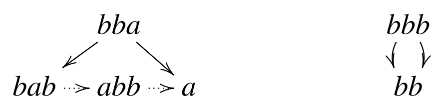
Presenting Finite Posets

Samuel Mimram
CEA, LIST / École Polytechnique*
samuel.mimram@cea.fr

We introduce a monoidal category whose morphisms are finite partial orders, with chosen minimal and maximal elements as source and target respectively. After recalling the notion of presentation of a monoidal category by the means of generators and relations, we construct a presentation of our category, which corresponds to a variant of the notion of bialgebra.

String rewriting systems have been originally introduced by Thue [21] in order to study word problems in monoids. A string rewriting system (Σ, R) consists of a set Σ, called the *alphabet*, and a set $R \subseteq \Sigma^* \times \Sigma^*$ of *rules*. The monoid $\Sigma^*/{\equiv_R}$, obtained by quotienting the free monoid Σ^* over Σ by the smallest congruence (wrt concatenation) containing R, is called the monoid *presented* by the rewriting system. The rewriting system can thus be thought of as a small description of the monoid, and the word problem consists in deciding whenever two words $u, v \in \Sigma^*$ represent the same word, i.e. are such that $u \equiv_R v$. Now, when the rewriting system is convergent, i.e. both terminating and confluent, normal forms provide canonical representatives of the equivalence classes: two words $u, v \in \Sigma^*$ are equivalent by the congruence $\equiv_R$ if and only if they have the same normal form, and the word problem can be thus be decided in this case.

Example 1. Consider the rewriting system (Σ, R) with $\Sigma = \{a, b\}$ and $R = \{(ba, ab), (bb, \varepsilon)\}$, where ε denotes the empty word. This rewriting system is easily shown to be terminating, and the two critical pairs can be joined:



the system is thus convergent. Normal forms are words of the form a^n or $a^n b$, with $n \in \mathbb{N}$, and from this it is easy to deduce that the rewriting system presents the additive monoid $\mathbb{N} \times (\mathbb{N}/2\mathbb{N})$: there is an obvious bijection between the set of normal forms and the elements of the monoid, and it remains to be checked that this bijection is compatible with product and units.

Starting from this point of view, it is natural to wonder whether the notion of rewriting system can be extended to more general settings, in particular to provide presentations for n-categories (a monoid can be seen as the particular case of a 1-category with only one object). A satisfactory answer to this question was provided by Street and Power's *computads* [19, 18], which were rediscovered as Burroni's *polygraphs* [4], and really deserve the name of *higher-dimensional rewriting systems* at the light of the preceding comments [15]. Those were the subject of my introductory invited presentation at the TERMGRAPH workshop in 2014, along with the way usual rewriting techniques generalize to this framework.

The aim of this article is to describe a particular example of a presentation of a monoidal category (i.e. a 2-category with one 0-cell) by a 2-dimensional rewriting system, whose morphisms are finite partial orders. This example was discovered during the author's PhD thesis [13] (Section 4.3.1) while

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studying asynchronous game semantics. We will focus here on this example only and introduce only the material which is necessary to handle it: we will only define presentations for monoidal categories (see [4] for the general definition of higher-dimensional rewriting system) and we will not mention the links to game semantics (see [13, 12, 14] for a more detailed account of the relationships between game semantics, partial orders, and presentations of categories). The proof has been split in a series of simple enough lemmas, which is why we have left out most proofs.

Related works. The general methodology used here is strongly inspired from the one developed by Lafont who studied many examples of 2-dimensional presentations [9]. A variant of this example was studied by Fiore and Devesas Campos [5] using similar techniques. Finally, a very interesting tool, based on distributive laws between monads, has been introduced by Lack in order to build presentations of monoidal categories by combining presentations of smaller (and simpler) monoidal subcategories [8]: a particularity of the present example is that this technique does not apply here.

Plan. We begin by formally introducing the category **P** of finite posets we will be interested in giving a presentation in Section 1, detail what we mean by a presentation of a monoidal category in Section 2, and give a few examples of presentations along with the one for **P** in Section 3. The rest of the article is devoted to proving that we actually have a presentation: we define a canonical factorization of morphisms of **P** in Section 4, which is used to finish the proof in Section 5. We conclude and hint at possible generalizations of this result in Section 6.

1 A category of finite posets

A *poset* is a pair $E = (\underline{E}, \leqslant_E)$ consisting of a set $\underline{E}$, whose elements are called *events*, and a relation $\leqslant_E \subseteq \underline{E} \times \underline{E}$ which is reflexive, antisymmetric and transitive. Two events $x, y \in \underline{E}$ are *independent* when neither $x \leqslant_E y$ nor $y \leqslant_E x$.

Definition 2. We write **Pos** for the category whose objects are posets and morphisms are increasing functions.

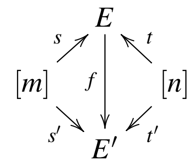
This category can be shown to be cocomplete [1], and in particular pushouts always exist. There is a full and faithful inclusion functor **Set** $\to$ **Pos**, such that the image of a set X is the poset $(X, =)$, and we will allow ourselves to implicitly see a set as a poset in this way: a poset in the image of this functor is called *discrete*. Given an integer $n \in \mathbb{N}$, we write $[n]$ for the set $\{0, 1, \ldots, n-1\}$ (or the associated discrete poset), and $[\vec{n}]$ for the totally ordered poset $([n], \leqslant)$. Given a poset E and a set $D \subseteq \underline{E}$, we write $E \setminus D$ for the poset such that $\underline{E \setminus D} = \underline{E} \setminus D$ and $\leqslant_{E \setminus D}$ is the restriction of $\leqslant_E$ to this set.

The main category of interest in this article is defined as follows:

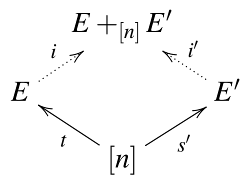
Definition 3. We write **P** for the category whose objects are natural numbers $n \in \mathbb{N}$, and morphisms $(s, E, t) : m \to n$ are equivalence classes of triples consisting of

- a finite poset E,
- a monomorphism $s : [m] \to E$, called *source*, whose images are minimal events in the poset,
- a monomorphism $t : [n] \to E$, called *target*, whose images are maximal events in the poset,

quotiented by the equivalence relation identifying two such triples (s,E,t) and (s',E',t') whenever there exists an isomorphism $f : E \to E'$ such that $f \circ s = s'$ and $f \circ t = t'$:



Given two composable morphisms $(s,E,t) : m \to n$ and $(s',E',t') : n \to p$, we consider the poset $E +_{[n]} E'$ defined by the following pushout in **Pos**



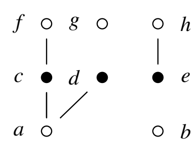
and define their composite as

$$(i \circ s, (E +_{[n]} E') \setminus [n], i' \circ t') \quad : \quad m \quad \to \quad p$$

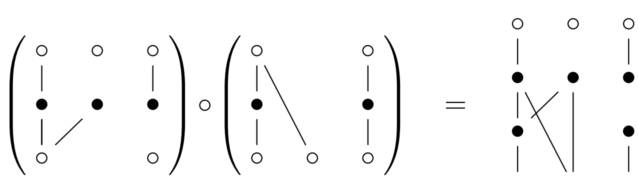
The identity on an object n is $(\mathrm{id}_n, [n], \mathrm{id}_n) : n \to n$. We leave to the reader as an easy exercise to check that composition is well-defined and that the axioms of categories are satisfied (and more generally, we will leave to the reader the check that constructions subsequently performed on the category are compatible with the equivalence relation). An event $x \in \underline{E}$ in a morphism $(s,E,t) : m \to n$ is called *external* if it is in the image of either s or t, i.e. $x \in s([m]) \cup t([n])$, and *internal* otherwise.

Remark 4. The above definition is a variant of the well-known construction of the bicategory of spans [3] inside a category with pullbacks such as **Pos**.

