COMPLEX PARITIES

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General remarks

- This was based on Street's article *Parity complexes*. Some definitions differ though, so blame me for mistakes!
- In the end, it turns out to be closer to Johson's *pasting schemes*.
- It can be also seen as a "set-theoretic version" of Steiner's augmented directed complexes.
- I had actually started implementing those as a variant of Steiner's ADC and Dimitri Ara recalled them to me.
- I tend to think while implementing, so not everything is proved here. . .
- . . . which saved me time since it turns out that most of my ideas were already thought of by other people.

http://www.lix.polytechnique.fr/Labo/Samuel.Mimram/rewr/

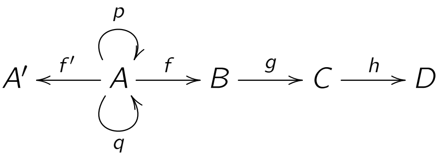
Diagrams

We have various formalizations of diagrams:

- *parity complexes* (Street)
- *pasting schemes* (Johnson)
- *pasting schemes* (Power)
- *augmented directed complexes* (Steiner)

Quizz

Consider the free category on the graph



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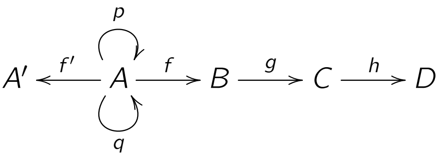

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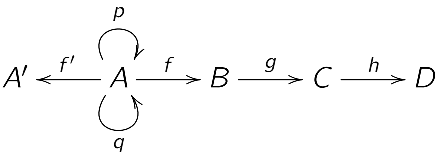

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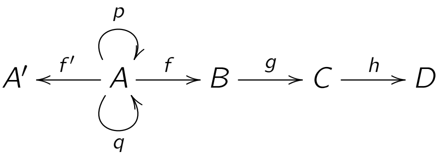

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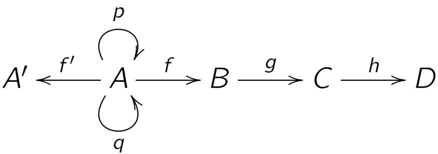

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- What is the morphism which has $\{f, f'\}$ as generators?
- What is the morphism which has $\{f, h\}$ as generators?

PARITY COMPLEXES

(my own version)

We first define our *signatures*:

Definition

A **pre-parity complex** S is a graded set

$$S = \coprod_{n\in\mathbb{N}} S_n$$

together with, for every $x \in S_{n+1}$ subsets

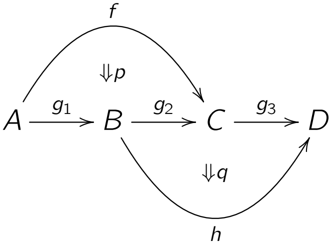
$$x^-, x^+ \subseteq S_n$$

An element $x \in S_n$ is called a *generator* of *dimension n*.

A **parity complex** is a pre-parity complex satisfying
suitable conditions.

Example

Consider the polygraph

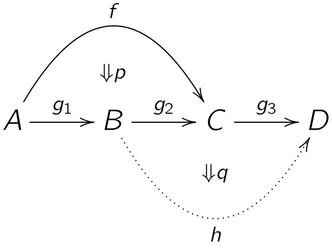


The corresponding parity complex is

$$S_0 = \{A, B, C, D\} \qquad f^- = \{A\} \qquad p^- = \{f\}$$
$$S_1 = \{f, g_i, h\} \qquad f^+ = \{C\} \qquad p^+ = \{g_1, g_2\}$$
$$S_2 = \{p, q\} \qquad \dots \qquad \dots$$

General idea

We are going to see cells as subsets of the signature, for instance the cell corresponding to $p *_0 g_3$ is



What conditions on signatures and cells allow us to ensure that the resulting ω-category is free on the generators?

