PRESENTING A CATEGORY MODULO A REWRITING SYSTEM

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Higher-dimensional rewriting

We can rewrite

- points (ARS)
- strings
- terms
- ...

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- morphisms in free *n*-categories

Unfortunately, the resulting notion of

higher-dimensional rewriting system

is sometimes too limited: we would like to rewrite in multiple dimensions at the same time.

We present here the case of dimension 1.

Presentations of monoids

Definition

A **presentation** $P = \langle P_1 \mid P_2 \rangle$ of a monoid M consists of

- a set P_1 of *generators*
- a set $P_2 \subseteq P_1^* \times P_1^*$ of *relations*

such that

$$M \quad \cong \quad P_1^* / \overset{*}{\underset{P_2}{\leftrightarrow}}$$

where

- P_1^* is the free monoid (of strings) over P_1
- $\overset{*}{\underset{P_2}{\leftrightarrow}}$ is the smallest congruence on P_1^* containing P_2

Example

- $\mathbb{N} \cong \langle a \mid \; \rangle$
- $\mathbb{N}/2\mathbb{N} \cong \langle a \mid (aa, 1) \rangle$
- $\mathbb{N} \times \mathbb{N} \cong \langle a, b \mid (ba, ab) \rangle$

Presentations of monoids

Definition

A **presentation** $P = \langle P_1 \mid P_2 \rangle$ of a monoid M consists of

- a set P_1 of *letters*
- a set $P_2 \subseteq P_1^* \times P_1^*$ of *rules*

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where

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Convergent presentations

presentation = string rewriting system

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Given $\langle P_1 \mid P_2 \rangle$ which is convergent (= terminating + confluent),

equivalence classes in P_1^* modulo $\overset{*}{\underset{P_2}{\leftrightarrow}}$ = normal forms

Convergent presentations

presentation = string rewriting system

Given $\langle P_1 \mid P_2 \rangle$ which is convergent (= terminating + confluent),

$$\begin{array}{c}\text{equivalence classes}\\ \text{in } P_1^* \text{ modulo } \underset{P_2}{\overset{*}{\leftrightarrow}}\end{array} \quad = \quad \text{normal forms}$$

and therefore showing $M = P_1^* / \underset{P_2}{\overset{*}{\leftrightarrow}}$ amounts to show

$$M \quad \cong \quad \mathrm{NF}(P_1^*)$$

(in a way compatible with mutiplication).

Convergent presentations

Example

Consider the system $P = \left\langle a \;\middle|\; aa \overset{\rho}{\to} 1 \right\rangle$:
we want to show that it presents $\mathbb{N}/2\mathbb{N} = \{0, 1\}$.

- it is terminating

Convergent presentations

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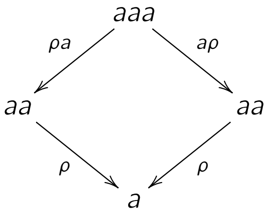
- it is locally confluent



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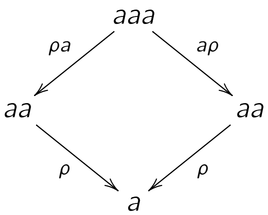

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- it is terminating
- it is locally confluent
- it is thus confluent
- normal forms are in bijection with elements of $\mathbb{N}/2\mathbb{N}$:

$$\mathrm{NF}(P) \quad = \quad \{1, a\} \quad \cong \quad \{0, 1\} \quad = \quad \mathbb{N}/2\mathbb{N}$$

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- the bijection is compatible with multiplication:

$$\begin{array}{ccc} aa & \longmapsto & 1+1 \\ {\scriptstyle \rho}\downarrow & & \| \\ 1 & \longmapsto & 0 \end{array}$$

Convergent presentations

Example

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$$\mathrm{NF}(P) \quad = \quad \{1, a\} \quad \cong \quad \{0, 1\} \quad = \quad \mathbb{N}/2\mathbb{N}$$