Example 5. The morphism $(s,E,t) : 2 \to 3$ with $E = \{a,b,c,d,e,f,g,h\}$ with $a < c < f$, $a < d$ and $e < h$, $s(0) = a$, $s(1) = b$, $t(0) = f$, $t(1) = g$, $t(2) = h$ will be drawn by its Hasse diagram, i.e. the graph of its immediate successor relation:



The bullets represent events of the poset, the filled ones denoting internal events and empty ones denoting external events, and events are increasing from bottom to top. Notice that the names of the bullets do not really matter, because of the quotient used when defining morphisms, and we will not figure them in the following. The image of the source (resp. target) is figured by the empty bullets at the bottom (resp. top), where the $s(i)$ (resp. $t(i)$) are always represented with i increasing from left to right. Composition is performed by "gluing" diagrams along the interface (by pushout) and erasing the external events of this interface:



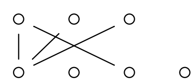
Notice that this category is self-dual, in the sense that there is an isomorphism $\mathbf{P} \cong \mathbf{P}^{\mathrm{op}}$, which is the identity on objects and respects the monoidal structure introduced in next section (Definition 9). Also, it contains the category of relations as a subcategory:

Definition 6. We write **Rel** for the category with integers as objects, such that a morphism $R : m \to n$ is a relation $R \subseteq [m] \times [n]$. The composite of two morphisms $R : m \to n$ and $S : n \to o$ is the relation $S \circ R : m \to o$ such that $(i,k) \in S \circ R$ if and only if there exists $j \in [n]$ such that $(i,j) \in R$ and $(j,k) \in S$. The identity relation $\mathrm{id}_n : n \to n$ is such that $(i,j) \in \mathrm{id}_n$ if and only if $i = j$.

The following lemma will allow us to implicitly see a relation as a morphism in the category **P**:

Lemma 7. *The following functor* $\mathbf{Rel} \to \mathbf{P}$ *is faithful. The functor is the identity on objects, and the image of a relation* $R : m \to n$ *is the morphism* $(s, E, t) : m \to n$*, with* $\underline{E} = [m] \uplus [n]$*, such that for any two elements* $i \in [m]$ *and* $j \in [n]$ *of* $\underline{E}$ *we have* $i \leqslant j$ *if and only if* $(i,j) \in R$*, and the maps* $s : [m] \to E$ *and* $t : [n] \to E$ *are the injections of the coproduct.*

Example 8. The relation $R : 4 \to 3$ with $R = \{(0,0),(0,1),(0,2),(2,0)\}$ can be seen as the following morphism of **P**:



2 Presenting monoidal categories

We recall that a *strict monoidal category* $(\mathscr{C}, \otimes, I)$ consists of a category $\mathscr{C}$ together with a functor $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ called *tensor product* and an object $I \in \mathscr{C}$ called *unit* such that the tensor is associative and admits I as unit: for every objects A, B and C, $(A \otimes B) \otimes C = A \otimes (B \otimes C)$ and $I \otimes A = A = A \otimes I$. In this article, we only consider monoidal categories which are strict. A *symmetry* in such a category consists of a natural transformation of components $\gamma_{A,B} : A \otimes B \to B \otimes A$ such that for every objects A, B, C we have

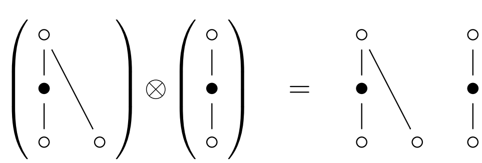
$$\gamma_{A \otimes B, C} = (\gamma_{A,C} \otimes \mathrm{id}_B) \circ (\mathrm{id}_A \otimes \gamma_{B,C}) \qquad \gamma_{I,A} = \mathrm{id}_A \qquad \gamma_{B,A} \circ \gamma_{A,B} = \mathrm{id}_{A \otimes B}$$

A functor between monoidal categories is *monoidal* when it respects the tensor product and the unit, and we moreover suppose that monoidal functors between symmetric monoidal categories preserve the symmetry. We write **MonCat** for the category of monoidal categories and monoidal functors.

Definition 9. The category **P** can be made into a monoidal category with 0 as unit, and tensor being defined by addition on objects ($m \otimes n = m + n$) and by disjoint union on morphisms. Moreover, a symmetry can easily be defined.

A monoidal category, such as **P**, with integers as objects and tensor product being given on objects by addition is often called a *PRO*, or a *PROP* when it is additionally equipped with a symmetry [10].

Example 10. Using the diagrammatic notations of Example 5, we have



In order to define presentations of monoidal categories, we first need to define an appropriate notion of signature for them. Since those contain objects and morphisms, a signature will consist of generators for both of them. For the sake of brevity, we follow here the formalization specific to monoidal categories (also sometimes called *tensor schemes* [7]), see [4] for the general case.

Definition 11. A *(monoidal) signature* $\Sigma = (\Sigma_1, s_1, t_1, \Sigma_2)$ consists of

– a set Σ_1 of *object generators*,

– a set Σ_2 of *morphism generators*,

– two functions $s_1, t_1 : \Sigma_2 \to \Sigma_1^*$, where Σ_1^* denotes the free monoid over Σ_1, assigning to a morphism generator its *source* and *target* respectively.

The category **MonSig** has signatures as objects and a morphism $f : (\Sigma_1, s_1, t_1, \Sigma_2) \to (\Sigma_1', s_1', t_1', \Sigma_2')$ is a pair $f = (f_1, f_2)$ of functions $f_1 : \Sigma_1 \to \Sigma_1'$ and $f_2 : \Sigma_2 \to \Sigma_2'$ such that $s_1' \circ f_2 = f_1^* \circ s_1$ and $t_1' \circ f_2 = f_1^* \circ t_1$, where $f_1^* : \Sigma_1^* \to {\Sigma_1'}^*$ is the extension of f_1 as a morphism of monoids.

A monoidal signature generates a free monoidal category in the following sense. Given a category $\mathscr{C}$, we write $\mathrm{Mor}(\mathscr{C})$ for the class of morphisms of $\mathscr{C}$.

Proposition 12. *Consider the forgetful functor U : **MonCat** $\to$ **MonSig** sending a monoidal category $(\mathscr{C}, \otimes, I)$ to the signature $\Sigma = (\Sigma_1, s_1, t_1, \Sigma_2)$, with Σ_1 being the set of objects of $\mathscr{C}$, Σ_2 consisting of triples $(A_1 \ldots A_p, f, B_1 \ldots B_q) \in \Sigma_1^* \times \mathrm{Mor}(\mathscr{C}) \times \Sigma_1^*$ such that $f \in \mathscr{C}(A_1 \otimes \ldots \otimes A_n, B_1 \otimes \ldots \otimes B_p)$, and functions s_1 and t_1 are given by first and third projection respectively. This functor admits a left adjoint F : **MonSig** $\to$ **MonCat**. Given a signature Σ, the monoidal category $F\Sigma$ is often denoted Σ^* and called the* free monoidal category *on the signature. More explicitly, Σ^* is the smallest category whose class of objects is Σ_1^*, for every morphism generator $f \in \Sigma_2$ there is a morphism $f \in \Sigma^*(s_1(f), t_1(f))$, for every object there is a formal identity on this object, morphisms are closed under formal composites and tensor products, and are quotiented by the axioms of monoidal categories.*

Remark 13. Suppose fixed a signature $\Sigma = (\Sigma_1, s_1, t_1, \Sigma_2)$ and a monoidal category $(\mathscr{C}, \otimes, I)$. Suppose moreover that we are given an object FA for every object generator $A \in \Sigma_1$, and a morphism $F\alpha : FA_1 \otimes \ldots \otimes FA_p \to FB_1 \otimes \ldots \otimes FB_q$ for every morphism generator $\alpha \in \Sigma_2$ with $s_1(\alpha) = A_1 \ldots A_p$ and $t_1(\alpha) = B_1 \ldots B_q$. Then, by the freeness property of Σ^*, the operation F extends uniquely as a monoidal functor $F : \Sigma^* \to \mathscr{C}$.

