1^*$ such that $f(\overline{u}) = \overline{a}$, and we have $f(u) \overset{\mathsf{Q}}{=} a$. Given $u, v \in \mathsf{P}_1^*$ such that $f(u) \overset{\mathsf{Q}}{=} f(v)$, we have $\overline{f}(\overline{u}) = \overline{f}(\overline{v})$, thus $\overline{u} = \overline{v}$ in $\overline{\mathsf{P}}$ since $\overline{f}$ injective, and finally $u \overset{\mathsf{P}}{=} v$.

Conversely, suppose given a monomorphism $f : \mathsf{P} \to \mathsf{Q}$. Given $a \in \mathsf{Q}_1$, by hypothesis, there exists $u_a \in \mathsf{P}_1^*$ such that $\overline{f}(\overline{u}_a) = \overline{a}$. Therefore, given $\overline{v} \in \overline{\mathsf{Q}}$, for some $v = a_1 \ldots a_n \in \mathsf{Q}_1^*$, we have

$$\overline{f}(\overline{u}_{a_1} \ldots \overline{u}_{a_n}) = \overline{f}(\overline{u}_{a_1}) \ldots \overline{f}(\overline{u}_{a_n}) = \overline{a}_1 \ldots \overline{a}_n = \overline{v}$$

and $\overline{f}$ is thus surjective. Suppose given $u, v \in \mathsf{P}_1^*$ such that $\overline{f}(\overline{u}) = \overline{f}(\overline{v})$: we have $f(u) \overset{\mathsf{Q}}{=} f(v)$, thus $u \overset{\mathsf{P}}{=} v$ and finally $\overline{u} = \overline{v}$. □

Lemma 33. The class of trivial cofibrations satisfies ${}^{\boxslash}((\mathcal{C} \cap \mathcal{W})^{\boxslash}) = \mathcal{C} \cap \mathcal{W}$.

Proof. By lemma 15 and lemma 24, the class $\mathcal{C} \cap \mathcal{W}$ is closed under sums, pushouts, countable compositions and retracts. We conclude by proposition 20. □

3.11 Fibrations. The class $\mathcal{F}$ of *fibrations* is determined by the two other classes: should there be a model structure, it is necessarily $\mathcal{F} = (\mathcal{C} \cap \mathcal{W})^{\boxslash}$. An explicit description of fibrant objects is given by lemma 51 and lemma 44.

3.12 A model structure. Finally, we have all the ingredients required to construct a model structure.

Theorem 34. There is a model structure on the category **rPres** of reflexive presentations with $\mathcal{W}$ as weak equivalences, $\mathcal{C} = {}^{\boxslash}(\mathcal{I}^{\boxslash})$ as cofibrations and $\mathcal{F} = (\mathcal{C} \cap \mathcal{W})^{\boxslash}$ as fibrations.

Proof. We apply proposition 21, with $\mathcal{J} = \mathcal{C} \cap \mathcal{W}$. The 2-out-of-3 property for $\mathcal{W}$ was shown in section 3.6. We first show $\mathcal{I}^{\boxslash} \subseteq \mathcal{J}^{\boxslash} \cap \mathcal{W}$. We have $\mathcal{J} = \mathcal{C} \cap \mathcal{W} \subseteq \mathcal{C}$ thus $\mathcal{C}^{\boxslash} \subseteq \mathcal{J}^{\boxslash}$. Moreover, by lemma 14 and lemma 30, we have $\mathcal{C}^{\boxslash} \subseteq \mathcal{W}$. Thus $\mathcal{C}^{\boxslash} \subseteq \mathcal{J} \cap \mathcal{W}$. Finally, by lemma 33, we have

$${}^{\boxslash}(\mathcal{J}^{\boxslash}) = \mathcal{J} = \mathcal{C} \cap \mathcal{W} = {}^{\boxslash}(\mathcal{I}^{\boxslash}) \cap \mathcal{W}$$

which concludes the proof. □

In fact, the situation considered here can be axiomatized as in the following theorem, due to Smith, see [3, Theorem 1.7]:

Theorem 35. In a locally presentable category, suppose given a subcategory $\mathcal{W}$ and a set $\mathcal{I}$ of morphisms such that

– $\mathcal{W}$ is closed under retracts and has the 2-out-of-3 property,

– $\mathcal{I}^{\boxslash} \subseteq \mathcal{W}$,

– ${}^{\boxslash}(\mathcal{I}^{\boxslash}) \cap \mathcal{W}$ is closed under pushouts and transfinite compositions,

– $\mathcal{W}$ satisfies the solution set condition at $\mathcal{I}$.