Axioms

We should first exclude “trivial loops”:

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For every generator $x \in S_{n+1}$, $x^- \neq \emptyset$ and $x^+ \neq \emptyset$.

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For every generator $x \in S_n$, $x^- \cap x^+ = \emptyset$.

A preorder on generators

Given $x, y \in S_n$, we write

$$x \triangleleft y$$

when

$$x^+ \cap y^- \quad \neq \quad \emptyset$$

and take the reflexive and transitive closure.

A preorder on generators

Given $x, y \in S_n$, we write

$$x \lhd y$$

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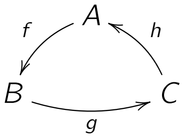
Axiom (acyclicity)

The preorder $\lhd$ should be acyclic.



Remark

Previous axioms ensure precisely that there is no non-trivial way for which $x \lhd x$.

Downward closure

Given $C \subseteq S$, we write $\downarrow C$ for its *downward closure*, i.e. the smallest graded set such that

$$x \in C_{n+1} \qquad \text{implies} \qquad x^- \cup x^+ \subseteq C_n$$

Well-formedness

Axiom

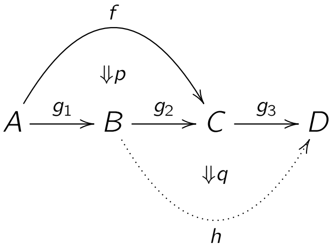
For every $x \in C_n$, $\downarrow x^-$ and $\downarrow x^+$ should be cells satisfying the globular identities:

$$\partial^-(\downarrow x^-) = \partial^-(\downarrow x^+) \qquad \partial^+(\downarrow x^-) = \partial^+(\downarrow x^+)$$

...for a good notion of **cell** and its **source** and **target**.

Definition

A **pre-cell** of dimension n consists of finite sets $(C_i)_{0 \leq i \leq n}$ with $C_i \subseteq S_i$.

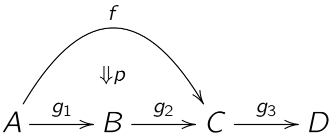


A **cell** is a pre-cell satisfying suitable conditions.

Source and target

Given a pre-cell C of dimension n, we define its **source** as the pre-cell obtained by

- removing generators in C_n,
- removing the downward closure of their targets,
- taking the downward closure,
- removing $C_n = \emptyset$.



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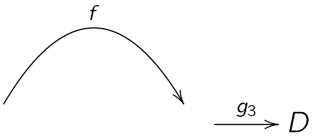

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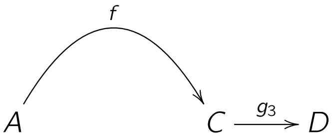

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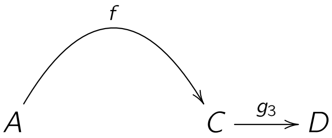

Given $X \subseteq S_n$, we write

$$d^+(X) \quad = \quad X \cup \bigcup_{x \in X} \partial^+(\downarrow x)$$

so that

$$\partial^-(C) = \downarrow \left(C \setminus d^+(C_n) \right)$$

Characterizing cells

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$$B' \xleftarrow{f'} A \xrightarrow{f} B$$

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4. the relation on C_0 such that $x \approx x'$ whenever there exists $y \in C_1$ with $\partial^-(y) = \{x\}$ and $\partial^+(y) = \{x'\}$ should be the full one (NB: this axiom will turn out to be superfluous)

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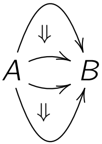
Remark

In Steiner's world, 2. is roughly *unital basis* and acyclicity is *loop-free*.

This is not enough!

The following is not excluded by our axioms:



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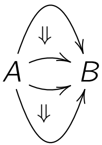
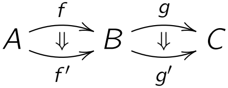


We cannot require the axiom 4. on 2-generators too:



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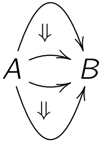

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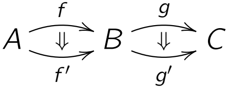

We could say that given two 2-generators, one should be either below or on the left of the other.