Remark 14. When the set $\Sigma_1 = \{1\}$ is reduced to one element, Σ_1^* is isomorphic to $\mathbb{N}$, and we denote by integers its elements. In this case, the monoidal category Σ^* is a PRO.

The following lemma is often useful in order to prove properties of morphisms of Σ^* by induction over the number of generators they consist in:

Lemma 15. *Every morphism f of Σ^* can be written as a finite composite $f = f_k \circ \ldots \circ f_1$ where each morphism f_i is of the form $f_i = \mathrm{id}_{A_i} \otimes \alpha_i \otimes \mathrm{id}_{B_i}$ for some objects A_i and B_i and morphism generator $\alpha_i \in \Sigma_2$.*

The notion of rewriting system, adapted to monoidal categories, can finally be defined as follows. We say that two morphisms in a category are *parallel* when they have the same source, and the same target.

Definition 16. A *(monoidal) rewriting system* is a pair (Σ, R) consisting of a signature Σ and a set $R \subseteq \mathrm{Mor}(\Sigma^*) \times \mathrm{Mor}(\Sigma^*)$ of pairs of morphisms which are parallel. The monoidal category *presented* by such a rewriting system is the monoidal category $\Sigma^*/{\equiv_R}$ obtained by quotienting the morphisms of Σ^* by the smallest congruence $\equiv_R$, wrt composition and tensor product, containing R.

Remark 17. Since $\equiv_R$ is the congruence generated by R, a functor $F : \Sigma^* \to \mathscr{C}$ such that $F(f) = F(g)$ for every $(f,g) \in R$ induces a quotient functor $(\Sigma^*/\equiv_R) \to \mathscr{C}$.

We often write $\alpha \Rightarrow \beta$ for a rule $(\alpha,\beta) \in R$. We recall below the most well-known example of presentation of a monoidal category: the simplicial category. It is fundamental since it is at the heart of simplicial algebraic topology.

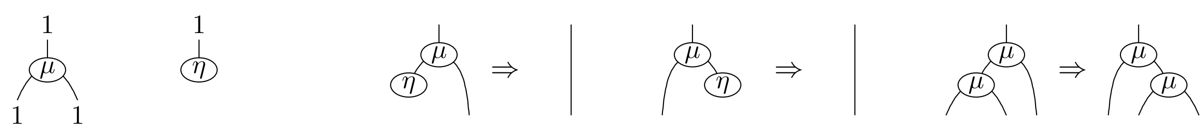
Example 18. Consider the signature $\Sigma = (\Sigma_1, s_1, t_1, \Sigma_2)$ with $\Sigma_1 = \{1\}$, $\Sigma_2 = \{\eta, \mu\}$, $s_1(\eta) = \varepsilon$ (where ε denotes the empty word), $s_1(\mu) = 11$, and $t_1(\eta) = t_1(\mu) = 1$. As explained in Remark 14, since the set Σ_1 contains only one generator, the monoid Σ_1^* is isomorphic to $\mathbb{N}$. The morphism generators can thus be denoted

$$\eta : 0 \to 1 \qquad \text{and} \qquad \mu : 2 \to 1$$

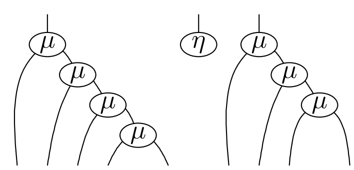
along with their source and target. We will use this notation for defining morphism generators in the following. We consider the set of rules defined by

$$R \quad = \quad \{ \quad \mu \circ (\eta \otimes \mathrm{id}_1) \Rightarrow \mathrm{id}_1 \quad , \quad \mu \circ (\mathrm{id}_1 \otimes \eta) \Rightarrow \mathrm{id}_1 \quad , \quad \mu \circ (\mu \otimes \mathrm{id}_1) \Rightarrow \mu \circ (\mathrm{id}_1 \otimes \mu) \quad \} \tag{1}$$

If we use the string diagrammatic notation for morphisms, and draw η and μ respectively as on the left below, the rules can be figured as on the right:



(notice that diagrams have to be read from bottom up, in order to be consistent with the notation for posets of Section 1). The rewriting system can be shown to be terminating and the five critical pairs can be joined, it is thus convergent. Normal forms are tensor products of "right combs", e.g.



(2)

Now, consider the *simplicial category* Δ, with integers as objects, and such that a morphism $f : m \to n$ is an increasing function $f : [m] \to [n]$. It is monoidal when equipped with the expected tensor product given on objects by addition, i.e. it is a PRO. With the above graphical representation in mind, it is easy to see that normal forms for the rewriting system are in bijection with morphisms of Δ, and that this bijection is compatible with composition and tensor product. For instance, the normal form (2) corresponds to the function $f : [9] \to [3]$ such that $f(i) = 0$ if $0 \leqslant i \leqslant 4$, and $f(i) = 2$ if $5 \leqslant i \leqslant 8$, see [9, 15] for details.

The rewriting system described in previous example corresponds to the theory of monoids: from the above presentation, we immediately deduce that, given a monoidal category $\mathscr{C}$, monoidal functors $\Delta \to \mathscr{C}$ are in bijection with monoids in $\mathscr{C}$, see Definition 19 and [11] (and moreover monoidal natural transformations correspond to morphisms of monoids). We will thus call the rewriting system the *theory* of monoids (the "the" is slightly abusive since there are multiple choices for orientations of rules and even of generators, but those will give the same presented category).

The proof that the theory of monoids is a presentation of the simplicial category, that we sketched in Example 18, is a direct adaptation of the classical rewriting argument in the case of monoids presented in Example 1. There are some theories for which no orientation of the rules gives rise to a convergent rewriting system. However, what matters here is only the fact that we have canonical representatives for equivalence classes of morphisms in Σ^* modulo the congruence generated by the rules: we can hope to define them "by hand" (in which case we call them *canonical forms*), instead of obtaining them as normal forms for a rewriting system. In order to show that a rewriting system (Σ, R) is a presentation for a monoidal category $\mathscr{C}$, one can thus apply the following recipe discovered by Lafont [9]:

1. define a functor $\Sigma^* \to \mathscr{C}$ which is bijective on objects by interpreting generators of Σ into $\mathscr{C}$, see Remark 13,
2. show that it induces a functor $F : (\Sigma^*/{\equiv_R}) \to \mathscr{C}$ by verifying that every pair of morphisms in R have the same image, see Remark 17,
3. define (e.g. inductively) a subset of morphisms in Σ^*, whose elements are called canonical forms,
4. show that every morphism in Σ^* is equivalent by $\equiv_R$ to a canonical form by induction, see Lemma 15,
5. show that the functor F restricted to canonical forms is a bijection.

The first two steps allow us to define a functor $F : (\Sigma^*/{\equiv_R}) \to \mathscr{C}$, interpreting the presented monoidal category into $\mathscr{C}$ and step 3 and 4 amount to choose at least one representative in each equivalence class of morphisms modulo $\equiv_R$. Finally, step 5 allows us to conclude. Namely, the functor F is full since every morphism of $\mathscr{C}$ is the image of the equivalence class of the corresponding canonical form. The functor is also faithful: given two morphisms f and g such that $Ff = Fg$, by step 4 there is a canonical form $\overline{f}$ associated to f and a canonical form $\overline{g}$ associated to g and those are necessarily equal by step 5, and therefore $f = g$. Notice that we can conclude *a posteriori* that each equivalence class contains exactly one canonical form. In the following, we use a variant of this methodology in order to build a presentation for the monoidal category **P**. We will define the presentation and interpret it in Section 5.

3 The theory of poalgebras

In this section, we first introduce some classical algebraic structures and provide the monoidal category presented by the associated theory (i.e. rewriting system), and then define the algebraic structure corresponding to the category **P**, which is a variant of those. We suppose fixed a monoidal category $(\mathscr{C}, \otimes, I)$ equipped with a symmetry γ.

