MECHANIZED TOOLS FOR HIGHER CATEGORIES

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Two recent works

Higher-dimensional weak categories have become a fundamental tool: algebraic topology / HoTT / ...

I will present two recent works in order to approach them with computers:

1. a definition of weak ω-categories as a **type theory** (joint with Thibaut Benjamin and Eric Finster): *check the validity of structural morphisms*

2. an extension of **rewriting** techniques to tricategories (joint with Simon Forest): *show coherence theorems*

HIGHER CATEGORIES

Higher categories

The definition of (strict) ω**-category** generalizes categories by taking higher cells into account.

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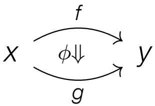
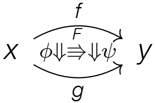
In such a category, you have

- 0-cells (objects): x
- 1-cells (morphisms): $x \xrightarrow{f} y$
- 2-cells:



- 3-cells:



- …

Higher categories

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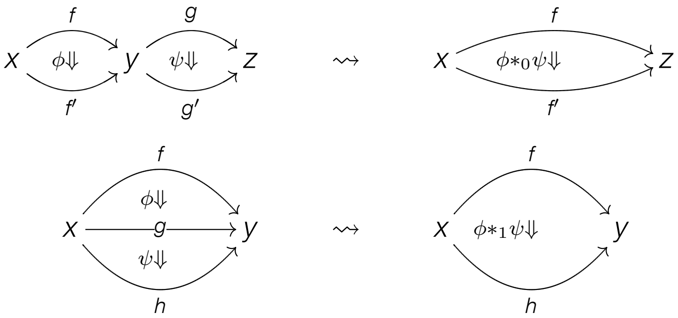
In such a category, you have **compositions**



More generally, n-cells can be composed in n ways.

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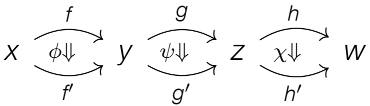
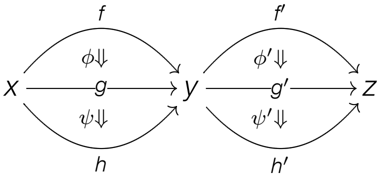
In such a category, you have **axioms**:

1. associativity of composition:



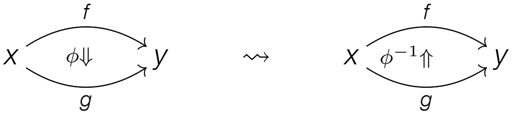
2. neutrality of identities
3. interchange laws:



Higher categories

The definition of (strict) ω**-category** generalizes categories by taking higher cells into account.

In the case where the orientation of arrows is not really relevant, you can consider (strict) ω**-groupoids** which are ω-categories in which all n-cells are invertible.



Weak ω-groupoids

It turns out that this definition is too strict.

Given a topological space X, one expects to be able to build an ω-groupoid whose

- 0-cells are the points of X,
- 1-cells are the paths in X,
  (we do have concatenation, constant paths, and inverses)
- 2-cells are homotopies,
- 3-cells are homotopies between homotopies,
- etc.

However,

- concatenation is only associative up to homotopy,
- interchange is not strict.

Grothendieck's homotopy hypothesis

A definition of ω-groupoids should satisfy **Grothendieck's homotopy hypothesis**:

weak ω-groupoids
$\approx$
topological spaces

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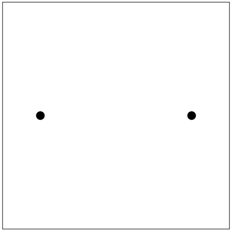

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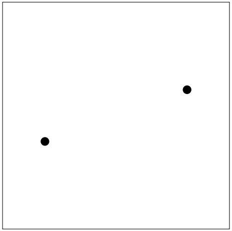

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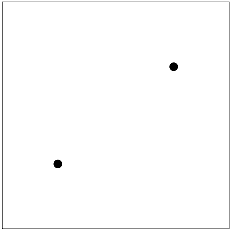

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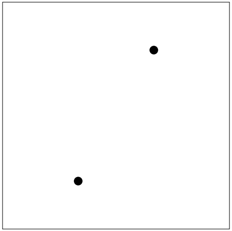

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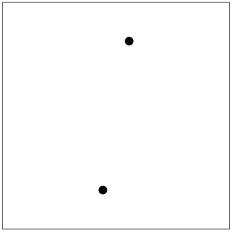

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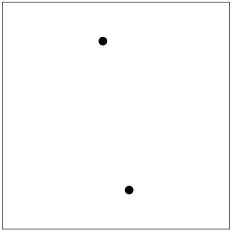

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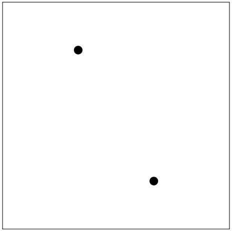

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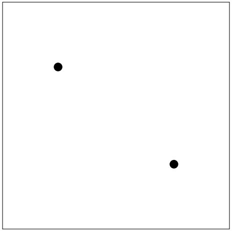

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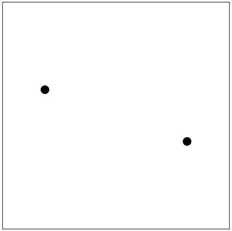

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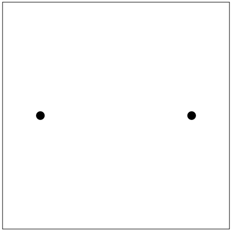