Then there is a cofibrantly generated model structure with ${}^{\boxslash}(\mathcal{I}^{\boxslash})$ as cofibrations, $\mathcal{W}$ as weak equivalences and $({}^{\boxslash}(\mathcal{I}^{\boxslash}) \cap \mathcal{W})^{\boxslash}$ as fibrations.

We do not detail the solution set condition and simply note here that it is always satisfied for categories which are small, which is the case for the categories considered here.

3.13 A Quillen functor. The category **Mon** can canonically be equipped with the *trivial model structure* where weak equivalences are isomorphisms and every morphism is both fibrant and cofibrant. The presentation functor **rPres** $\to$ **Mon** described in section 1.4 is a left adjoint (lemma 3 and remark 6) which trivially preserves cofibrations and trivial cofibrations, and is thus a Quillen functor. Moreover, this functor reflects weak equivalences and, given a presentation P, the counit $\mathsf{P} \to \langle \overline{\mathsf{P}} \rangle$ of the adjunction is a weak equivalence: by [7, Corollary 1.3.16], the presentation functor is thus a Quillen equivalence. By [7, Proposition 1.3.13], this means that the derived functor induces, as expected, an equivalence of categories between the localization of **rPres** under weak equivalences and the one of **Mon** (which is **Mon** itself):

$$\mathrm{Ho}(\mathbf{rPres}) \cong \mathrm{Ho}(\mathbf{Mon}) \simeq \mathbf{Mon}.$$

4 Tietze transformations as trivial cofibrations

In section 4.1 below, we introduce a class $\mathcal{J}$ of morphisms of reflexive presentations such that pushouts of morphisms in this class correspond to elementary Tietze transformations. Contrarily to what one could expect, this family does not generate all trivial cofibrations: we have a strict inclusion ${}^{\boxslash}(\mathcal{J}^{\boxslash}) \subsetneq \mathcal{C} \cap \mathcal{W}$. However, we show that the two classes coincide for morphisms with fibrant codomain: we thus say that the class $\mathcal{J}$ is pseudo-generating, following the terminology of Simpson [11, Section 8.7].

4.1 Pseudo-generating trivial cofibrations. We write $\mathcal{J}$ for the class of morphisms of **rPres**, called *pseudo-generating trivial cofibrations*

$$\begin{gathered}
\langle a_1, \ldots, a_m \mid \rangle \hookrightarrow \langle a_1, \ldots, a_m, a_{m+1} \mid u \Rightarrow a_{m+1} \rangle \\
\langle a_1, \ldots, a_m \mid \rangle \hookrightarrow \langle a_1, \ldots, a_m \mid u \Rightarrow u \rangle \\
\langle a_1, \ldots, a_{m+n} \mid u \Rightarrow v \rangle \hookrightarrow \langle a_1, \ldots, a_{n+m} \mid u \Rightarrow v, v \Rightarrow u \rangle \\
\langle a_1, \ldots, a_{m+n+p} \mid u \Rightarrow v, v \Rightarrow w \rangle \hookrightarrow \langle a_1, \ldots, a_{n+m+p} \mid u \Rightarrow v, v \Rightarrow w, u \Rightarrow w \rangle \\
\langle a_1, \ldots, a_{m+n+p+q} \mid u \Rightarrow v \rangle \hookrightarrow \langle a_1, \ldots, a_{m+n+p+q} \mid wuw' \Rightarrow wvw' \rangle
\end{gathered}$$

for some $m, n, p \in \mathbb{N}$ with

$$\begin{aligned}
u &= a_1 \ldots a_m & w &= a_{m+n+1} \ldots a_{m+n+p} \\
v &= a_{m+1} \ldots a_{m+n} & w' &= a_{m+n+p+1} \ldots a_{m+n+p+q}
\end{aligned}$$