- the bijection is compatible with multiplication:
- therefore we do have a presentation:

$$\mathbb{N}/2\mathbb{N} \quad \cong \quad P_1^* / \underset{P_2}{\overset{*}{\leftrightarrow}}$$

Presentations of monoids

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A **presentation** $P = \langle P_1 \mid P_2 \rangle$ consists of

- a set P_1 of *generators*
- a set P_2 of *relations*

$$P_2 \quad \subseteq \quad P_1^* \times P_1^*$$

Presentations of monoids

Definition

A **presentation** $P = \langle P_1 \mid P_2 \rangle$ consists of

- a set P_1 of *generators*
- a set P_2 of *relations* with two functions

$$s_1, t_1 \quad : \quad P_2 \quad \to \quad P_1^*$$

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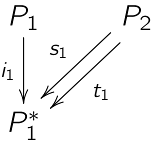
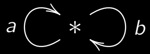
i.e. a diagram in **Set**



a, b



monoid = category with one object

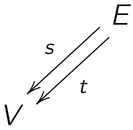
PRESENTING CATEGORIES

Graphs

Definition

A **graph** $G = (V, s, t, E)$ consists of

- a set V of *vertices*
- a set E of *edges*
- *source* and *target* functions $s, t : E \to V$



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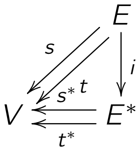
The **free category** generated by G has

- objects: vertices V
- morphisms: paths E^* (with concatenation as composition)

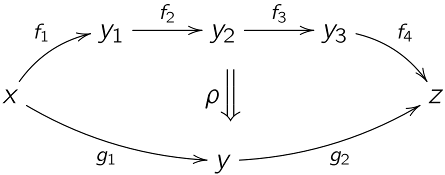


Presentations of categories

Definition

A **presentation** P of category consists of

- a graph (the *signature*)
- a set of rules rewriting a path into another path with same source and target



The *presented category* $\|P\|$ is the free category on the graph with paths taken modulo the congruence generated by rules.

Presentations of categories (formally)

Definition
A **presentation** P of category consists of

P_0

- a set P_0 of *object generators*

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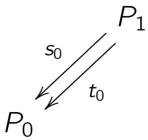

- a set P_0 of *object generators*
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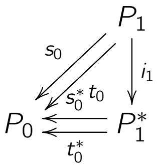

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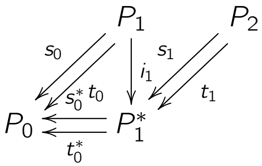


- a set P_0 of *object generators*
- a set P_1 of *morphism generators*
- a set P_2 of *relations*
  with $s_0^* \circ s_1 = s_0^* \circ t_1$ and $t_0^* \circ s_1 = t_0^* \circ t_1$

Presenting the dihedral group

Definition

The dihedral group D_n is the group of isometries of the plane preserving a regular polygon with n faces. This group admits the presentation

$$P = \left\langle r, s \;\middle|\; r^n = 1, s^2 = 1, rsr = s \right\rangle$$

where

- r corresponds to a rotation of $2\pi/n$
- s corresponds to a symmetry

Example

$D_8 =$

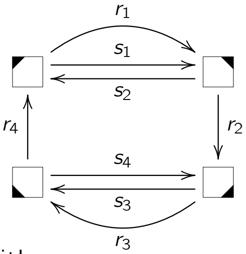
Presenting the dihedral category

Definition

The **dihedral category** $D_n^\bullet$ is the variant with a vertex of the polygon is distinguished.