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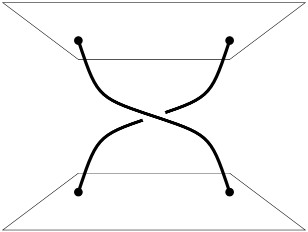

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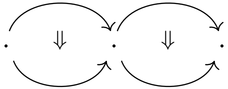

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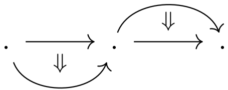

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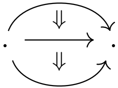

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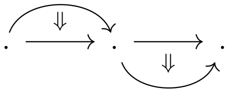

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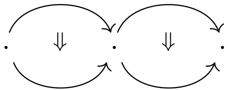

Type-theoretic weak ω-categories

Here, we fill the following gap:

	groupoids	categories
category theory	Grothendieck	Maltsiniotis
type theory	Brunerie	**Finster-Mimram**

Why is this useful

- We have a **simple definition**
  (no advanced categorical concepts, a few inference rules)
- We have a **syntax**
  (we can reason by induction, etc.)
- We have **tools**
  (we can have the machine check our terms)
- A step toward **directed homotopy type theory**?
  (we are still far from handling variance, univalence, etc.)

A TYPE-THEORETIC DEFINITION OF CATEGORIES

Judgments in type-theory

- Γ is a well-formed context:

$$\Gamma \vdash$$

- A is a well-formed type in context Γ:

$$\Gamma \vdash A$$

- t is a term of type A in context Γ:

$$\Gamma \vdash t : A$$

- t and u are equal terms of type A in context Γ:

$$\Gamma \vdash t = u : A$$

A type-theoretic definition of categories

Cartmell, 1984:

- type constructors:

$$\frac{\Gamma \vdash}{\Gamma \vdash \star} \qquad \frac{\Gamma \vdash x : \star \qquad \Gamma \vdash y : \star}{\Gamma \vdash x \to y}$$

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- term constructors:

$$\frac{}{x : \star \vdash \mathrm{id}(x) : x \to x}$$

$$\frac{}{x : \star, y : \star, f : x \to y, z : \star, g : y \to z \vdash \mathrm{comp}(f, g) : x \to z}$$

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$$\frac{\Gamma \vdash f : x \to y}{\Gamma \vdash \mathrm{comp}(\mathrm{id}(x), f) = f} \qquad \frac{\Gamma \vdash f : x \to y}{\Gamma \vdash \mathrm{comp}(f, \mathrm{id}(y)) = f} \qquad \ldots$$

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- plus “standard rules” (contexts, weakening, substitutions, …)

Models of the type theory

A **model** of the type theory consists in interpreting

- closed types as sets,
- closed terms as elements of their type,

in such a way that axioms are satisfied.

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A model of the previous type theory consists of

- a set $[\![\star]\!]$
- for each $x, y \in [\![\star]\!]$, a set $[\![\rightarrow]\!]_{x,y}$
- for each $x \in [\![\star]\!]$, an element $[\![\mathrm{id}]\!]_x \in [\![\rightarrow]\!]_{x,x}$
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- …

In other words, a model of the type theory is precisely a **category** (and a morphism is a functor).

Going higher

We could gradually implement weak n-categories:

- bicategories
- tricategories
- tetracategories
- pentacategories
- ...

The problem is that

- the number of axioms is exploding
- nobody knows the definition excepting in low dimensions
- we would like to have a "uniform" definition

(note: it might still be a good idea in low dimensions!)

Unbiased definition

Since the composition is associative for categories, the composite of any diagram like

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} x_n$$

is uniquely defined.

So, instead of having a binary composition and identities, we could have a more general rule

$$\overline{x_0 : \star, x_1 : \star, f_1 : x_0 \to x_1, \dots, x_n : \star, f_n : x_{n-1} \to x_n \vdash \mathrm{comp}(f_1, \dots, f_n) : x_0 \to x_n}$$

Unbiased definition

We can axiomatize categories with n-ary composition.

- This is very redundant, for instance

  $$\mathrm{comp}(\mathrm{comp}(f,g),h) = \mathrm{comp}(f,g,h) = \mathrm{comp}(f,\mathrm{comp}(g,h))$$

  or even

  $$\mathrm{comp}(f) \quad = \quad f$$

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- We have to characterize what we want to compose exactly.
  For instance, should be able to compose

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \ldots \xrightarrow{f_n} x_n$$

  but not

$$x \underset{g}{\overset{f}{\rightleftarrows}} y \qquad z \qquad\qquad \text{or} \qquad\qquad x \xrightarrow{f} y \xleftarrow{g} z$$

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- However, this generalizes nicely in higher dimensions!

A TYPE-THEORETIC DEFINITION OF GLOBULAR SETS

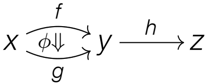
Globular sets

Definition

A **globular set** consists of

- a set G, and
- for every $x, y \in G$, a globular set G_y^x.

Example



corresponds to

$$G = \{x, y, z\} \qquad G_y^x = \{f, g\} \qquad (G_y^x)_g^f = \{\phi\} \qquad ((G_y^x)_g^f)_\phi^\phi = \emptyset \qquad \ldots$$

Globular sets

Definition

A **globular set** consists of

- a set G, and
- for every $x, y \in G$, a globular set G_y^x.

Alternatively, this can be defined as

- a sequence of sets G_n of n-cells for $n \in \mathbb{N}$,
- with source and target maps

$$s_n, t_n : G_{n+1} \to G_n$$

satisfying suitable axioms.