Lemma 36. Given a pseudo-generating cofibrations $j : \mathsf{P} \to \mathsf{Q}$ and a morphism of presentations $f : \mathsf{P} \to \mathsf{P}'$, consider the pushout $j' : \mathsf{P}' \to \mathsf{Q}'$ of j along f:

$$\begin{tikzcd} \mathsf{P} \arrow[r, "f"] \arrow[d, "j"'] & \mathsf{P}' \arrow[d, dotted, "j'"] \\ \mathsf{Q} \arrow[r, dotted] & \mathsf{Q}' \end{tikzcd}$$

then there is an elementary Tietze transformation from P' to Q', and conversely every elementary Tietze transformation arises in this way.

Proof. Pushout of the five kinds of morphisms in $\mathcal{J}$ precisely give rise to the five kinds of Tietze transformations (T1), (T2r), (T2s), (T2t) and (T2c). $\square$

We are thus tempted to call *generalized Tietze transformation* a morphism in $\mathcal{J}$-cell. In particular, every element of $\mathcal{J}$ is itself a Tietze transformation and thus, by theorem 10,

Lemma 37. Generating trivial cofibrations are weak equivalences: $\mathcal{J} \subseteq \mathcal{W}$.

Moreover, those morphisms are monomorphisms and thus, by lemma 26,

Lemma 38. The pseudo-generating trivial cofibrations are cofibrations: $\mathcal{J} \subseteq {}^{\boxslash}(\mathcal{I}^{\boxslash})$.

Remark 39. By general properties [7, Proposition 2.1.18], we have that morphisms in ${}^{\boxslash}(\mathcal{J}^{\boxslash})$ are retracts of Tietze transformations. We do not know whether the morphisms in ${}^{\boxslash}(\mathcal{J}^{\boxslash})$ are precisely Tietze transformations or not.

4.2 Morphisms in ${}^{\boxslash}(\mathcal{J}^{\boxslash})$. The following lemmas show that the morphisms in the class ${}^{\boxslash}(\mathcal{J}^{\boxslash})$ are trivial cofibrations. We will however see in section 4.4 that not every trivial cofibration is in this class, i.e., the inclusion is strict.

Lemma 40. We have ${}^{\boxslash}(\mathcal{J}^{\boxslash}) \subseteq {}^{\boxslash}(\mathcal{I}^{\boxslash})$.

Proof. By lemma 38, we have that $\mathcal{J} \subseteq {}^{\boxslash}(\mathcal{I}^{\boxslash})$. Thus, by lemma 14, we have

$$ {}^{\boxslash}(\mathcal{J}^{\boxslash}) \subseteq {}^{\boxslash}(({}^{\boxslash}(\mathcal{I}^{\boxslash}))^{\boxslash}) = {}^{\boxslash}(\mathcal{I}^{\boxslash}). $$ $\square$

Lemma 41. We have ${}^{\boxslash}(\mathcal{J}^{\boxslash}) \subseteq \mathcal{W}$.

Proof. By lemma 36, a pushout of an element in $\mathcal{J}$ is an elementary Tietze transformation and thus a weak equivalence by lemma 8. By proposition 20, any element of ${}^{\boxslash}(\mathcal{J}^{\boxslash})$ is a countable composition of elementary Tietze transformations, and thus a weak equivalence by lemma 24. $\square$

Lemma 42. We have ${}^{\boxslash}(\mathcal{J}^{\boxslash}) \subseteq \mathcal{C} \cap \mathcal{W}$.

Proof. By Lemmas 40 and 41. $\square$

4.3 Pseudo-fibrations. The morphisms in $\mathcal{J}^{\boxslash}$ are called *pseudo-fibrations.* A *pseudo-fibrant object* P is one such that the terminal morphism $\mathsf{P} \to 1$ is a pseudo-fibration.

Lemma 43. A presentation P is pseudo-fibrant when

– for every word $u \in \mathsf{P}_1^*$, there is a generator $a \in \mathsf{P}_1$ such that $u \Rightarrow a \in \mathsf{P}_2$,

– the relation P_2 on P_1^* is a congruence.