Definition 19. A *monoid* (M, η, μ) in $\mathscr{C}$ consists of an object M together with two morphisms $\eta : I \to M$ and $\mu : M \otimes M \to M$ such that $\mu \circ (\mu \otimes \mathrm{id}_M) = \mu \circ (\mathrm{id}_M \otimes \mu)$ and $\mu \circ (\eta \otimes \mathrm{id}_M) = \mathrm{id}_M = \mu \circ (\mathrm{id}_M \otimes \eta)$. Such a monoid is *commutative* when $\mu \circ \gamma_{M,M} = \mu$. A *comonoid* (M, ε, δ) is a monoid in $\mathscr{C}^{\mathrm{op}}$.

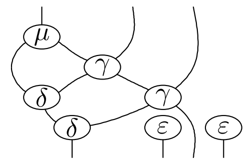
Proposition 20 ([11, 9])**.** *The theory for monoids presents the simplicial category Δ, see Example 18. The theory for commutative monoids presents the PROP* **F** *such that a morphism $f : m \to n$ is a function $f : [m] \to [n]$.*

Definition 21. A *bialgebra* $(B, \eta, \mu, \varepsilon, \delta)$ consists of a monoid (B, η, μ) and a comonoid (B, ε, δ) such that $\delta \circ \mu = (\mu \otimes \mu) \circ (\mathrm{id}_B \otimes \gamma_{B,B} \otimes \mathrm{id}_B) \circ (\delta \otimes \delta)$ and $\varepsilon \circ \eta = \mathrm{id}_I$. Such a bialgebra is *bicommutative* when both the monoid and the comonoid structure are commutative, and *qualitative* when moreover $\mu \circ \delta = \mathrm{id}_B$.

Proposition 22 ([10, 6, 17, 9, 8, 14])**.** *The theory for bicommutative bialgebras presents the PROP* $\mathbf{Mat}_{\mathbb{N}}$ *such that a morphism $f : m \to n$ is an $(m \times n)$-matrix with coefficients in $\mathbb{N}$ together with usual composition. The theory for qualitative bicommutative bialgebras presents the PROP* **Rel***, see Definition 6.*

Example 23. Consider the rewriting system (Σ, R) corresponding to the theory for qualitative bicommutative bialgebras. The relation of Example 8 corresponds to the following morphism of $\Sigma^*/\equiv_R$:

$$(\mu \otimes \mathrm{id}_2) \circ (\mathrm{id}_1 \otimes \gamma \otimes \mathrm{id}_1) \circ (\delta \otimes \gamma) \circ (\delta \otimes \varepsilon \otimes \mathrm{id}_1 \otimes \varepsilon)$$

which can be represented graphically as  .

In the above propositions, we have been considering presentations of PROPs and not PROs as earlier. In order to present them we could have used a variant of rewriting systems in order to freely generate symmetries. However, thanks to the following proposition [4], there is a way to explicitly incorporate an explicit symmetry into the presentations without changing the notion of presentation. This is the one we have been implicitly using.

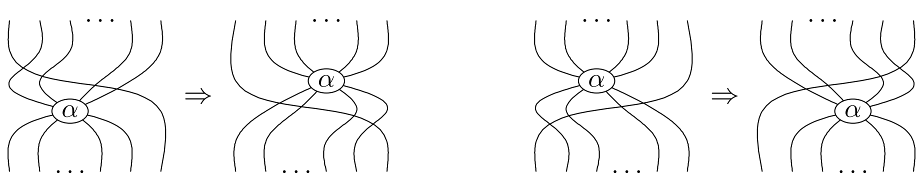
Proposition 24. *Suppose that a PRO $\mathscr{C}$ is presented by a rewriting system (Σ, R). Then the rewriting system (Σ', R') presents the free PROP on the PRO $\mathscr{C}$ where Σ' is obtained by adding a new morphism generator $\gamma : 2 \to 2$ to Σ, and R' is obtained from R by adding the rule $\gamma \circ \gamma \Rightarrow \mathrm{id}_2$ together with, for every morphism generator $\alpha : m \to n$, two rules*

$$\gamma_{n,1} \circ (\alpha \otimes \mathrm{id}_1) \Rightarrow (\mathrm{id}_1 \otimes \alpha) \circ \gamma_{m,1} \qquad (\alpha \otimes \mathrm{id}_1) \circ \gamma_{1,m} \Rightarrow \gamma_{1,n} \circ (\mathrm{id}_1 \otimes \alpha) \tag{3}$$

where the morphism $\gamma_{m,n} : m+n \to n+m$ is defined by induction on $(m,n) \in \mathbb{N} \times \mathbb{N}$ by

$$\gamma_{1,0} = \mathrm{id}_1 \qquad \gamma_{1,n+1} = (\mathrm{id}_1 \otimes \gamma_{1,n}) \circ (\gamma \otimes \mathrm{id}_n) \qquad \gamma_{0,n} = \mathrm{id}_n \qquad \gamma_{m+1,n} = (\gamma_{m,n} \otimes \mathrm{id}_n) \circ (\mathrm{id}_m \otimes \gamma_{1,n})$$

Graphically, a representation of the rules (3) *is*



Example 25. The PROP corresponding to monoids is presented by the rewriting system (Σ, R) with $\Sigma_1 = \{1\}$, and $\Sigma_2 = \{\eta : 0 \to 1, \mu : 2 \to 1, \gamma : 2 \to 2\}$, and R consisting of rules (1) together with

$$\gamma \circ (\eta \otimes \mathrm{id}_1) \Rightarrow \mathrm{id}_1 \otimes \eta \qquad \gamma \circ (\mu \otimes \mathrm{id}_1) \Rightarrow (\mathrm{id}_1 \otimes \mu) \circ (\gamma \otimes \mathrm{id}_1) \circ (\mathrm{id}_1 \otimes \gamma)$$

$$\eta \otimes \mathrm{id}_1 \Rightarrow \gamma \circ (\mathrm{id}_1 \otimes \eta) \qquad (\mu \otimes \mathrm{id}_1) \circ (\mathrm{id}_1 \otimes \gamma) \circ (\gamma \otimes \mathrm{id}_1) \Rightarrow \gamma \circ (\mathrm{id}_1 \otimes \mu)$$

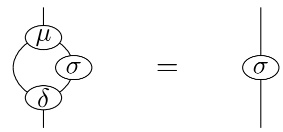
$$\gamma \circ \gamma \Rightarrow \mathrm{id}_2 \qquad (\gamma \otimes \mathrm{id}_1) \circ (\mathrm{id}_1 \otimes \gamma) \circ (\gamma \otimes \mathrm{id}_1) \Rightarrow (\mathrm{id}_1 \otimes \gamma) \circ (\gamma \otimes \mathrm{id}_1) \circ (\mathrm{id}_1 \otimes \gamma)$$

A presentation for the PROP $\mathbf{F}$ corresponding to commutative monoids is obtained by further adding the rule $\mu \circ \gamma \Rightarrow \mu$, see Proposition 20.

The aim of this article is to show that the category $\mathbf{P}$, defined in Section 1, admits the following theory as presentation.