Example

$D_4^\bullet \quad =$



admits the presentation P with

$P_0 \quad = \quad \{$  $\}$

$$P_1 \quad = \quad \{r_i, s_i \mid i = 1, \ldots, 4\}$$

$$P_2 \quad = \quad \{\ldots\}$$

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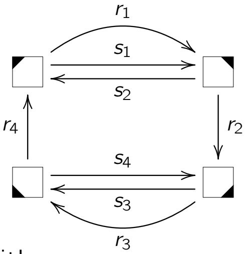
$D_4^\bullet \quad =$ 

admits the presentation P with

$$r_{i+3} \circ r_{i+2} \circ r_{i+1} \circ r_i = \mathrm{id} \quad s_{j+1} \circ s_j = \mathrm{id} \quad r_j \circ s_{j+1} \circ r_j = s_j$$
$$s_j \circ s_{j+1} = \mathrm{id} \quad r_{j+3} \circ s_{j+2} \circ r_{j+1} = s_{j+1}$$

for $i \in \{1, \ldots, 4\}$ and $j \in \{1, 3\}$, where the indices are to be taken modulo 4 so that they lie in $\{1, \ldots, 4\}$.

PRESENTING
MODULO

Presentations modulo

Presentations of categories start from a graph and quotient paths.

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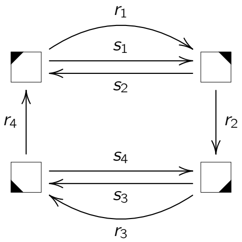

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Example

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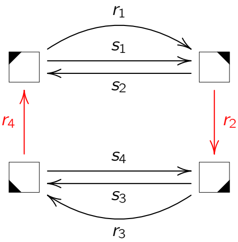
$$D_4^\bullet / \{r_2, r_4\} \quad =$$

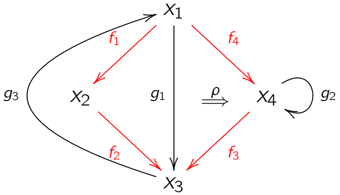


What happens if we set  =  and  =  by imposing that r_2 and r_4 should "be considered as identities"?

Definition

A **presentation modulo** $(P, \tilde{P}_1)$ of category consists of

- a presentation of category P,
- a set $\tilde{P}_1 \subseteq P_1$ of *equational generators*.



$$
\begin{aligned}
P_0 &= \{x_i\} \\
P_1 &= \{f_i, g_i\} \\
\tilde{P}_1 &= \{f_i\} \\
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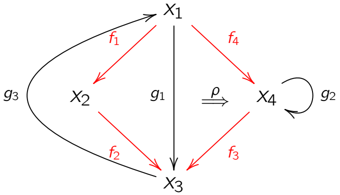

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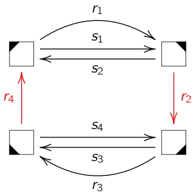
$$

Since, we want to consider objects modulo relations in $\tilde{P}_1$, it is natural to suppose that

Assumption

The abstract rewriting system $(P_0, \tilde{P}_1)$ is convergent.

The category presented modulo



Given a presentation modulo $(P, \tilde{P}_1)$, we have three possible ways of defining the presented category from $\|P\|$:

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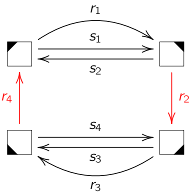
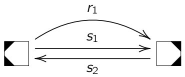
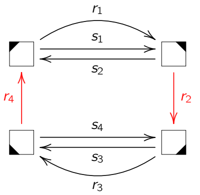




Given a presentation modulo $(P, \tilde{P}_1)$, we have three possible ways of defining the presented category from $\|P\|$:

1. **quotient** by equational generators: turn them into identities,

The category presented modulo



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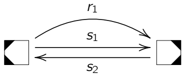

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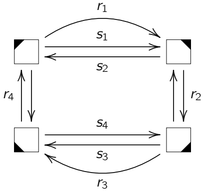
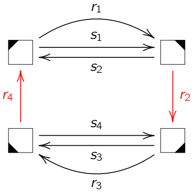


2. **localize** by equational generators: turn them into isomorphisms,

The category presented modulo



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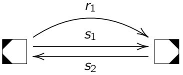

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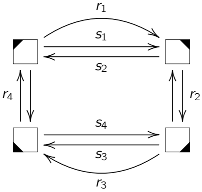

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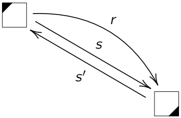
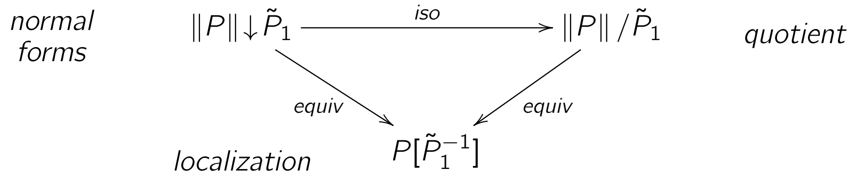


3. **restrict** to objects which are normal forms wrt equational generators.