Proposition

Globular sets are precisely the models of the type theory

$$\frac{\Gamma \vdash}{\Gamma \vdash \star} \qquad \frac{\Gamma \vdash t : A \qquad \Gamma \vdash u : A}{\Gamma \vdash t \underset{A}{\to} u} \qquad \ldots$$

Globular sets

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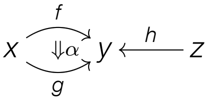
Globular sets are precisely the models of the type theory

$$\frac{\Gamma \vdash}{\Gamma \vdash \star} \qquad \frac{\Gamma \vdash t : A \qquad \Gamma \vdash u : A}{\Gamma \vdash t \underset{A}{\to} u} \qquad \dots$$

Remark

A finite globular set

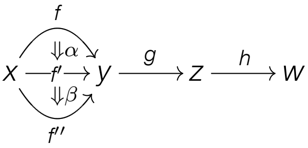


can be encoded as a context

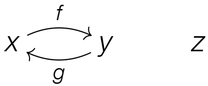
$$x : \star, y : \star, z : \star, f : x \underset{\star}{\to} y, g : x \underset{\star}{\to} y, h : z \underset{\star}{\to} y, \alpha : f \underset{x \underset{\star}{\to} y}{\to} g$$

PASTING SCHEMES

We now want to define **pasting schemes** which are diagrams for which we expect to have a composition. For instance,



is a pasting scheme, but not



or

$$x \xrightarrow{f} y \xleftarrow{g} z$$

Disks

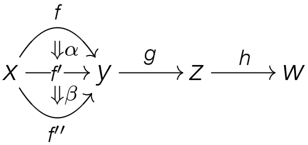
Given $n \in \mathbb{N}$, the ***n*-disk** D_n is the globular set corresponding to a general n-cell:

D_0 D_1 D_2 D_3

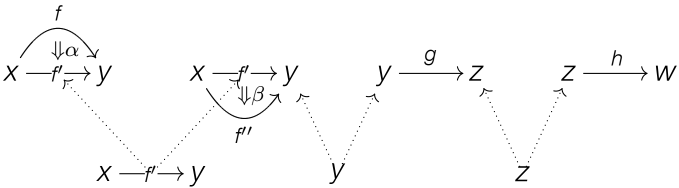
(these are the representable globular sets)

Pasting schemes

A **pasting scheme** is a globular set



- *Grothendieck*: which can be obtained as a particular colimit of disks



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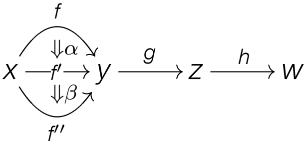
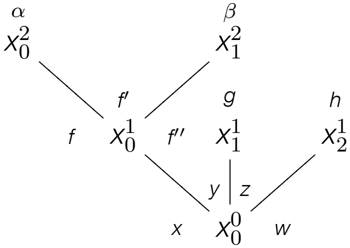


- *Batanin*: which is described by a particular tree



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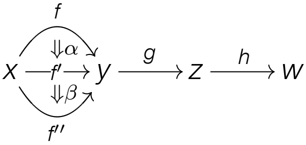


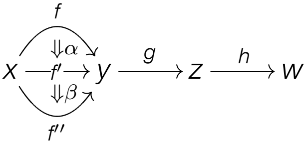
- *Finster-Mimram*: which is “totally ordered”

Order relation

We can define a preorder $\triangleleft$ on the cells of a globular set by

$$\mathrm{source}(x) \triangleleft x \qquad \text{and} \qquad x \triangleleft \mathrm{target}(x)$$

For the globular set



we have

$$x \triangleleft f \triangleleft \alpha \triangleleft f' \triangleleft \beta \triangleleft f'' \triangleleft y \triangleleft g \triangleleft z \triangleleft h \triangleleft w$$

Characterization of pasting schemes

Theorem

*A globular set is a **pasting scheme** if and only if it is*

- *non-empty,*
- *finite, and*
- *the relation $\triangleleft$ is a total order.*

Construction of pasting schemes

A *pointed globular set* is a globular set with a distinguished cell.

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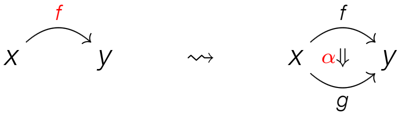
Theorem

*A **pasting scheme** is a pointed globular set which can be constructed as follows:*

- *we start from a 0-cell* x
- *we can add a new $(n+1)$-cell and its new target, its source being the distinguished n-cell*



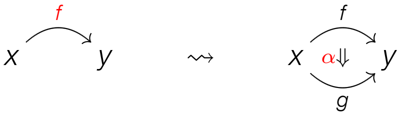
Construction of pasting schemes

A *pointed globular set* is a globular set with a distinguished cell.

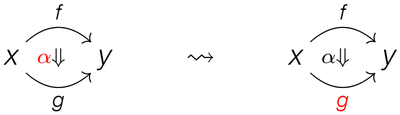
Theorem

A ***pasting scheme*** *is a pointed globular set which can be constructed as follows:*

- *we start from a* 0*-cell* x
- *we can add a new* $(n+1)$*-cell and its new target, its source being the distinguished n-cell*