This suggests considering the relation $\blacktriangleleft$ on all generators of C generated by $x \blacktriangleleft y$ whenever

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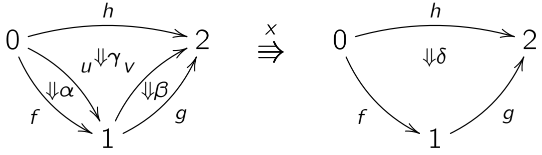
Problem (Pratt)

The relation can have cycles:



We have:

$$f \blacktriangleleft 1 \blacktriangleleft v \blacktriangleleft \beta \blacktriangleleft x \blacktriangleleft \delta \blacktriangleleft f$$

Global order

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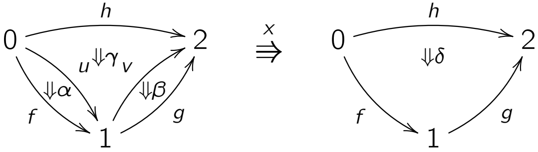

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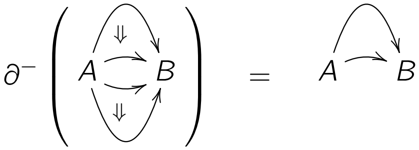
Those for which $\blacktriangleleft$ is acyclic (for Steiner: *strongly acyclic*) are very nice, but this restriction is clearly too strong.

Note that the source of our pre-cell is



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So we also require our cells to have cells as boundaries.

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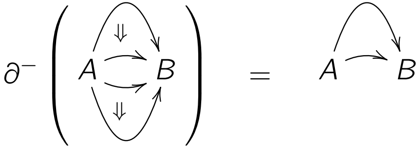

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So we also require our cells to have cells as boundaries.

This also handles our connexity condition:

$$\partial^- \left(A \xrightarrow{f} B \qquad A' \xrightarrow{f'} B' \right) \quad = \quad A \qquad\qquad A'$$

To sum up

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A **cell** of dimension n is a pre-cell which

- is closed under faces: $C = \downarrow C$,
- has non-conflicting n-generators: $x^\epsilon \cap y^\epsilon = \emptyset$ for $x \in C_n$
- is a singleton if $n = 0$
- is s.t. $\partial^\epsilon(C)$ is a cell
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There might be simpler (to check) axioms, but it should be enough for now.

Canonicity of the definition

I do not think that those restrictions are “canonical”. In particular, I believe that we could also have the same results if we restricted to **opetopes**, with

- x^- arbitrary (including empty!)
- x^+ a singleton

Composition

We can make an ω-category with n-cells as n-cells, where

- composition is given by (graded) union,
- identities amount to add $C_{n+1} = \emptyset$,
- the axioms are satisfied (to be checked...)

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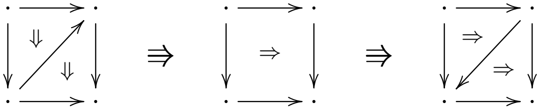
Conjecture

This ω-category is freely generated by cells of the form $\downarrow x$ for some generator $x \in S$.

(Street has actually proved this, but axioms here differ from his)
(in fact, we need a bit more see next slide)

Acyclicity and composition

It was noticed by Power that acyclicity is not preserved by composition:



(note the loop in the diagonal in the underlying graph)

DECOMPOSING CELLS

Decomposition

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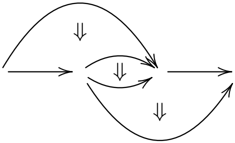
$$f_C \quad = \quad x$$

- If $C = (C_0, \ldots, C_n)$ with $C_n = \emptyset$ it is the identity

$$f_C \quad = \quad \mathrm{id}\left(f_{(C_0,\ldots,C_{n-1})}\right)$$

Decomposition

If there are multiple top-dimensional cells, we are going to split the cell in slices:



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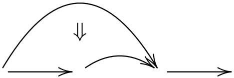
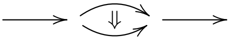
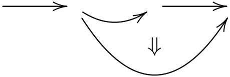
 $*_1$  $*_1$ 

Decomposition

- If $C_n = \{x_1, \ldots, x_p\}$ with $p > 1$ and $x_1 \triangleleft \ldots \triangleleft x_p$. For each $1 \leq i \leq p$, we write

$$C^i = \downarrow\left(C \setminus \left(d^-\left(\{x_1, \ldots, x_{i-1}\}\right) \cup d^+\left(\{x_{i+1}, \ldots, x_n\}\right)\right)\right)$$

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which is such that $C^i_n = \{x_i\}$ and we have

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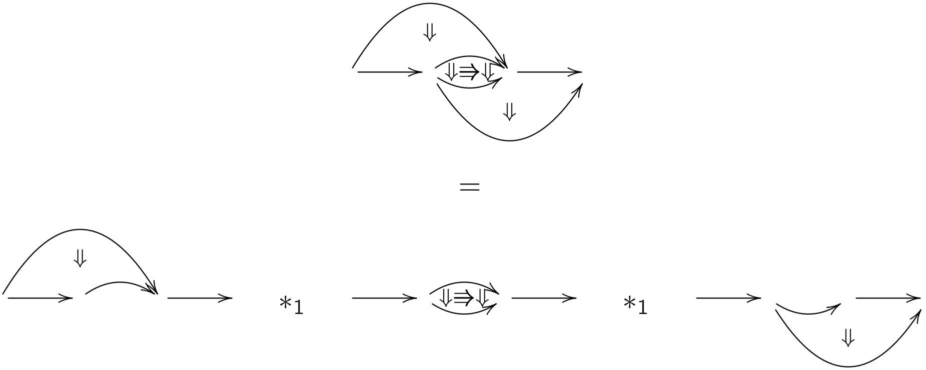
  $$f_C \quad = \quad f_{C^1} *_{n-1} f_{C^2} *_{n-1} \ldots *_{n-1} f_{C^n}$$

  Note that each C^i has only one top-dimensional generator.

Decomposition

If there is only one top-dimensional cell, we remove one layer of whiskers:



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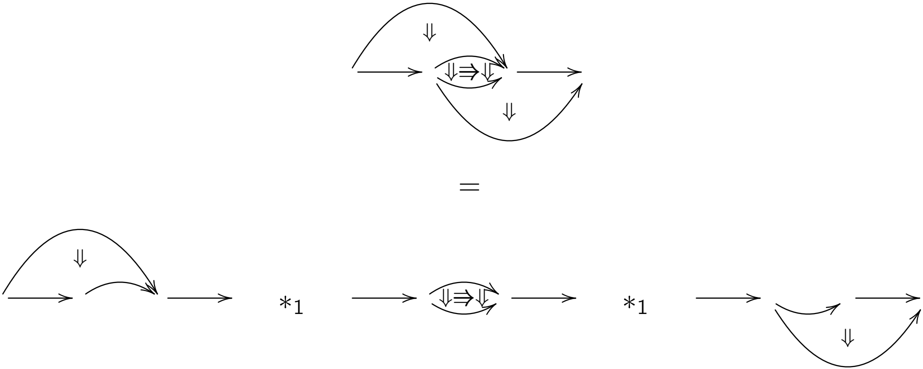

Note that the morphism in the middle has lower-dimensional whiskers.

Decomposition

- If $C_n = \{x\}$ then

$$C = (C_0, \ldots, C_k, \emptyset, \ldots, \emptyset, \{x\})$$

with k minimal such.