In particular, we have $u \overset{\mathsf{P}}{=} v$ if and only if $u \Rightarrow v \in \mathsf{P}_2$.

More generally, pseudo-fibrations can be described as follows:

Lemma 44. A morphism $f : \mathsf{P} \to \mathsf{Q}$ is a pseudo-fibration when

- for every $u \in \mathsf{P}_1^*$ and $b \in \mathsf{Q}_1$ such that $f(u) \Rightarrow b \in \mathsf{Q}_2$, there exists $a \in \mathsf{P}_1$ with $f^*(a) = b$ and $u \Rightarrow a \in \mathsf{P}_2$,
- for every $u \in \mathsf{P}_1^*$,
$$f^*(u) \Rightarrow f^*(u) \in \mathsf{Q}_2 \qquad \text{implies} \qquad u \Rightarrow u \in \mathsf{P}_2,$$
- for every $u, v \in \mathsf{P}_1^*$ with $u \Rightarrow v \in \mathsf{P}_2$,
$$f^*(v) \Rightarrow f^*(u) \in \mathsf{Q}_2 \qquad \text{implies} \qquad v \Rightarrow u \in \mathsf{P}_2,$$
- for every $u, v, w \in \mathsf{P}_1^*$ with $u \Rightarrow v \in \mathsf{P}_2$ and $v \Rightarrow w \in \mathsf{P}_2$,
$$f^*(u) \Rightarrow f^*(w) \in \mathsf{Q}_2 \qquad \text{implies} \qquad u \Rightarrow w \in \mathsf{P}_2,$$
- for every $u, v, w, w' \in \mathsf{P}_1^*$ with $u \Rightarrow v \in \mathsf{P}_2$,
$$f^*(wuw') \Rightarrow f^*(wvw') \in Q_2 \qquad \text{implies} \qquad wuw' \Rightarrow wvw' \in \mathsf{P}_2.$$

Lemma 45. Any fibration is a pseudo-fibration: $\mathcal{F} = (\mathcal{C} \cap \mathcal{W})^\boxslash \subseteq \mathcal{J}^\boxslash$.

Proof. By lemma 42, we have ${}^\boxslash(\mathcal{J}^\boxslash) \subseteq \mathcal{C} \cap W$. Therefore, by lemma 14,
$$\mathcal{F} = (\mathcal{C} \cap \mathcal{W})^\boxslash \subseteq ({}^\boxslash(\mathcal{J}^\boxslash))^\boxslash = \mathcal{J}^\boxslash.$$
□

Lemma 46. For any object P, there exists a pseudo-fibrant object $\tilde{\mathsf{P}}$, called a *pseudo-fibrant replacement* of P, together with a map $\mathsf{P} \to \tilde{\mathsf{P}}$ in ${}^\boxslash(\mathcal{J}^\boxslash)$.

Proof. Use the small object argument (proposition 20) to factor the terminal morphism $\mathsf{P} \to 1$ as a morphism in ${}^\boxslash(\mathcal{J}^\boxslash)$ followed by a morphism in $\mathcal{J}^\boxslash$. □

4.4 $\mathcal{J}$ is not generating. Contrarily to what one might expect, the class $\mathcal{J}$ is not generating trivial cofibrations. This can be seen by observing that the following inclusion does not hold:
$$\mathcal{J}^\boxslash \cap \mathcal{W} \subseteq \mathcal{I}^\boxslash$$
For instance, consider the inclusion
$$\langle a \,|\, \rangle \to \langle a, b \,|\, b \Rightarrow bb, 1 \Rightarrow bb \rangle$$
which corresponds to the example developed section 1.10. This morphism is both a pseudo-fibration since the only relations to lift are the reflexivity relations (which are not noted here, see section 1.6) and a weak equivalence since both presented monoids are $\mathbb{N}$. However, it is not a trivial fibration since it is not surjective on generators. The same example can be used to show that the inclusion
$${}^\boxslash(\mathcal{I}^\boxslash) \cap \mathcal{W} \subseteq {}^\boxslash(\mathcal{J}^\boxslash)$$
does not hold either: the map above is a trivial cofibration since it is both a monomorphism and a weak equivalence, but it cannot be obtained as a retract of a composite of pushouts of sums of elements of $\mathcal{J}$. Namely, the generator b has to be added using a Tietze transformation (T1), but the relations are not of the right form. Intuitively, the relation $1 \Rightarrow b$ has to be added first, see section 1.10.