Definition 26. A *poalgebra* $(P,\eta,\mu,\varepsilon,\delta,\sigma)$ in a symmetric monoidal category $\mathscr{C}$ consists of a qualitative bicommutative bialgebra $(P,\eta,\mu,\varepsilon,\delta)$ in Σ together with a morphism $\sigma : P \to P$ which is satisfying the following axiom, called *transitivity*:

$$\mu \circ (\mathrm{id}_P \otimes \sigma) \circ \delta \quad = \quad \sigma \tag{4}$$



Remark 27. The additional axiom seems to break the horizontal symmetry of the structure, but this is not the case since the the dual axiom is implied:

$$\mu \circ (\sigma \otimes \mathrm{id}_P) \circ \delta = \mu \circ \gamma \circ (\sigma \otimes \mathrm{id}_P) \circ \delta = \mu \circ (\mathrm{id}_P \otimes \sigma) \circ \gamma \circ \delta = \mu \circ (\mathrm{id}_P \otimes \sigma) \circ \delta = \sigma$$

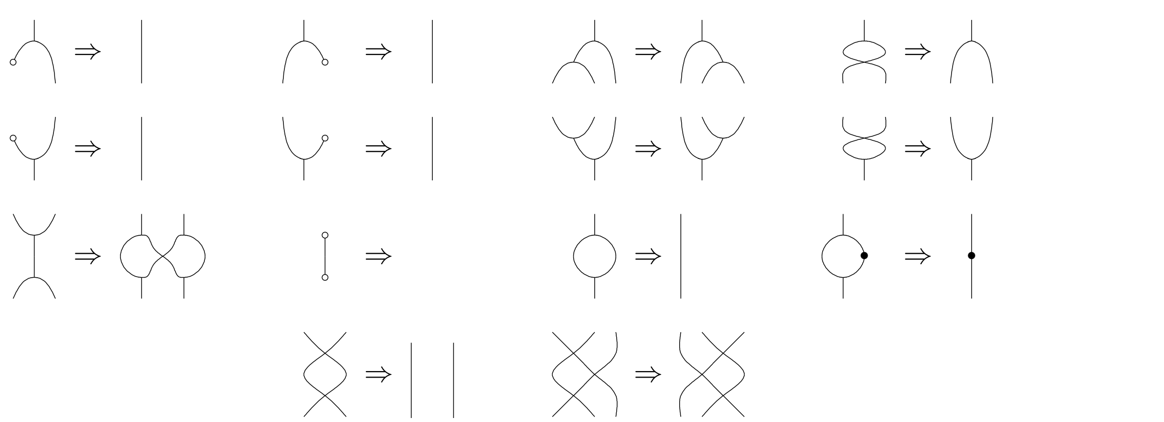
To sum up the axioms introduced in previous section, the theory of poalgebras is the rewriting system (Σ, R) with $\Sigma_1 = \{1\}$ as object generators, and the morphism generators in Σ_2 are, together with their diagrammatic notation,

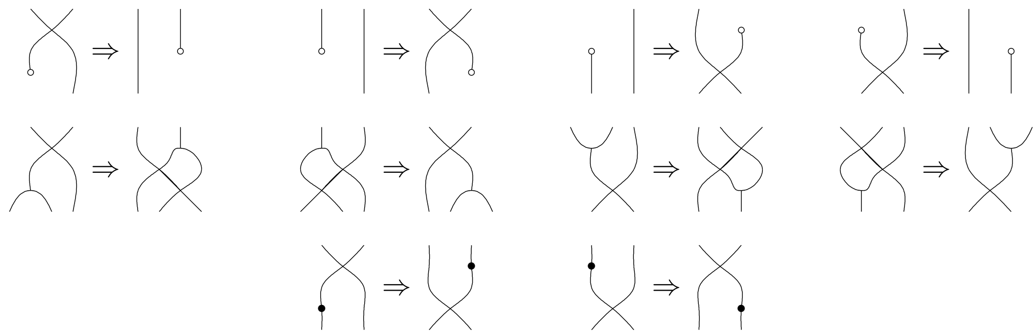
$$\eta : 0 \to 1 \qquad \mu : 2 \to 1 \qquad \varepsilon : 1 \to 0 \qquad \delta : 1 \to 2 \qquad \sigma : 1 \to 1 \qquad \gamma : 2 \to 2$$

and the rules in R are

$$\mu \circ (\eta \otimes \mathrm{id}_1) \Rightarrow \mathrm{id}_1 \qquad \mu \circ (\mathrm{id}_1 \otimes \eta) \Rightarrow \mathrm{id}_1 \qquad \mu \circ (\mu \otimes \mathrm{id}_1) \Rightarrow \mu \circ (\mathrm{id}_1 \otimes \mu) \qquad \mu \circ \gamma \Rightarrow \mu$$
$$(\varepsilon \otimes \mathrm{id}_1) \circ \delta \Rightarrow \mathrm{id}_1 \qquad (\mathrm{id}_1 \otimes \varepsilon) \circ \delta \Rightarrow \mathrm{id}_1 \qquad (\delta \otimes \mathrm{id}_1) \circ \delta \Rightarrow (\mathrm{id}_1 \otimes \delta) \circ \delta \qquad \gamma \circ \delta \Rightarrow \delta$$
$$\delta \circ \mu \Rightarrow (\mu \otimes \mu) \circ (\mathrm{id}_1 \otimes \gamma \otimes \mathrm{id}_1) \circ (\delta \otimes \delta) \qquad \varepsilon \circ \eta \Rightarrow \mathrm{id}_0 \qquad \mu \circ \delta \Rightarrow \mathrm{id}_1 \qquad \mu \circ (\mathrm{id}_1 \otimes \sigma) \circ \delta \Rightarrow \sigma$$
$$\gamma \circ \gamma \Rightarrow \mathrm{id}_2 \qquad (\gamma \otimes \mathrm{id}_1) \circ (\mathrm{id}_1 \otimes \gamma) \circ (\gamma \otimes \mathrm{id}_1) \Rightarrow (\mathrm{id}_1 \otimes \gamma) \circ (\gamma \otimes \mathrm{id}_1) \circ (\mathrm{id}_1 \otimes \gamma)$$
$$\gamma \circ (\eta \otimes \mathrm{id}_1) \Rightarrow \mathrm{id}_1 \otimes \eta \qquad \eta \otimes \mathrm{id}_1 \Rightarrow \gamma \circ (\mathrm{id}_1 \otimes \eta) \qquad \varepsilon \otimes \mathrm{id}_1 \Rightarrow (\mathrm{id}_1 \otimes \varepsilon) \circ \gamma \qquad (\varepsilon \otimes \mathrm{id}_1) \circ \gamma \Rightarrow \mathrm{id}_1 \otimes \varepsilon$$
$$\gamma \circ (\mu \otimes \mathrm{id}_1) \Rightarrow (\mathrm{id}_1 \otimes \mu) \circ (\gamma \otimes \mathrm{id}_1) \circ (\mathrm{id}_1 \otimes \gamma) \qquad (\mu \otimes \mathrm{id}_1) \circ (\mathrm{id}_1 \otimes \gamma) \circ (\gamma \otimes \mathrm{id}_1) \Rightarrow \gamma \circ (\mathrm{id}_1 \otimes \mu)$$
$$(\delta \otimes \mathrm{id}_1) \circ \gamma \Rightarrow (\mathrm{id}_1 \otimes \gamma) \circ (\gamma \otimes \mathrm{id}_1) \circ (\mathrm{id}_1 \otimes \delta) \qquad (\gamma \otimes \mathrm{id}_1) \circ (\mathrm{id}_1 \otimes \gamma) \circ (\delta \otimes \mathrm{id}_1) \Rightarrow (\mathrm{id}_1 \otimes \delta) \circ \gamma$$
$$\gamma \circ (\sigma \otimes \mathrm{id}_1) \Rightarrow (\mathrm{id}_1 \otimes \sigma) \circ \gamma \qquad (\sigma \otimes \mathrm{id}_1) \circ \gamma \Rightarrow \gamma \circ (\mathrm{id}_1 \otimes \sigma)$$

or graphically,





This rewriting system is not confluent and whether the rules can be oriented and completed in order to obtain a convergent rewriting system is still an open question, although some progress have been made in this direction [13] (Section 5.5). The main result of this article is the following theorem, and the rest of this article is devoted to its proof.

Theorem 28. *The category* **P** *is the PROP presented by the theory of poalgebras.*

4 Canonical factorizations

In this section, we further study the category **P**, and define a canonical way of writing a morphism in **P** as a composite from a simple family of morphisms. This will be used in Section 5 in order to show that canonical forms of the presentation are in bijection with morphisms of **P**. We have omitted the proofs which can generally be done by easy (but not necessarily short) induction.