Theorem

Given a presentation modulo $(P, \tilde{P}_1)$ satisfying suitable assumptions, the three constructions are related by

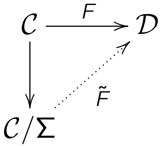


Quotient and localization

Suppose given a category $\mathcal{C}$ and a set Σ of morphisms of $\mathcal{C}$.

Definition

The **quotient** of $\mathcal{C}$ by Σ is the category $\mathcal{C}/\Sigma$ such that



for any category $\mathcal{D}$ there is a bijection between

- functors $F : \mathcal{C} \to \mathcal{D}$ sending elements of Σ to identities
- functors $\tilde{F} : \mathcal{C}/\Sigma \to \mathcal{D}$

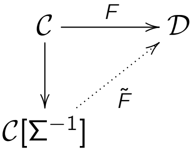
It always exists for abstract reasons.

Quotient and localization

Suppose given a category $\mathcal{C}$ and a set Σ of morphisms of $\mathcal{C}$.

Definition

The **localization** of $\mathcal{C}$ by Σ is the category $\mathcal{C}[\Sigma^{-1}]$ such that



for any category $\mathcal{D}$ there is a bijection between

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It always exists for abstract reasons.

Counter-examples

Without the *suitable assumptions*, the theorem is false.

Theorem

Given a presentation modulo $(P, \tilde{P}_1)$ *satisfying* *suitable assumptions, the three constructions are related by*

normal forms

$$\begin{array}{ccc} \|P\| \downarrow \tilde{P}_1 & \xrightarrow{\textit{iso}} & \|P\| / \tilde{P}_1 \\ & {\scriptstyle \textit{equiv}} \searrow \quad \swarrow {\scriptstyle \textit{equiv}} & \\ & P[\tilde{P}_1^{-1}] & \end{array}$$

quotient

localization

Counter-examples

Without the *suitable assumptions*, the theorem is false.
Consider the category

$$\mathcal{C} \quad = \quad x \underset{g}{\overset{f}{\rightrightarrows}} y$$

with $\Sigma = \{f, g\}$:

- the *quotient* is

$$\mathcal{C}/\Sigma \quad = \quad \overline{x} \circlearrowright \mathrm{id}$$

- the *localization* is equivalent to

$$\mathcal{C}[\Sigma^{-1}] \quad = \quad \star \circlearrowright n \in \mathbb{Z}$$

They are not equivalent!

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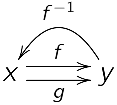
with $\Sigma = \{f\}$:

- the *category of normal forms* is

- the *localization* is



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Consider the category

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with $\Sigma = \{f\}$:

- the *category of normal forms* is

$$y \circlearrowright \mathrm{id}$$

- the *quotient* is

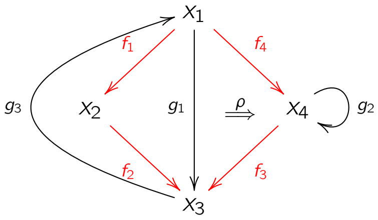
$$\overline{y} \circlearrowright g^n$$

They are not isomorphic!

Assumption 1: convergence

Assumption

The abstract rewriting system $(P_0, \tilde{P}_1)$ is convergent.