Remark 47. As a simpler (but less illuminating) example, consider the inclusion

$$\langle a \mid \rangle \to \langle a, b \mid b \Rightarrow aa \rangle$$

which is not an elementary Tietze transformation, because of the chosen orientation for the relation (T1). Similarly, the inclusion

$$\langle a, b, c, d \mid aa \Rightarrow bb, bb \Rightarrow cc, cc \Rightarrow dd \rangle \to \langle a, b, c, d \mid aa \Rightarrow bb, bb \Rightarrow cc, cc \Rightarrow dd, aa \Rightarrow dd \rangle$$

is a pseudo-fibration and a weak equivalence, but not a trivial fibration one since the relation $aa \Rightarrow dd$ cannot be lifted.

4.5 $\mathcal{J}$ is pseudo-generating. It is interesting to note that the inclusions of previous section are satisfied if we restrict to fibrations whose codomain is fibrant. We begin by a reciprocal to lemma 42:

Lemma 48. Any trivial cofibration $f : \mathsf{P} \to \mathsf{Q}$ with pseudo-fibrant codomain Q belongs to $\mathcal{J}$-cell, and thus to ${}^{\boxslash}(\mathcal{J}^{\boxslash})$.

Proof. Since i is a trivial cofibration, it is an injection and we have $\overline{\mathsf{P}} = \overline{\mathsf{Q}}$. For simplicity, we suppose that i is an inclusion. For every generator in $a \in \mathsf{Q}_1 \setminus \mathsf{P}_1$, there is a word $u_a \in \mathsf{P}_1^*$ such that $u_a \overset{\mathsf{Q}}{=} a$ and therefore $u_a \Rightarrow a \in \mathsf{Q}_2$ since Q is pseudo-fibrant (Q_2 is a congruence). Writing P^0 for P with the generator a and a relation $u_a \Rightarrow a$ added, for every $a \in \mathsf{Q}_1 \setminus \mathsf{P}_1$, we have a morphism $\mathsf{P} \to \mathsf{P}^0$ in $\mathcal{J}$-cell factoring f (the inclusion $\mathsf{P} \to \mathsf{P}^0$ can be expressed as a pushout of a coproduct of pseudo-generating trivial cofibrations of the first form). We write P^{i+1} for the presentation obtained from P^i by adding

- a relation $u \Rightarrow u$ for every word u over P_1^i,
- a relation $v \Rightarrow u$ for every relation $u \Rightarrow v \in P_2^i$,
- a relation $u \Rightarrow w$ for every relations $u \Rightarrow v, v \Rightarrow w \in P_2^i$,
- a relation $wuw' \Rightarrow wvw'$ for every relation $u \Rightarrow v \in P_2^i$ and words w, w' over P_1^i.

There is a morphism $\mathsf{P}^i \to \mathsf{P}^{i+1}$ in $\mathcal{J}$-cell. Every generator of Q gets added at the first step and every relation of Q gets added at some step. Therefore $\mathsf{Q} = \operatorname{colim}_i \mathsf{P}^i$ and f belongs to $\mathcal{J}$-cell. □

Remark 49. The above proof essentially consists in using the small object argument to construct a factorization $f = h \circ g$ with $g \in \mathcal{J}$-cell and $h \in \mathcal{J}^{\boxslash}$, and observing that h can be chosen to be an identity when Q is pseudo-fibrant.

Lemma 50. Any pseudo-fibration $p : \mathsf{P} \to \mathsf{Q} \in \mathcal{J}^{\boxslash}$ with pseudo-fibrant target Q is a fibration, i.e., $p \in (\mathcal{C} \cap \mathcal{W})^{\boxslash}$.