- *or the distinguished cell becomes the target of the previous one*



Construction of pasting schemes

The construction of the pasting scheme

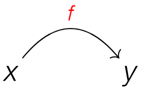
X

corresponds to its order

X

Construction of pasting schemes

The construction of the pasting scheme

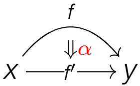


corresponds to its order

$x \quad \triangleleft \quad f$

Construction of pasting schemes

The construction of the pasting scheme

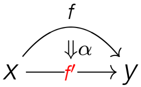


corresponds to its order

$$x \quad \lhd \quad f \quad \lhd \quad \alpha$$

Construction of pasting schemes

The construction of the pasting scheme

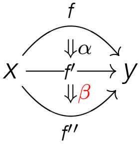


corresponds to its order

$$x \quad \triangleleft \quad f \quad \triangleleft \quad \alpha \quad \triangleleft \quad f'$$

Construction of pasting schemes

The construction of the pasting scheme

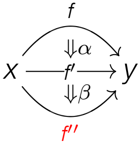


corresponds to its order

$$x \quad \triangleleft \quad f \quad \triangleleft \quad \alpha \quad \triangleleft \quad f' \quad \triangleleft \quad \beta$$

Construction of pasting schemes

The construction of the pasting scheme

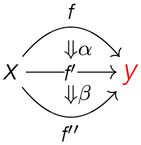


corresponds to its order

$$x \triangleleft f \triangleleft \alpha \triangleleft f' \triangleleft \beta \triangleleft f''$$

Construction of pasting schemes

The construction of the pasting scheme

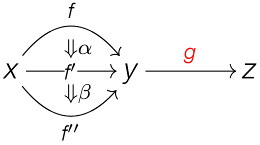


corresponds to its order

$$x \quad \triangleleft \quad f \quad \triangleleft \quad \alpha \quad \triangleleft \quad f' \quad \triangleleft \quad \beta \quad \triangleleft \quad f'' \quad \triangleleft \quad y$$

Construction of pasting schemes

The construction of the pasting scheme

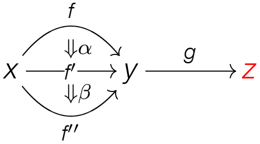


corresponds to its order

$$x \lhd f \lhd \alpha \lhd f' \lhd \beta \lhd f'' \lhd y \lhd g$$

Construction of pasting schemes

The construction of the pasting scheme

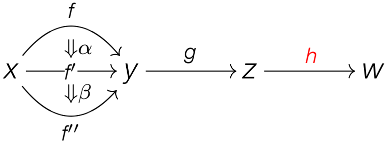


corresponds to its order

$$x \;\triangleleft\; f \;\triangleleft\; \alpha \;\triangleleft\; f' \;\triangleleft\; \beta \;\triangleleft\; f'' \;\triangleleft\; y \;\triangleleft\; g \;\triangleleft\; z$$

Construction of pasting schemes

The construction of the pasting scheme

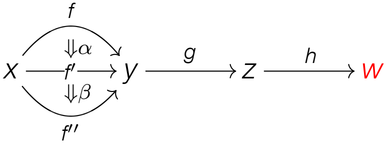


corresponds to its order

$$x \triangleleft f \triangleleft \alpha \triangleleft f' \triangleleft \beta \triangleleft f'' \triangleleft y \triangleleft g \triangleleft z \triangleleft h$$

Construction of pasting schemes

The construction of the pasting scheme

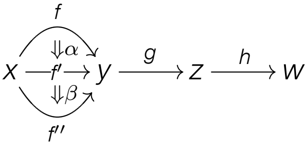


corresponds to its order

$$x \lhd f \lhd \alpha \lhd f' \lhd \beta \lhd f'' \lhd y \lhd g \lhd z \lhd h \lhd w$$

Type-theoretic pasting schemes

Now, recall that a pasting scheme



can be seen as a context

$$x : \star, y : \star, f : x \to y, f' : x \to y,$$
$$\alpha : f \to f', f'' : x \to y, \beta : f' \to f'',$$
$$z : \star, g : y \to z, w : \star, h : z \to w$$

Type-theoretic pasting schemes

A context Γ (seen as a globular set) is a **pasting scheme** iff

$$\Gamma \vdash_{\mathsf{ps}}$$

is derivable with the rules

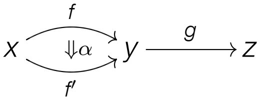
$$\frac{}{x : \star \vdash_{\mathsf{ps}} x : \star} \qquad \frac{\Gamma \vdash_{\mathsf{ps}} x : \star}{\Gamma \vdash_{\mathsf{ps}}}$$

$$\frac{\Gamma \vdash_{\mathsf{ps}} x : A}{\Gamma, y : A, f : x \underset{A}{\to} y \vdash_{\mathsf{ps}} f : x \underset{A}{\to} y} \qquad \frac{\Gamma \vdash_{\mathsf{ps}} f : x \underset{A}{\to} y}{\Gamma \vdash_{\mathsf{ps}} y : A}$$

Type-theoretic pasting schemes

Note that with those rules

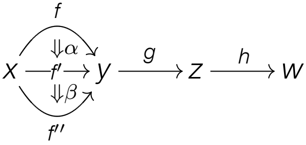
- the order of cells matters:



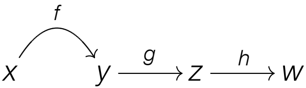
- because of this we can easily check
- proofs are canonical

Source and targets

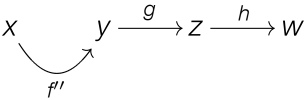
A pasting scheme Γ has



- a **source** $\partial^-(\Gamma)$:



- a **target** $\partial^+(\Gamma)$:



both of which can be defined by induction on contexts.