The factorization of a morphism that we are going to introduce will not actually depend on the morphism only, but on a linearization of it, in the following sense: given a poset $E \in \mathbf{Pos}$, a *linearization* ℓ of E is a morphism $\ell : E \to [\vec{n}]$ in **Pos**, for some $n \in \mathbb{N}$, which is both epi and mono. While this definition is nice from a categorical point of view, it is sometimes cumbersome to work with. It is easy to see that a morphism $\ell : E \to [\vec{n}]$ is a linearization of E if and only if the underlying function $\ell : \underline{E} \to [n]$ is a bijection (and thus n is the cardinal of E). A linearization of E thus amounts to an enumeration $\underline{E} = \{x_0, \ldots, x_{n-1}\}$ of the events of E (writing $\ell^{-1}(i)$ as x_i), in way compatible with the partial order, and it turns out to be simpler to axiomatize the inverse function. We will thus use the following definition in the rest of the article:

Definition 29. A *linearization* of a poset E is a bijective function $x : [n] \to \underline{E}$ such that, for every $i, j \in [n]$ with $i < j$, we have $x(i) \not\geq x(j)$.

The following lemma can be shown by induction on the cardinal of finite posets. It is actually still true for non-finite posets if we assume the axiom of choice [20], but we will not need this here.

Lemma 30. *Every finite poset admits a linearization.*

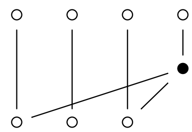
In order to characterize when two morphisms are linearizations of a given poset E, we define a relation $\sim$ on linearizations of E, as follows. Given two linearizations $x, x' : [n] \to \underline{E}$ of E, we have $x \sim x'$ when there exists $i \in [n-1]$ such that $x'(i) = x(i+1)$, $x'(i+1) = x(i)$ and $x'(j) = x(j)$ for $j \neq i$ and $j \neq i+1$, i.e. x and x' only differ by swapping two consecutive independent events; otherwise said, writing $\tau_i^n : [n] \to [n]$ for the transposition exchanging i and $i+1$ in the set $[n]$, we have $x' = x \circ \tau_i^n$.

Lemma 31. *Any two linearizations of a poset E are equivalent by the equivalence relation generated by* $\sim$.

In the following, by a *linearization of the internal events* of a morphism $(s,E,t) : m \to n$ in **P**, we mean a linearization of the poset $E \setminus (s([m]) \cup t([n]))$, or equivalently an injective function $[k] \to \underline{E}$ whose image is the set of internal events of E and which satisfies the condition of Definition 29.

We now introduce a notion of "canonical factorization" for morphisms of **P**, which will depend on a linearization of the internal events of the morphism. Given $n \in \mathbb{N}$ and $I \subseteq [n]$, we write $X^n_I : n \to n+1$ for the morphism with one internal node x, so that $\underline{X}^n_I = [n] \uplus \{x\} \uplus [n+1]$, with the canonical injections as source $s : [n] \to X^n_I$ and target $t : [n+1] \to X^n_I$, and whose partial order is such that

$$\forall i \in [n],\ s(i) < t(i) \qquad \text{and} \qquad \forall i \in I,\ s(i) < x \qquad \text{and} \qquad x < t(n)$$

Example 32. For instance, the morphism $X^3_{\{0,2\}} : 3 \to 4$ is  .

These morphisms can be used as basic "building blocks" for morphisms as follows.

Lemma 33. *Suppose given*

$$m,k,n \in \mathbb{N}, \qquad \textit{sets } I_j \subseteq [m+j] \textit{ indexed by } j \in [k], \qquad \textit{and a relation } R : m+k \to n. \tag{5}$$

The composite

$$R \circ X^{m+k-1}_{I_{k-1}} \circ \ldots \circ X^{m+1}_{I_1} \circ X^m_{I_0} \tag{6}$$

is the morphism $(s,E,t) : m \to n$ with k internal events, that we denote by $x(i)$ with $i \in [k]$, which is the smallest partial order such that $x(i) < x(j)$ when $m+i \in I_j$, $s(i) < x(j)$ when $i \in I_j$, $x(i) < t(j)$ when $(m+i, j) \in R$, and $s(i) < t(j)$ when $(i, j) \in R$. Moreover, the function $x : [k] \to \underline{E}$ thus defined is a linearization of the internal events of E.

The data (5) is called a *factorization* of the morphism (6), the morphism (6) is called the *composite* of the factorization (5), and the above linearization of the internal events is said to be *induced* by the factorization. From the preceding lemma, it is easy to deduce that any morphism of P can be put into this form:

Lemma 34. *Suppose given a morphism $(s,E,t) : m \to n$ of* **P** *with k internal events, together with a linearization $x : [k] \to \underline{E}$ of the internal events of E. We have*

$$E \quad = \quad R \circ X^{m+k-1}_{I_{k-1}} \circ \ldots \circ X^{m+1}_{I_1} \circ X^m_{I_0} \tag{7}$$

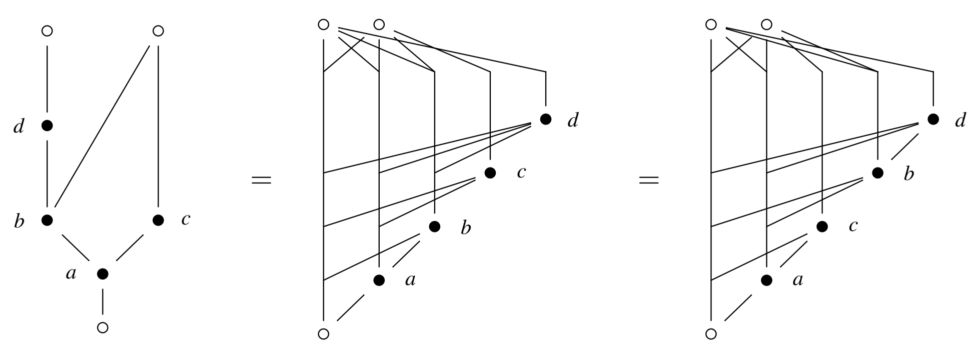
with

$$I_j \quad = \quad \{i \in [m] \mid s(i) < x(j)\} \cup \{m+i \in [m+k] \mid x(i) < x(j)\}$$

and $R : m+k \to n$ is the relation defined by

$$R \quad = \quad \{(i, j) \in [m] \times [n] \mid s(i) < t(j)\} \cup \{(m+i, j) \mid i \in [k], j \in [n], x(i) < t(j)\}$$

Example 35. The poset $E : 1 \to 2$ of the left of



admits the factorization

$$R \circ X^4_{\{0,1,2\}} \circ X^3_{\{0,1\}} \circ X^2_{\{0,1\}} \circ X^1_{\{0\}} \quad \text{with} \quad R = \{(0,0),(1,0),(2,0),(4,0),(0,1),(1,1),(2,1),(3,1)\}$$

as shown in the middle. Another possible factorization of the same morphism is depicted on the right. Because partial orders are transitive, by using Lemma 33 the factorizations (7) of morphisms of **P** provided by Lemma 34 can easily be shown to be transitive, in the following sense:

Definition 36. A factorization of a morphism of **P** of the form (7) is *transitive* when

– given $i \in [m+k]$ and $i', i'' \in [k]$ such that $i \in I_{i'}$ and $m+i' \in I_{i''}$, we have $i \in I_{i''}$,

– given $i \in [m+k]$, $i' \in [k]$ and $i'' \in [n]$ such that $i \in I_{i'}$ and $(m+i', i'') \in R$ we have $(i, i'') \in R$.

A transitive factorization of a morphism is called a *canonical factorization*.

Lemma 37. *The factorizations* (7) *of a morphism of* **P** *provided by Lemma 34 are transitive. Moreover, the factorization associated to the composite of a transitive factorization, with the induced linearization of the internal events, is itself.*

To sum up, we have just shown that the data (5) of a factorization (7) is a way to faithfully encode a morphism of **P** together with a linearization of its internal events:

Proposition 38. *Pairs* (E,x)*, constituted of a morphism* E *of* **P** *together with a linearization* x *of its internal events, are in bijection with factorizations of the form* (7) *which are transitive.*

Because a morphism might admit multiple linearizations of its internal events, a morphism generally admits multiple factorizations (7). However, we know by Lemma 31 that any two of them are related by $\sim$, and the following lemma characterizes how replacing a linearization by another one related by $\sim$ affects the factorization.