Proof. Suppose given a trivial cofibration $i : \mathsf{P}' \to \mathsf{Q}' \in \mathcal{C} \cap \mathcal{W}$ and two morphisms $f : \mathsf{P}' \to \mathsf{P}$ and $g : \mathsf{Q}' \to \mathsf{Q}'$ such that $p \circ f = g \circ i$. By lemma 46, we can consider a pseudo-fibrant replacement $\tilde{\mathsf{Q}}'$ of Q' together with the associated morphism $j : \mathsf{Q}' \to \tilde{\mathsf{Q}}'$ in ${}^{\boxslash}(\mathcal{J}^{\boxslash})$, and thus in $\mathcal{C} \cap \mathcal{W}$ by lemma 42. By orthogonality, there is a map $k : \tilde{\mathsf{Q}}' \to \mathsf{Q}$ such that $k \circ j = g$. Finally, by lemma 48 $(j \circ i) \boxslash p$, from which we deduce the existence of $h : \tilde{\mathsf{Q}}' \to \mathsf{P}$ such that $h \circ j \circ i = f$ and $p \circ h = k$.

$$
\begin{tikzcd}
\mathsf{P}' \arrow[r, "f"] \arrow[d, "\mathcal{C}\cap\mathcal{W}\ni i"'] & \mathsf{P} \arrow[d, "p\in\mathcal{J}^{\boxslash}"] \\
\mathsf{Q}' \arrow[r, "g"] \arrow[d, "\mathcal{C}\cap\mathcal{W}\supseteq{}^{\boxslash}(\mathcal{J}^{\boxslash})\ni j"'] & \mathsf{Q} \arrow[d, "\in\mathcal{J}^{\boxslash}"] \\
\tilde{\mathsf{Q}}' \arrow[r, "\in\mathcal{J}^{\boxslash}"'] \arrow[ur, "k"] \arrow[uur, "h"] & 1
\end{tikzcd}
$$

Therefore the morphism $h \circ j : \mathsf{Q}' \to \mathsf{P}$ is a filler and thus $i \boxslash p$. □

Lemma 51. Pseudo-fibrant and fibrant objects coincide.

Proof. By lemma 45, any fibrant object is pseudo-fibrant. Conversely, by lemma 50, it suffices to check that the terminal object is pseudo-fibrant, which can be verified directly. □

Lemma 52. Given a monoid M, its standard presentation $\langle M \rangle$ is fibrant.

Proof. The presentation $\langle M \rangle$ satisfies the conditions of lemma 43 and is thus pseudo-fibrant and thus fibrant by lemma 51. □

5 Tietze equivalences as cospans

In this section we reconstruct the proof of the Tietze theorem by showing that any two presentations of the same monoid can be related by a cospan of generalized Tietze transformations.

5.1 Coproduct. We often write $\iota_0, \iota_1 : X \sqcup X \to X$ for the canonical injections into a coproduct.

Lemma 53. In a model category, when X is cofibrant, the canonical injections $\iota_0 : Y \to Y \sqcup X$ and $\iota_1 : Y \to X \sqcup Y$ are cofibrations.

Proof. We have a pushout diagram

$$
\begin{tikzcd}
\varnothing \arrow[r] \arrow[d] & Y \arrow[d, "\iota_1"] \\
X \arrow[r, "\iota_0"'] & X \sqcup Y \arrow[ul, phantom, very near start, "\ulcorner"]
\end{tikzcd}
$$

When X is cofibrant, the initial map into X is a cofibration, and the map ι_1 is thus also a cofibration, as a pushout of a cofibration. The other case is similar. □

5.2 Weak equivalences as cospans. We now recall the contents of the proof of the celebrated Ken Brown lemma, which shows that every weak equivalence between cofibrant objects factors as a cospan of trivial cofibrations.