A TYPE-THEORETIC DEFINITION OF ω-CATEGORIES

Type-theoretic ω-groupoids

We expect that in an ω-category every pasting scheme has a composite:

$$\frac{\Gamma \vdash_{\mathsf{ps}} \qquad \Gamma \vdash A}{\Gamma \vdash \mathsf{coh}_{\Gamma,A} : A}$$

Type-theoretic ω-groupoids

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$$\frac{\Gamma \vdash_{\mathsf{ps}} \qquad \Gamma \vdash A}{\Gamma \vdash \mathsf{coh}_{\Gamma,A} : A}$$

You can derive expected operations, such as composition:

$$x : \star, y : \star, f : x \underset{\star}{\to} y, z : \star, g : y \underset{\star}{\to} z \vdash \mathsf{coh} : x \underset{\star}{\to} z$$

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We expect that in an ω-category every pasting scheme has a composite:

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You can derive expected operations, such as composition:

$$x : \star, y : \star, f : x \underset{\star}{\rightarrow} y, z : \star, g : y \underset{\star}{\rightarrow} z \vdash \mathsf{coh} : x \underset{\star}{\rightarrow} z$$

However, you can derive too much:

$$x : \star, y : \star, f : x \underset{\star}{\rightarrow} y \vdash \mathsf{coh} : y \underset{\star}{\rightarrow} x$$

We have in fact a definition of ω-**groupoids** (close to Brunerie's).

Type-theoretic ω-groupoids

We need to take care of side-conditions and in fact split the rule in two:

- operations:

$$\frac{\Gamma \vdash_{\mathsf{ps}} \qquad \Gamma \vdash t \underset{A}{\to} u \qquad \partial^-(\Gamma) \vdash t : A \qquad \partial^+(\Gamma) \vdash u : A}{\Gamma \vdash \mathsf{coh}_{\Gamma, t \underset{A}{\to} u} : t \underset{A}{\to} u}$$

  whenever

$$FV(t) = FV(\partial^-(\Gamma)) \qquad \text{and} \qquad FV(u) = FV(\partial^+(\Gamma))$$

- coherences:

$$\frac{\Gamma \vdash_{\mathsf{ps}} \qquad \Gamma \vdash A}{\Gamma \vdash \mathsf{coh}_{\Gamma, A} : A}$$

  whenever

$$FV(A) = FV(\Gamma)$$

Type-theoretic ω-groupoids

Definition

An ω**-category** is a model of this type theory.

Type-theoretic ω-groupoids

Definition

An ω**-category** is a model of this type theory.

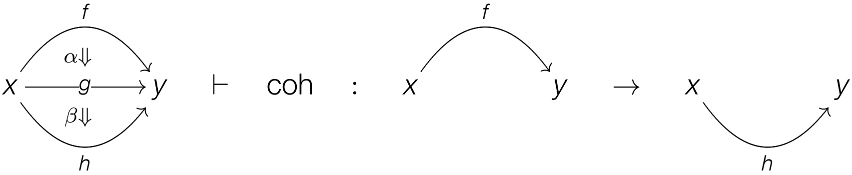
Theorem

This definition coincides with Grothendieck-Maltsiniotis'.

(conjecture recently proved by Thibaut Benjamin)

Type-theoretic ω-groupoids

A typical example of **operation** is *composition*



(this coherence is noted "comp" in the following).

Type-theoretic ω-groupoids

A typical example of **coherence** is *associativity*

$$x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} w$$

$$\vdash$$

$$\text{coh} \quad : \quad x \xrightarrow{\text{comp}(\text{comp}(f,g),h)} w \quad \to \quad x \xrightarrow{\text{comp}(f,\text{comp}(g,h))} w$$

Implementation(s)

There are currently three implementations:

- `https://github.com/ericfinster/catt`
  - follows closely the rules of the article
- `https://github.com/smimram/catt`
  - has support for implicit arguments
  - has support for (some) Π-types
  - has support for “Hom” type variables:

    ```
    let comp (X : Hom) =
      coh (x : X) (y : X) (f : x -> y)  (z : X) (g : y -> z)
          : (x -> z)
    ```
  - has a web interface
- `https://github.com/ThiBen/catt`
  - best of both worlds

In practice,

- you simply enter a list of coherences
  (there is no reduction, etc.),
- if the program does not complain then they are valid
  operations in weak ω-categories.

“Demo”

- identity 1-cells

```
coh id (x : *) : * | x -> x ;
```

“Demo”

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```
coh id (x : *) : * | x -> x ;
```

- composition of 1-cells:

```
coh comp (x : *) (y : *) (f : * | x -> y)
         (z : *) (g : * | y -> z)
         : * | x -> z ;
```

"Demo"

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```
coh id (x : *) : * | x -> x ;
```

- composition of 1-cells:

```
coh comp (x : *) (y : *) (f : * | x -> y)
         (z : *) (g : * | y -> z)
         : * | x -> z ;
```

- associativity of composition of 1-cells:

```
coh assoc
    (x : *) (y : *) (f : * | x -> y) (z : *)
    (g : * | y -> z) (w : *) (h : * | z -> w)
    : * | x -> w
        | comp x z (comp x y f z g) w h ->
          comp x y f w (comp y z g w h) ;
```

“Demo”

- identity 1-cells

```
coh id (x : *) : * | x -> x ;
```

- composition of 1-cells:

```
coh comp (x : *) (y : *) (f : * | x -> y)
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    (x : *) (y : *) (f : * | x -> y) (z : *)
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    : * | x -> w
        | comp x z (comp x y f z g) w h ->
          comp x y f w (comp y z g w h) ;
```

- …

“Demo”

Only defining the Eckmann-Hilton morphism takes 300 lines

because you have to

- define usual operations and coherences,
- explicitly insert and remove identities,
- take care of bracketing of composites.