Assumption 2: residuation

Assumption

For every pair of distinct coinitial generators

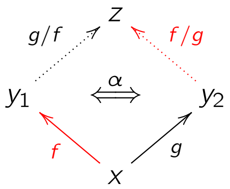
$$f : x \to y_1 \in \tilde{P}_1 \qquad \text{and} \qquad g : x \to y_2 \in P_1$$

there exist a fixed pair of cofinal morphisms

$$g/f : y_1 \to z \in P_1^* \qquad \text{and} \qquad f/g : y_2 \to z \in \tilde{P}_1^*$$

and a relation

$\alpha \in P_2$ with



The morphism g/f is called **residual** *of* g *after* f, idem for f/g.

Assumption 3: cylinder property

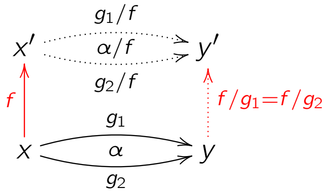
Assumption

For every

$$f : x \to x' \in \tilde{P}_1 \qquad \text{and} \qquad \alpha : g_1 \Rightarrow g_2 : x \to y \in P_2$$

we have

- $f/g_1 = f/g_2$
- $\alpha/f : g_1/f \overset{*}{\Leftrightarrow} g_2/f$ exists



Assumption 3: cylinder property

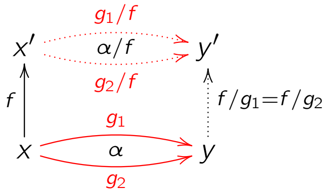
Assumption

For every

$$f : x \to x' \in P_1 \qquad \text{and} \qquad \alpha : g_1 \Rightarrow g_2 : x \to y \in P_2$$

we have

- $f/g_1 = f/g_2$
- $\alpha/f : g_1/f \overset{*}{\Leftrightarrow} g_2/f$ exists



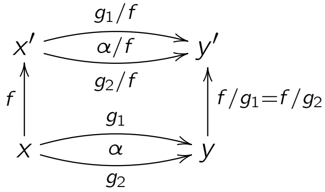
Assumption 4: termination

Assumption

Given $f : x \to x'$ and $\alpha : g_1 \Rightarrow g_2 : x \to y$, we have

$$|\alpha/f| \quad < \quad |\alpha|$$

for some function $|-| : P_2 \to \mathbb{N}$.



Assumption 5: opposite

Assumption

The *opposite presentation modulo* $(P^{\mathrm{op}}, \tilde{P}_1^{\mathrm{op}})$ with

- $P^{\mathrm{op}} = (P_0, P_1^{\mathrm{op}}, P_2^{\mathrm{op}})$
- $P_1^{\mathrm{op}} = \{f^{\mathrm{op}} : y \to x \mid f : x \to y \in P_1\}$
- $P_2^{\mathrm{op}} = \{\alpha^{\mathrm{op}} : f^{\mathrm{op}} \Rightarrow g^{\mathrm{op}} \mid \alpha : f \Rightarrow g\}$ with $f^{\mathrm{op}} = f_1^{\mathrm{op}} \circ ... \circ f_k^{\mathrm{op}}$ for $f = f_k \circ \ldots \circ f_1$
- $\tilde{P}_1^{\mathrm{op}}$ is the subset of P_1^{op} corresponding to $\tilde{P}_1$

also satisfies previous assumptions

Proofs

Theorem

Given a presentation modulo $(P, \tilde{P}_1)$ satisfying the five assumptions, the three constructions are related by

$$\|P\|\downarrow\tilde{P}_1 \xrightarrow{\text{iso}} \|P\|/\tilde{P}_1$$

$$\|P\|\downarrow\tilde{P}_1 \xrightarrow{\text{equiv}} P[\tilde{P}_1^{-1}] \xleftarrow{\text{equiv}} \|P\|/\tilde{P}_1$$

normal forms ($\|P\|\downarrow\tilde{P}_1$), *quotient* ($\|P\|/\tilde{P}_1$), *localization* ($P[\tilde{P}_1^{-1}]$)

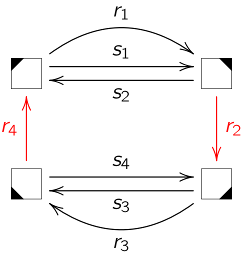
Proof. See the article!