Lemma 54 (Ken Brown's lemma). In a model category, every weak equivalence $w : X \to Y$ between cofibrant objects X and Y factors as $w = p \circ i$ where i is a trivial cofibration and p a trivial fibration which admits a section by a trivial cofibration j:

$$\begin{tikzcd} & Z \arrow[dr, bend left, two heads, "p"] & \\ X \arrow[ur, hook, "i"] \arrow[rr, "w"'] & & Y. \arrow[ul, hook, "j"] \end{tikzcd}$$

Proof. We can factor the map $(w, \mathrm{id}_Y) : X \sqcup Y \to Y$ as a cofibration $k : X \sqcup Y \to Z$ followed by a trivial fibration $p : Z \to Y$. Since X and Y are cofibrant, by lemma 53, the injections into $X \sqcup Y$ are cofibrations. We define $i = k \circ \iota_0$ and $j = k \circ \iota_1$:

$$\begin{tikzcd} X \arrow[dr, hook, "\iota_0"] \arrow[drr, bend left, "i"] \arrow[drrr, bend left=40, "w"] & & & \\ & X \sqcup Y \arrow[r, hook, "k"] & Z \arrow[r, two heads, "p"] & Y \\ Y \arrow[ur, hook, "\iota_1"'] \arrow[urr, bend right, "j"'] \arrow[urrr, bend right=40, "\mathrm{id}_Y"'] & & & \end{tikzcd}$$

The maps i and j are cofibrations as composites of cofibrations and are weak equivalences by the 2-out-of-3 property. ☐

Remark 55. In the previous lemma, the cospan (i, j) can be considered as a factorization of w, in the sense that we have $j \circ w = j \circ p \circ i = i$.

Remark 56. In a model category where monomorphisms are cofibrations (such as the case of interest here, see lemma 27), a simpler argument can be given: since Y is cofibrant and p is a trivial fibration, the diagram

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d, hook] & Z \arrow[d, two heads, "p"] \\ Y \arrow[r, "\mathrm{id}_Y"'] \arrow[ur, dotted, "j"] & Y \end{tikzcd}$$

admits a filler $j : Y \to Z$, which is a section of p; moreover, since j is a monomorphism, it is a cofibration, and it is a weak equivalence by the 2-out-of-3 property.

Theorem 57. In a model category $\mathcal{M}$ in which every object is cofibrant, every isomorphism in $\mathrm{Ho}(\mathcal{M})$ is the localization of a cospan of trivial cofibrations.

Proof. Consider an isomorphism $f : X \to Y$ in $\mathrm{Ho}(\mathcal{M})$. We write $\mathcal{M}'$ for the full subcategory of $\mathcal{M}$ whose objects are fibrant. The fibrant replacement functor $F : \mathcal{M} \to \mathcal{M}'$ induces an equivalence between the homotopy categories [7, Proposition 1.2.3]. Moreover, $\mathrm{Ho}(\mathcal{M}')$ is a quotient of $\mathcal{M}'$ by homotopy equivalences [7, Theorem 1.2.10], the map Ff is thus a homotopy equivalence and thus a weak equivalence [7, Proposition 1.2.8]. The map f is thus the localization of a span of weak equivalences

$$\begin{tikzcd} X \arrow[d, hook, "i_X"'] & Y \arrow[d, hook, "i_Y"] \\ FX \arrow[r, "Ff"'] & FY \end{tikzcd}$$

where $i_X : X \to FX$ is the trivial cofibration associated to the fibrant replacement. By lemma 54, we thus have two cospans of trivial cofibrations

$$
\begin{tikzcd}
 & X' & & Y' & \\
X \arrow[ur, hook] & & FY \arrow[ul, hook'] \arrow[ur, hook] & & Y \arrow[ul, hook']
\end{tikzcd}
$$

and we conclude to the existence of one cospan of trivial cofibrations using the fact that trivial cofibrations are closed under pushouts. □

5.3 Tietze equivalences. We can now conclude with the abstract proof of the Tietze theorem.

Theorem 58. In the category **rPres**, two presentations P and Q are such that $\overline{\mathsf{P}} \simeq \overline{\mathsf{Q}}$ if and only if there is a cospan of generalized Tietze transformations (of morphisms in $\mathcal{J}$-cell) from P to Q.