"Demo"

```
let eh (X : Hom) (x : X)
       (a : id x -> id x) (b : id x -> id x)
     : (comp' a b -> comp' b a)
     =
    comp11
    (comp' (unitl'- a) (unitr'- b))
    (assoc3 _ _ _ _)
    (compl2r' _ _ (unitlr x) _)
    (compl2' _ _ (comp3 (assoc- _ _ _)
    (comp' (unitr+- (id x)) (id _)) (unitl _)))
    (compl' _ (assoc- _ _ _)) (complr' _ (ich b a) _)
    (complr' _ (compr' (comp (unitr- _) (compl' _ (unitr+--
    (comp (complr' _ (assoc3 _ _ _ _) _) (compl' _ (assoc4
    (comp' (unitlr- x) (compl' _ (compl' _ (comp' (unitrl-
    (assoc3- _ _ _ _)
    (comp' (unitr' b) (unitl' a))
```

“Demo”

- no inverses:

```
coh inv (x : *) (y : *) (f : * | x -> y)
        : * | y -> x ;
```

  produces

```
Checking coherence: inv
Valid tree context
Src/Tgt check forced
Source context: (x : *)
Target context: (y : *)
Failure: Source is not algebraic for y : *
```

SEMI-STRICT CATEGORIES

The theory we presented is the *canonical* globular theory of higher-categories.

It works well, but we need many “very small steps” in proofs.

It seems that the interchange law is the “culprit”, so let’s try to keep this weak and everything else strict.

It turns out that this works (at least in low dimensions) and provides a nice framework for rewriting.

Higher-categories

A strict higher category is

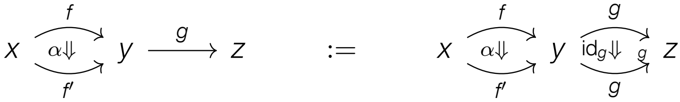
- a globular set
- with compositions (*n* for *n*-cells) and identities

such that

- composition is associative
- identities are units
- interchange laws are satisfied

Identities

Given a 2-cell α and a 1-cell g, we can define



$$\alpha *_0 g \quad := \quad \alpha *_0 \mathrm{id}_g$$

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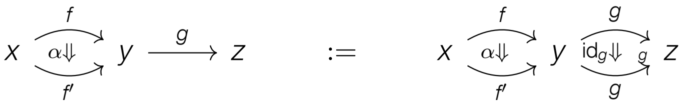


$$\alpha *_0 g \quad := \quad \alpha *_0 \mathrm{id}_g$$

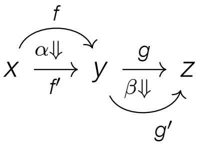
More generally, given an m-cell ϕ and an n-cell ψ, we can define an $(m \vee n)$-cell

$$\phi \quad *_i \quad \psi$$

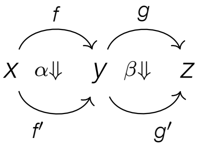
for $0 \leq i < m \wedge n$.

Compositions

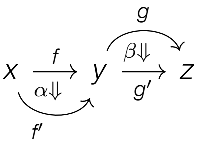
Given 2-cells α and β there are many ways to express the same composition:



$(\alpha *_0 g) *_1 (f' *_0 \beta)$



$\alpha *_0 \beta$



$(f *_0 \beta) *_1 (\alpha *_0 g')$

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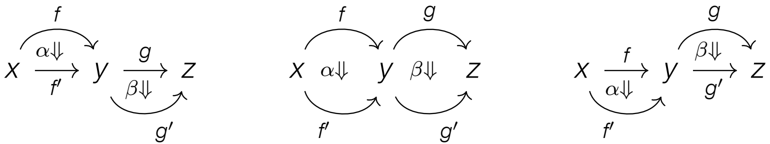

$(\alpha *_0 g) *_1 (f' *_0 \beta)$ $\qquad$ $\alpha *_0 \beta$ $\qquad$ $(f *_0 \beta) *_1 (\alpha *_0 g')$

We can restrict to compositions

$$\phi \quad *_i \quad \psi$$

with $i = \dim(\phi) \wedge \dim(\psi) - 1$.

Precategories

A **precategory** C is a globular set

$$C_0 \underset{t_0}{\overset{s_0}{\leftleftarrows}} C_1 \underset{t_1}{\overset{s_1}{\leftleftarrows}} C_2 \underset{t_2}{\overset{s_2}{\leftleftarrows}} \cdots$$

together with

- compositions

$$* \quad : \quad C_i \underset{i \wedge j - 1}{\times} C_j \quad \to \quad C_{i \vee j}$$

- identities

$$\mathrm{id} \quad : \quad C_i \quad \to \quad C_{i+1}$$

such that compositions and identities have expected source and target and are (suitably) associative.

Precategories

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such that compositions and identities have expected source and target and are (suitably) associative.

An n**-precategory** is a precategory C in which $C_i = \emptyset$ for $i > n$.

A **Gray category** is a 3-precategory equipped for every 2-cells

$$x \underset{u'}{\overset{u}{\Downarrow \phi}} y \qquad \text{and} \qquad y \underset{v'}{\overset{v}{\Downarrow \psi}} z$$

with an invertible **interchanger** 3-cell

$$X_{\phi,\psi} \quad : \quad (\phi * v) * (u' * \psi) \quad \Rrightarrow \quad (u * \psi) * (\phi * v')$$

$$x \underset{u'}{\overset{u}{\Downarrow \phi}} y \xrightarrow{\;v\;} z \quad \underset{v'}{\Downarrow \psi} \qquad \Rrightarrow \qquad x \xrightarrow{\;u\;} y \underset{v'}{\overset{v}{\Downarrow \psi}} z \quad \underset{u'}{\Downarrow \phi}$$

such that

- interchangers are compatible with compositions and id,
- interchangers are “natural”,
- the interchange law is strict for 3-cells.