- Termination ensures global properties.
- The cylinder property is close to the usual "cube identity" for residuals, it ensures that every equational morphism is epi and has pushout along other morphisms.
- We use the description of the localization as a category of fractions.

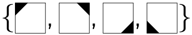
□

The dihedral example

What is the category presented by the following presentation modulo?



with

$P_0 \quad = \quad$ 

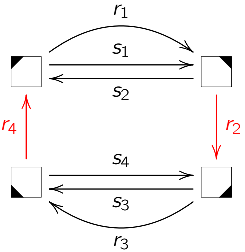
$$P_1 \quad = \quad \{r_i, s_i \mid i = 1, \ldots, 4\}$$

$$\tilde{P}_1 \quad = \quad \{r_2, r_4\}$$

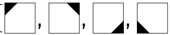
$$P_2 \quad = \quad \{\ldots\}$$

The dihedral example

What is the category presented by the following presentation modulo?



with

$P_0 = \{$  $\}$

$P_1 = \{r_i, s_i \mid i = 1, \ldots, 4\}$

$\tilde{P}_1 = \{r_2, r_4\}$

$P_2 = \{\ldots\}$

Problem: *it does not satisfy our hypothesis!* $(r_2/s_2 = ?)$

Tietze transformations

Definition

Given a presentation P, a **Tietze transformation** consists in

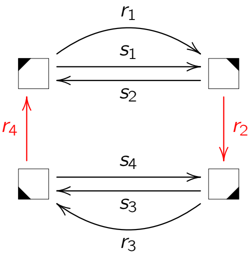
- adding / removing a **definable generator**:
  a generator $f \in P_1$ together with a relation $\alpha : f \Rightarrow g \in P_2$ such that $g \in (P_1 \setminus \{f\})^*$,
- adding / removing a **derivable relation**:
  a relation $\alpha : f \Rightarrow g \in P_2$ such that f and g are equivalent wrt the congruence generated by the relations in $P_2 \setminus \{\alpha\}$.

Proposition

Two presentations P and P' are related by a finite sequence of Tietze transformations if and only if they present the same category, i.e. $\|P\| \cong \|P'\|$.

The dihedral example

Consider the presentation



$r_2 / s_2 = ?$

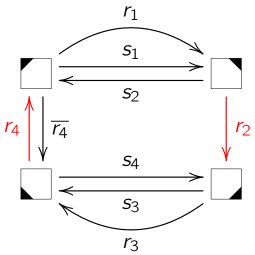
with relations

$$r_{i+3} \circ r_{i+2} \circ r_{i+1} \circ r_i = \mathrm{id} \quad s_{j+1} \circ s_j = \mathrm{id} \quad r_j \circ s_{j+1} \circ r_j = s_j$$

$$s_j \circ s_{j+1} = \mathrm{id} \quad r_{j+3} \circ s_{j+2} \circ r_{j+1} = s_{j+1}$$

The dihedral example

Consider the presentation



$r_2 / s_2 = ?$

with relations

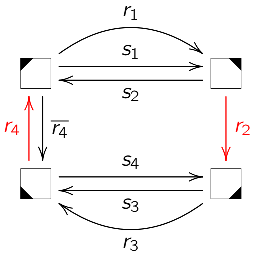
$$r_{i+3} \circ r_{i+2} \circ r_{i+1} \circ r_i = \mathrm{id} \quad s_{j+1} \circ s_j = \mathrm{id} \quad r_j \circ s_{j+1} \circ r_j = s_j$$

$$s_j \circ s_{j+1} = \mathrm{id} \quad r_{j+3} \circ s_{j+2} \circ r_{j+1} = s_{j+1}$$