Proof. Suppose given two presentations $\mathsf{P}, \mathsf{Q} \in \mathbf{rPres}$ such that $\overline{\mathsf{P}} \simeq \overline{\mathsf{Q}}$. With the model structure introduced in section 3, this can be rewritten ans $\mathrm{Ho}(\mathsf{P}) \simeq \mathrm{Ho}(\mathsf{Q})$, and therefore we deduce that there is a cospan of trivial cofibrations

$$
\begin{tikzcd}
 & \mathsf{R} & \\
\mathsf{P} \arrow[ur, hook] & & \mathsf{Q}. \arrow[ul, hook']
\end{tikzcd}
$$

Up to taking a fibrant replacement of R and suppose that R is fibrant and thus pseudo-fibrant by lemma 51. We deduce that this is a span of Tietze transformations by lemma 48. Conversely, Tietze transformations are weak equivalences by lemma 41 and thus P and Q become isomorphic after localizing under weak equivalences. □

6 Variants and extensions

Many variants of the situation considered here could be thought of and are left for future work.

6.1 Non-reflexive presentations. If we consider the category **Pres** of (non-necessarily reflexive) presentations, many of the constructions performed in previous section can still be carried over. However, lemma 30 does not hold anymore, preventing the construction of a model category: the elements of $\mathcal{I}^{\boxslash}$ are not necessarily weak equivalences. As a counter-example consider the morphism

$$\langle a, b \mid \rangle \to \langle c \mid \rangle.$$

It belongs to $\mathcal{I}^{\boxslash}$ since it satisfies the conditions of lemma 29 (which still holds): it is surjective on generators and lifts every required relations since there are none. It is however not a weak equivalence since the monoids presented by the source and the target are respectively $\mathbb{N} * \mathbb{N}$ and $\mathbb{N}$ which are not isomorphic (the first one is not commutative for instance). We expect that there is however a right semi-model structure in the sense of [2], whose cofibrations are generated by $\mathcal{I}$.

6.2 Multisets of relations. The notion of presentation can be modified in order to allow multiple relations with the same source and the same target: such a presentation P consists of a set P_1 of generators together with a set P_2 of relations equipped with source and target maps $\mathsf{s}, \mathsf{t} : \mathsf{P}_2 \to \mathsf{P}_1$. Here, an element $\alpha \in \mathsf{P}_2$ with $\mathsf{s}(\alpha) = u$ and $\mathsf{t}(\alpha) = v$ encodes a relation $u \Rightarrow v$. We expect that this modification does not significantly changes the situation studied here.

6.3 Presentations of categories. As a further generalization, one can consider presentations of categories. Such a presentation P of a category consists of a set P_0 of objects, a set P_1 of generators for morphisms equipped with source and target maps $\mathsf{s}_0, \mathsf{t}_0 : \mathsf{P}_1 \to \mathsf{P}_0$, and a set P_2 of relations equipped with source and target maps $\mathsf{s}_1, \mathsf{t}_1 : \mathsf{P}_2 \to \mathsf{P}_1^*$ such that $\mathsf{s}_0^* \circ \mathsf{s}_1 = \mathsf{s}_0^* \circ \mathsf{t}_1$ and $\mathsf{t}_0^* \circ \mathsf{s}_1 = \mathsf{t}_0^* \circ \mathsf{t}_1$. Here, P_1^* denotes the morphisms of the free category over the graph $(\mathsf{P}_0, \mathsf{P}_1)$ and the category presented by P is obtained by quotienting the morphisms of this free category under the congruence generated by P_2. The notion of presentation of monoid of section 6.2, is the particular case where $\mathsf{P}_0 = \{\star\}$ is reduced to one element. We expect the proof of this paper to generalize to this setting.

6.4 Presentations of n-categories. This notion of presentation sketched in previous section, is a particular case of the notion of *polygraph*, see [4], which generalizes to present n-categories. It would be interesting to see whether the model structure extends to this case.

6.5 Presentations of groupoids. The notion of Tietze transformation was originally developed for presentations of groups. It would be interesting to generalize the model structure to this case, as well as generalizations of presentations of groupoids.

6.6 Coherent presentations. A notion of Tietze transformation for coherent presentations of categories is introduced in [5]. We would like to investigate this case, as well as, more generally, develop a notion of Tietze transformation for resolutions of categories by $(\infty, 1)$-polygraphs.

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