Gray categories

In other words, a Gray category is

- a (strict) 3-category in which interchange equality of 2-cells has been replaced by an invertible 3-cell,
- a tricategory in which all coherence morphisms are strict excepting interchangers.

Coherence for tricategories

Theorem (Gordon-Power-Street)

Every tricategory is (suitably) equivalent to a Gray category.

A proof assistant for Gray categories

We could therefore think of making a proof-assistant for Gray categories (or higher semi-strict categories).

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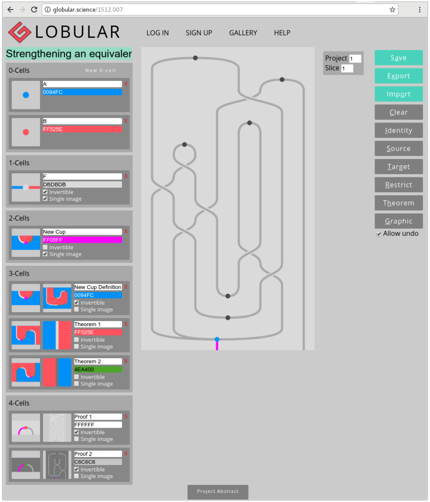


http://globular.science

Pros and cons

Vicary had this nice idea that semi-strict categories could be useful in practice.

Pros:

- most of the boring manipulations is taken care of for us
- very nice graphical interface

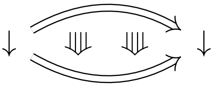
Cons:

- we do not know whether it properly generalizes to dimension $n > 3$ (and how)

Can we automate some of the proofs?

Coherence

Typically, we want to show **coherence** theorems: in a given Gray category, there is at most one 3-cell between any pair of parallel 2-cells.



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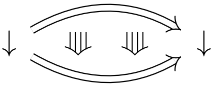

General idea: when generalizing an algebraic structure to higher dimensions, we want to replace equality by unique isomorphism.

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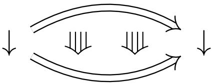

General idea: when generalizing an algebraic structure to higher dimensions, we want to replace equality by unique isomorphism.

We are generally interested in **Gray groupoids**, i.e., Gray categories with invertible 3-cells.

COHERENT PRESENTATIONS OF GRAY CATEGORIES

Signatures

A **signature** P consists of sets

- P_0: generating 0-cells
- P_1: generating 1-cells
- P_2: generating 2-cells

together with their source and targets.

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A **signature** P consists of sets

- P_0: generating 0-cells
- P_1: generating 1-cells
- P_2: generating 2-cells

together with their source and targets.

We have a 2-precategory P^* with sets of *i*-cells

$$P_i^*$$

obtained by formally composing *i*-generators.

Generated morphisms

Given a signature P,

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Note: this is a *canonical form* modulo the theory of precategories!

Monoids

For instance the signature for monoids is

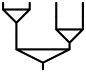
$$\mathsf{P}_0 = \{\star\} \qquad \mathsf{P}_1 = \{a : \star \to \star\} \qquad \mathsf{P}_2 = \{\mu : 2 \Rightarrow 1, \eta : 0 \Rightarrow 1\}$$

(where $2 = aa$, etc.)

We draw

$\mu =$  $\qquad\qquad \eta =$ 

We have the following morphism in P_2^*:

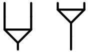
$(\mu * 2) * (1 * \mu) * \mu \quad =$ 

Compositions

Note that we have

$$(\mu * 2) * (1 * \mu)$$

and



$$(2 * \mu) * (\mu * 1)$$

but

does not make sense.

Rewriting systems

A **rewriting system** consists of

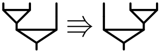
- a presentation $(\mathsf{P}_0, \mathsf{P}_1, \mathsf{P}_2)$,
- a set of **rewriting rules** P_3 between elements of P_2^*.

A **rewriting step** is a rewriting rule “in context”.

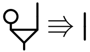
We write P_3^* for the rewriting paths.

For monoids, the rules are

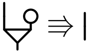
$$A : (\mu * 1) * \mu \Rrightarrow (1 * \mu) * \mu$$



$$L : (\eta * 1) * \mu \Rrightarrow \mu$$



$$R : (1 * \eta) * \mu \Rrightarrow \mu$$



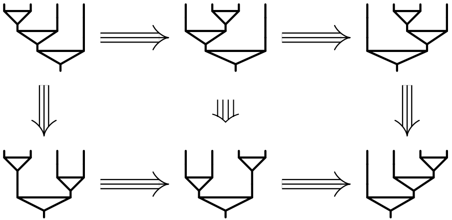
Note that in models we have in mind, these are invertible.

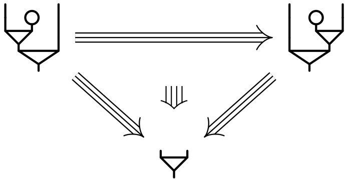
Coherent presentations

A **coherent presentation** P consists of

- a rewriting system (P_0, P_1, P_2, P_3),
- a set P_4 of **relations** between elements of P_3^*.