$$r_3 \circ r_2 \circ r_1 = \overline{r_4}$$

The dihedral example

Consider the presentation



$r_2 / s_2 = \,?$

with relations

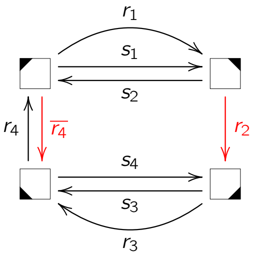
$$r_{i+3} \circ r_{i+2} \circ r_{i+1} \circ r_i = \mathrm{id} \quad s_{j+1} \circ s_j = \mathrm{id} \quad r_j \circ s_{j+1} \circ r_j = s_j$$

$$s_j \circ s_{j+1} = \mathrm{id} \quad r_{j+3} \circ s_{j+2} \circ r_{j+1} = s_{j+1}$$

$$r_4 \circ \overline{r_4} = \mathrm{id} \quad \overline{r_4} \circ r_4 = \mathrm{id} \quad r_3 \circ r_2 \circ r_1 = \overline{r_4}$$

The dihedral example

Consider the presentation



$$r_2 / s_2 = \overline{r_4}$$

with relations

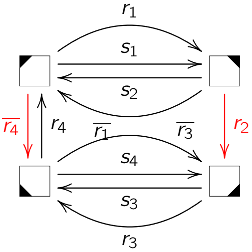
$$r_{i+3} \circ r_{i+2} \circ r_{i+1} \circ r_i = \mathrm{id} \quad s_{j+1} \circ s_j = \mathrm{id} \quad r_j \circ s_{j+1} \circ r_j = s_j$$

$$s_j \circ s_{j+1} = \mathrm{id} \quad r_{j+3} \circ s_{j+2} \circ r_{j+1} = s_{j+1}$$

$$r_4 \circ \overline{r_4} = \mathrm{id} \quad \overline{r_4} \circ r_4 = \mathrm{id} \quad r_3 \circ r_2 \circ r_1 = \overline{r_4}$$

The dihedral example

Consider the presentation



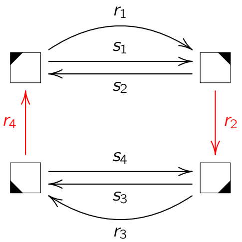
with relations

$$
\begin{array}{llll}
s_{j+1} \circ s_j = \mathrm{id} & r_1 \circ s_2 \circ r_1 = s_1 & r_k \circ \overline{r}_k = \mathrm{id} & r_2 \circ r_1 = \overline{r}_3 \circ \overline{r}_4 \\
s_j \circ s_{j+1} = \mathrm{id} & \overline{r}_3 \circ s_3 \circ \overline{r}_3 = s_4 & \overline{r}_k \circ r_k = \mathrm{id} & r_3 \circ r_2 = \overline{r}_4 \circ \overline{r}_1 \\
 & & & s_3 \circ r_2 = \overline{r}_4 \circ s_2 \\
 & & & r_2 \circ s_1 = s_4 \circ \overline{r}_4
\end{array}
$$

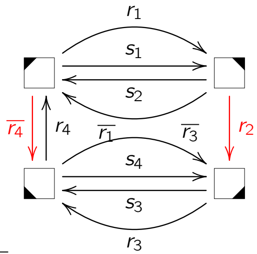
and all residuals can be suitably defined...

The dihedral example

The category presented modulo by

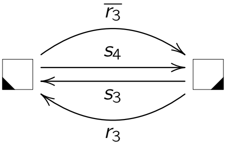


or



is

$D_2^{\bullet} \quad =$



and we have that $D_2^{\bullet}$

- is isomorphic to the quotient $D_4^{\bullet}/\{r_2, r_4\}$,
- embeds fully and faithfully into the category $D_4^{\bullet}$,
- is equivalent to the localization $D_4^{\bullet}[\{r_2, r_4\}^{-1}]$.

Conclusion and future works

We have

- defined a presentation of a category modulo an abstract rewriting system,
- shown that it comes with a decent notion of presented category,
- generalized well-known techniques in rewriting (residuation) and group theory (Ore theorem).

Next step is to go higher in dimensions where really interesting examples occur, e.g. we could present the cartesian product of monoidal categories!