Lemma 39. *Suppose that* $x : [k] \to E$ *is a linearization of* $E : m \to n$ *and* $i \in [k]$ *is such that* $x' = x \circ \tau^k_i$ *is another linearization of* E*. Suppose that*

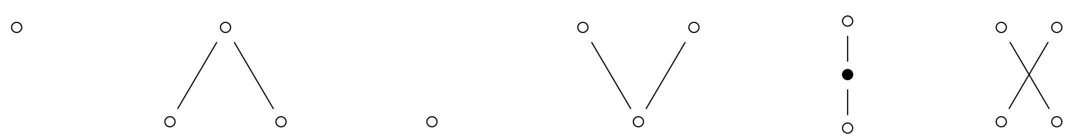
$$R \circ X^{m+k-1}_{I_{k-1}} \circ \ldots X^m_{I_0} \qquad \text{and} \qquad R' \circ X^{m+k-1}_{I'_{k-1}} \circ \ldots \circ X^m_{I'_0}$$

are the factorizations associated by Proposition 38 to (E,x) *and* (E,x') *respectively. Then we have* $I'_j = I_j$ *for* $0 \leqslant j < i+1$, $I'_j = \tau^{m+j}_{m+i}(I_j)$ *for* $i+1 \leqslant j < k$*, and* $R' = R \circ \tau^{m+k}_{m+i}$.

Example 40. The two factorizations of the morphism given in Example 35 only differ by a swapping of b and c. It can easily be checked that the relation in the second factorization can be deduced from the one of the first by precomposing by the transposition τ^5_2.

5 A presentation of the monoidal category of finite posets

As announced earlier, we use the general methodology explained at the end of Section 2 in order to show this. We first define an interpretation $[\![-]\!] : \Sigma^* \to \mathbf{P}$ by defining the interpretation of the generators, as per Remark 13. The object generator $1 \in \Sigma_1$ is interpreted as the object 1 of **P**, and the interpretations of the morphism generators are the following morphisms of **P**:



$[\![\eta]\!]$ $[\![\mu]\!]$ $[\![\varepsilon]\!]$ $[\![\delta]\!]$ $[\![\sigma]\!]$ $[\![\gamma]\!]$

Moreover, the interpretation leaves the rules invariant:

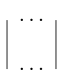
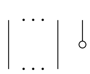
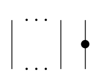
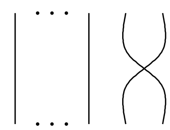
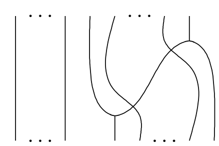
Lemma 41. *For any rule $\alpha \Rightarrow \beta$ of R, we have $[\![\alpha]\!] = [\![\beta]\!]$. Otherwise said,* $(1, [\![\eta]\!], [\![\mu]\!], [\![\varepsilon]\!], [\![\delta]\!], [\![\sigma]\!])$ *is a poalgebra in* $\mathbf{P}$.

We write $\mathbb{P}$ for the category $\mathbb{P} = (\Sigma^*/\equiv_R)$. As noticed in Remark 17, by the above lemma the interpretation functor $[\![-]\!] : \Sigma^* \to \mathbf{P}$ induces a quotient functor $[\![-]\!] : \mathbb{P} \to \mathbf{P}$, which is bijective on objects. Our aim in this section is to show that it is an isomorphism, i.e. $\mathbb{P} \cong \mathbf{P}$.

Given $n \in \mathbb{N}$ and $i \in [n]$, we define notations for the following morphisms in $\mathbb{P}$:

- $I^n = \mathrm{id}_{n+1} : n+1 \to n+1$
- $H^n = (\mathrm{id}_n \otimes \eta) : n \to n+1$
- $S^n = (\mathrm{id}_n \otimes \sigma) : n+1 \to n+1$
- $G^n = \mathrm{id}_n \otimes \gamma_{1,1} : n+2 \to n+2$
- $W_i^n = (\mathrm{id}_n \otimes \mu) \circ (\mathrm{id}_{i+1} \otimes \gamma_{1,n-i-1} \otimes \mathrm{id}_1) \circ (\mathrm{id}_i \otimes \delta \otimes \mathrm{id}_{n-i}) : n+1 \to n+1$

Graphically,

$I^n =$  $H^n =$  $S^n =$  $G^n =$  $W_i^n =$ 

Using the relations in R it is easy to show that

Lemma 42. *For every $n \in \mathbb{N}$, and $i, j \in [n]$, we have $W_j^n \circ W_i^n = W_i^n \circ W_j^n$ and $W_i^n \circ W_i^n = W_i^n$.*

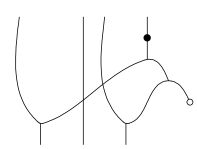
The above lemma implies that given $I = \{i_0, \ldots, i_{k-1}\} \subseteq [n]$, it makes sense to define the morphism

$$W_I^n \quad = \quad W_{i_{k-1}}^n \circ \ldots \circ W_{i_1}^n \circ W_{i_0}^n \quad : \quad n+1 \quad \to \quad n+1$$

with, by convention, $W_\emptyset^n = I^n$. We also define for $n \in \mathbb{N}$ and $I \subseteq [n]$, $X_I^n = S^n \circ W_I^n \circ H^n : n \to n+1$. The following lemma justifies that we use the same notation for this morphism of $\mathbb{P}$ and the morphism of $\mathbf{P}$ introduced in Section 4.

Lemma 43. *Given $n \in \mathbb{N}$ and $I \subseteq [n]$, we have $[\![X_I^n]\!] = X_I^n$.*

Example 44. For instance $X_{0,2}^3$ is as follows (notice the resemblance with Example 32):

$X_{0,2}^3 \quad =$ 

We say that a morphism R is a *relation* when it can be obtained by composing and tensoring generators not involving σ. The name is justified by Proposition 22 and the following lemma.

Lemma 45. *The canonical embedding* **Rel** $\to \mathbb{P}$ *is faithful.*

Proof. By definition of $\mathbb{P}$, this category contains a qualitative bicommutative bialgebra and therefore there is a canonical embedding functor **Rel** $\to \mathbb{P}$ by Proposition 22, whose image consists of morphisms which can be obtained by composing and tensoring generators not involving σ. Moreover, the property of "not containing a generator σ" in a morphism of $\mathbb{P}$ is preserved by the axioms of poalgebras and the axioms of poalgebras not involving σ are precisely the axioms of qualitative bicommutative bialgebras. □

Lemma 46. *The canonical functor* $[\![-]\!] : \mathbb{P} \to \mathbf{P}$ *is full.*

Proof. Suppose given a morphism $(s, E, t) : m \to n$. By Lemma 34, the morphism admits a factorization of the form (7). Since for any relation R in $\mathbb{P}$, $[\![R]\!]$ is the corresponding relation in $\mathbf{P}$, and we have $[\![X^n_I]\!] = X^n_I$, the morphism E is in the image of $[\![-]\!]$. □

Lemma 47. *Every morphism* $f : m \to n$ *of* $\mathbb{P}$ *can be factored as*

$$R \circ X^{m+k-1}_{I_{k-1}} \circ \ldots \circ X^{m+1}_{I_1} \circ X^m_{I_0} \tag{8}$$

for some $k \in \mathbb{N}$ *with* $I_j \subseteq [n+j]$ *for every* $j \in [k]$ *and* $R : m+k \to n$ *is a relation.*

Proof. We proceed by induction on the number of generators occurring in f. If f is the identity, it has no internal event, i.e. it is a relation as per Lemma 7; it is thus of the form (8) with $k = 0$. Otherwise, by Lemma 15, the morphism f can be factored as $f = f' \circ (\mathrm{id}_i \otimes \alpha \otimes \mathrm{id}_j)$ with $i, j \in \mathbb{N}$ and α a morphism generator. By induction hypothesis, we know that f' admits a factorization of the form (8), and we show that f then admits a factorization (8) by case analysis on i and α. □

We say that a morphism of the form (8) is *transitive* when it satisfies the conditions for transitivity of Definition 36.