For monoids, we want to put the relations

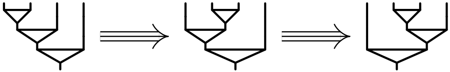




Generated categories

Given a coherent presentation, we write

- P^* for the 3-precategory with cells $(\mathsf{P}^*_0, \mathsf{P}^*_1, \mathsf{P}^*_2, \mathsf{P}^*_3)$



Generated categories

Given a coherent presentation, we write

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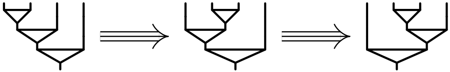
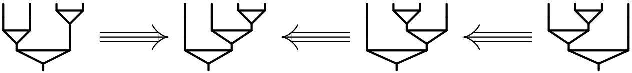


- $P^\top$ for the 3-precategory P^* with 3-cells formally inverted



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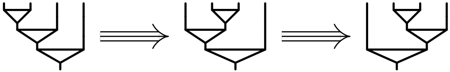

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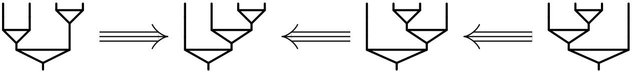


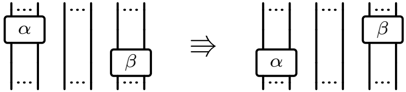
- $\overline{\mathsf{P}}$ for the 3-precategory $\mathsf{P}^\top$ with 3-cells quotiented by the congruence generated by P_4: the **presented precategory**.

Lemma

$\overline{\mathsf{P}}$ *is a Gray category when* P *is such that*

- *for each 2-generators* $\alpha, \beta \in \mathsf{P}_2$ *and 1-cell* $v \in \mathsf{P}_1^*$ *there is an* ***interchange*** *3-cell*

  $X_{\alpha,v,\beta}$:



Presenting Gray categories

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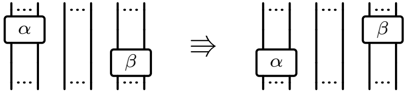
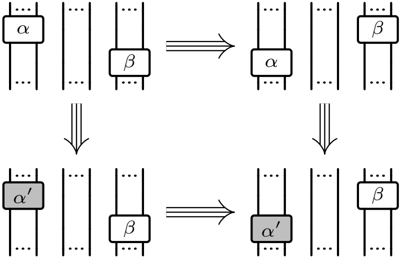
  

- *for each* $A : \alpha \Rrightarrow \alpha' \in \mathsf{P}_3$*,* $u \in \mathsf{P}_1^*$ *and* $\beta \in \mathsf{P}_2$ *there is a relation*

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  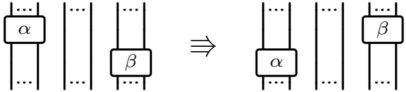

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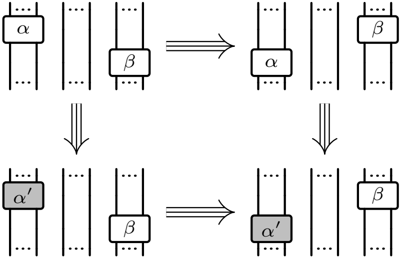
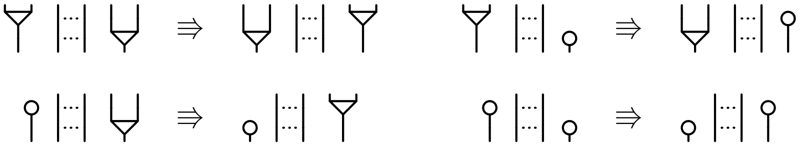
  

- *relations generating interchange of 3-cells*

The theory of monoids

We consider the theory of monoids with the additional

- rewriting rules



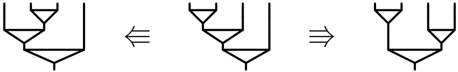
- and relations

…

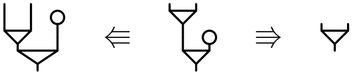
Rewriting

A **critical branching** is a minimal and non-trivial[1] overlapping of two rewriting rules.

For instance,



or

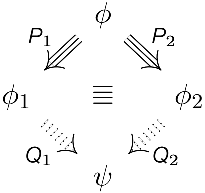


[1] Including according to the relations of interchange!

Theorem

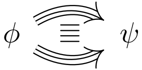
Suppose given a presentation

- *which is* terminating *(no infinite sequence of rewriting steps),*
- *has* confluent critical pairs, *i.e.,*



where $\equiv$ *is the congruence generated by* P_4*.*

Then $\overline{\mathrm{P}}$ *is coherent:*



Proof.

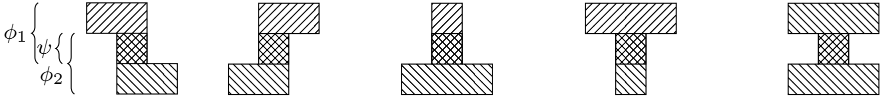
Variants of critical pairs + Newman + Church-Rosser lemmas. □

Critical branchings

Theorem

A presentation with finite number of rewriting rules (excepting for interchangers) has a finite number of critical branchings.

Proof.



$\phi_1 = \phi'_1 * (u * \psi)$, $\phi_2 = (\psi * v) * \phi'_2$

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$\phi_1 = \phi'_1 * (u * \psi * v) * \phi''_1$, $\phi_2 = \psi$

□

Note:

- even though there is an infinite number of interchangers!
- this is not true for 3-categories

Results

We have been able to show coherence for various classical structures in tricategories, including:

- pseudomonoids,
- adjunctions,
- etc.

THANKS!