Decomposition

- If $C_n = \{x\}$ then

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with k minimal such. We write $x_k = (\downarrow x)_k$,

$$C'_k = C_k \setminus x_k \qquad \text{and} \qquad C'_k = C_k^- \sqcup C_k^+$$

where

  - $y \in C_k^-$ if $y \lhd z$ for some $z \in x_k$ and
  - $y \in C_k^+$ if $\neg(y \lhd z)$ for every $z \in x_k$.

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We write

$$\begin{aligned} C^- &= \downarrow \left(C \setminus \left(\{x\} \cup d^-(C_k^-) \right) \right) \\ C^x &= \downarrow \left(C \setminus \left(d^- \left(C_k^- \right) \cup d^+ \left(C_k^+ \right) \right) \right) \end{aligned}$$

and C^+ similar to C^-,

Decomposition

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with k minimal such. We write $x_k = (\downarrow x)_k$,

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and C^+ similar to C^-, and

$$f_C = f_{C^-} *_{k-1} f_{C^x} *_{k-1} f_{C^+}$$

EXAMPLES OF PARITY COMPLEXES

Globes

The n-**globe** G_n is the signature with

$$
\begin{array}{lll}
S_i = \{x_i, y_i\} & x_{i+1}^- = y_{i+1}^- = x_i & z^- = x_{n-1} \\
S_n = \{z\} & x_{i+1}^+ = y_{i+1}^+ = y_i & z^+ = y_{n-1}
\end{array}
$$

In low dimensions:

$$
\begin{array}{cccc}
z & x_0 \xrightarrow{z} y_0 & x_0 \underset{y_1}{\overset{x_1}{\rightrightarrows}} y_0 \quad (\Downarrow z) & \cdots \\
0 & 1 & 2 & 3
\end{array}
$$

Let's see that in rewr!

`http://www.lix.polytechnique.fr/Labo/Samuel.Mimram/rewr/`

Globes

```
# globe 4
# sigcat
x0
y0

x1 : x0 -> y0
y1 : x0 -> y0

x2 : x1 -> y1
y2 : x1 -> y1

x3 : x2 -> y2
y3 : x2 -> y2

z : x3 -> y3
```

Orientals

The *n*-**simplex** Δ_n has, as generators of dimension n, increasing sequences of integers

$$x \quad = \quad 0 \leq x_0 < x_1 < \ldots < x_n \leq n+1$$

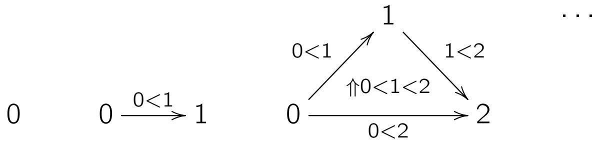
writing $\partial_i(x)$ for the sequence with the i-th element removed, we have

$$\begin{aligned} x^- &= \{\partial_{2i+1}(x)\} \\ x^+ &= \{\partial_{2i}(x)\} \end{aligned}$$

We recover the usual formulas for orientals!

Orientals

In low dimensions, we have



0 1 2 3

Orientals

```
# simplex 3
# sigcat
01 : 0 -> 1
02 : 0 -> 2
03 : 0 -> 3
12 : 1 -> 2
13 : 1 -> 3
23 : 2 -> 3
```

012 : 02 -> 01 $*_0$ 12
013 : 03 -> 01 $*_0$ 13
023 : 03 -> 02 $*_0$ 23
123 : 13 -> 12 $*_0$ 23

0123 : 023 $*_1$ (012 $*_0$ 23) -> 013 $*_1$ (01 $*_0$ 123)

Chain complexes

A parity complex S induces a chain complex $\mathbb{Z}S$ of free abelian groups

$$\dots \xrightarrow{\partial_2} \mathbb{Z}S_2 \xrightarrow{\partial_1} \mathbb{Z}S_1 \xrightarrow{\partial_0} \mathbb{Z}S_0$$

with $\partial_i(x) = \sum x^+ - \sum x^-$.

Chain complexes

A parity complex S induces a chain complex $\mathbb{Z}S$ of free abelian groups

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with $\partial_i(x) = \sum x^+ - \sum x^-$.