Lemma 48. *Every morphism* $f : m \to n$ *of* $\mathbb{P}$ *admits a factorization of the form* (8) *which is transitive. Such a factorization is called* canonical.

Proof. Suppose given a morphism $f : m \to n$. By Lemma 47, we know that it admits a factorization of the form (8). This factorization can be shown to be equal to a transitive one by progressively adding "transitivity edges" (in the sense of Definition 36), by using the fact that the $\mathbb{P}$ contains a poalgebra, which in particular satisfies the transitivity axiom (4). □

Similarly to Section 4 (see Lemma 39), the canonical factorizations are not unique but one can go from one to another by "switching" two consecutive X^n_I in the factorization, which will result in precomposing the relation of the factorization by a transposition. The following lemmas formalize this.

Lemma 49. *Given* $n \in \mathbb{N}$*, and* $I, J \subseteq [n]$*, we have* $X^{n+1}_J \circ X^n_I = G^n \circ X^{n+1}_I \circ X^n_J$.

Proof. With the notations $\overline{H}^n = H^n \otimes \mathrm{id}_1$, $\overline{S}^n = S^n \otimes \mathrm{id}_1$, $\overline{W}^n_I = W^n_I \otimes \mathrm{id}_1$, we have

$$\begin{aligned}
X^{n+1}_J X^n_I &= S^{n+1} W^{n+1}_J H^{n+1} S^n W^n_I H^n & &= \overline{S}^n \overline{W}^n_I S^{n+1} W^{n+1}_J \overline{H}^n H^n \\
&= S^{n+1} W^{n+1}_J \overline{S}^n H^{n+1} W^n_I H^n & &= \overline{S}^n \overline{W}^n_I S^{n+1} \overline{H}^n W^n_J H^n \\
&= S^{n+1} W^{n+1}_J \overline{S}^n \overline{W}^n_I H^{n+1} H^n & &= \overline{S}^n \overline{W}^n_I S^{n+1} \overline{H}^n W^n_J H^n \\
&= S^{n+1} W^{n+1}_J \overline{S}^n \overline{W}^n_I \overline{H}^n H^n & &= \overline{S}^n \overline{W}^n_I \overline{H}^n S^n W^n_J H^n \\
&= S^{n+1} \overline{S}^n W^{n+1}_J \overline{W}^n_I \overline{H}^n H^n & &= \overline{S}^n \overline{W}^n_I G^n H^{n+1} S^n W^n_J H^n \\
&= \overline{S}^n S^{n+1} W^{n+1}_J \overline{W}^n_I \overline{H}^n H^n & &= \overline{S}^n G^n W^{n+1}_I H^{n+1} S^n W^n_J H^n \\
&= \overline{S}^n S^{n+1} \overline{W}^n_I W^{n+1}_J \overline{H}^n H^n & &= G^n X^{n+1}_I X^n_J
\end{aligned}$$

Each of the equalities involved are direct to show by using the definition of the morphisms and the axioms symmetric monoidal categories. For instance, one can show that $H^{n+1} S^n = \overline{S}^n H^{n+1}$ using the exchange law of monoidal categories: $H^{n+1} S^n = (n+1 \otimes \eta) \circ (n \otimes \sigma) = (n \otimes \sigma \otimes 1) \circ (n \otimes \eta) = \overline{S}^n H^{n+1}$. Other steps are similar. □

Lemma 50. *We write $G_i^n = i \otimes \gamma \otimes (n-i)$ (so that $G^n = G_n^n$). Given $n \in \mathbb{N}$, $i \in [n+1]$ and $I \subseteq [n+1]$, we have $X_I^{n+2} \circ G_i^n = G_i^n \circ X_{\tau_i^{n+1}(I)}^{n+1}$.*

As a direct corollary of the two previous lemmas, we have

Lemma 51. *Given $k \in \mathbb{N}$, sets $I_j \subseteq [n+j]$ indexed by $j \in [k]$, a relation $R : m+k \to n$, and $i \in [k-1]$ such that $m+i \notin I_{i+1}$, we have $R \circ X_{I_{k-1}}^{m+k-1} \circ \ldots \circ X_{I_0}^m = R' \circ X_{I'_{k-1}}^{m+k-1} \circ \ldots \circ X_{I'_0}^m$ with $I'_j = I_j$ for $0 \leqslant j < i+1$, $I'_j = \tau_{m+i}^{m+j}(I_j)$ for $i+1 \leqslant j < k$, and $R' = R \circ \tau_{m+i}^{m+k}$.*

Finally, we are now able to show that the axioms of poalgebras are a presentation of the category **P**.

Theorem 52. *The categories $\mathbb{P}$ and* **P** *are isomorphic as symmetric monoidal categories: the category* **P** *is presented by the theory of poalgebras.*

Proof. The canonical functor $[\![-]\!] : \mathbb{P} \to \mathbf{P}$ is bijective on objects and was shown to be full in Lemma 46. In order to conclude, we have to show that it is faithful. Suppose given two morphisms $f, g : m \to n$ of $\mathbb{P}$, such that $[\![f]\!] = [\![g]\!]$. By Lemma 48, we can suppose that they are written as a canonical factorization, and their images by $[\![-]\!]$ are thus canonical factorizations in **P**, in the sense of Definition 36. They therefore correspond to two linearizations of the same poset. By Lemma 31, we can transform one permutation into the other by a series of commutations of consecutive elements, each one corresponding to transforming the canonical form in **P** in the way described in Lemma 39. We write $[\![f]\!] = E_1 = E_2 = \ldots = E_n = [\![g]\!]$ for this sequence of canonical forms. By Lemma 51, we can mimic these transformations on f, obtaining a sequence of canonical factorizations $f = f_1 = f_2 = \ldots = f_n = g$ in $\mathbb{P}$ such that for every i we have $[\![f_i]\!] = E_i$. Therefore $f = g$ and the functor $[\![-]\!]$ is faithful. □

6 Conclusion

We have shown that the monoidal category **P** of partial orders, is the theory of poalgebras. This monoidal category contains as subcategories various well-known categories such as the simplicial category Δ, the category **F** of functions, the category **Rel** of relations, etc. and the presentations of those categories (see Section 3) can thus be recovered as particular cases of our result.

Many interesting variants of this result can be obtained along the same lines. For instance, by adding the axioms $\varepsilon \circ \sigma = \varepsilon$ and $\sigma \circ \eta = \eta$ to the theory of poalgebra, we obtain a presentation of the subcategory of **P** whose morphisms are of the form (s, E, t) with s and t surjective. Similarly, by removing the transitivity axiom (4) we can present a variant of the category where morphisms are partial orders which are not supposed to be transitive, which can be seen as directed acyclic graph, and thus recover the result of Fiore and Devesas Campos [5]. A variant of this theory, where the morphism σ has been replaced by a family of generators with various number of inputs and outputs, has also been considered in [2] in order to provide an axiomatic definition of the notion of "network".

This work should be considered as a first step and can be generalized in various promising ways. This presentation can be extended from posets to *event structures* [22], which are partial orders equipped with a notion of incompatibility, thus allowing us to hope for algebraic methods in order to study concurrent processes, which those structures naturally model. Another very interesting direction consists in exploring higher-dimensional structures: for instance, we would like to extend this work to construct a presentation of the monoidal bicategory of integers, partial orders and increasing functions. In particular the full monoidal subbicategory whose 1-cells are forests (i.e. partial orders such that two incomparable elements do not have an upper bound) is expected to be presented by (a variant of) the theory of commutative strong monads, and to be closely related to Moerdijk's dendroidal sets [16].

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