Can we define a tensor product satisfying the following?

$$\mathbb{Z}(S \otimes T) \quad = \quad \mathbb{Z}S \otimes \mathbb{Z}T$$

Products

The **tensor product** of two signatures S and T is defined

$$(S \otimes T)_n \quad = \quad \coprod_{i+j=n} S_i \times T_j$$

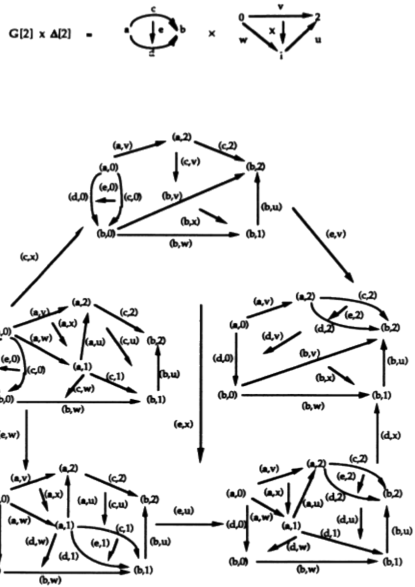
with faces

$$(x, y)^- \quad = \quad (x^-, y^{\sigma(x)})$$

where $\sigma(x) = -$ iff $\dim(x)$ is even and similarly for targets.

Products

R. STREET. PARITY COMPLEXES



Products

```
# street_gs 2
# sigcat
...
x0|012 : x0|02 -> x0|01 *_0 x0|12
y0|012 : y0|02 -> y0|01 *_0 y0|12
x1|01 : x0|01 *_0 x1|1 -> x1|0 *_0 y0|01
x1|02 : x0|02 *_0 x1|2 -> x1|0 *_0 y0|02
...

x1|012 : (x0|012 *_0 x1|2) *_1 (x0|01 *_0 x1|12) *_1 (x1|01 *_0
y1|012 : (x0|012 *_0 y1|2) *_1 (x0|01 *_0 y1|12) *_1 (y1|01 *_0
z|01 : x1|01 *_1 (z|0 *_0 y0|01) -> (x0|01 *_0 z|1) *_1 y1|01
z|02 : x1|02 *_1 (z|0 *_0 y0|02) -> (x0|02 *_0 z|2) *_1 y1|02
z|12 : x1|12 *_1 (z|1 *_0 y0|12) -> (x0|12 *_0 z|2) *_1 y1|12

z|012 : (x1|012 *_1 (z|0 *_0 y0|01 *_0 y0|12)) *_2 (z|02 *_1 (y1
```

Remark

Street notes that the tensor product of signatures shall not necessarily be so, because the resulting $\triangleleft$ might not be acyclic.

Can someone come up with an explicit example?

Cubes

The ***n*-cube** is $I^{\otimes n}$ where I is the standard interval (the 1-globe or the 1-simplex):

$$I \quad = \quad - \xrightarrow{0} +$$

For instance, the 2-cube is

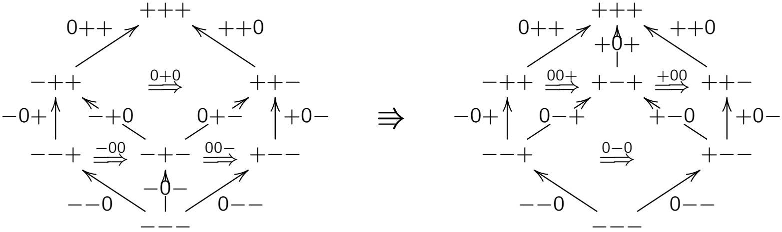
$$\begin{array}{ccc} -- & \xrightarrow{-0} & +- \\ {\scriptstyle 0-}\downarrow & \Downarrow 00 & \downarrow{\scriptstyle 0+} \\ -+ & \xrightarrow[+0]{} & ++ \end{array}$$

Cubes

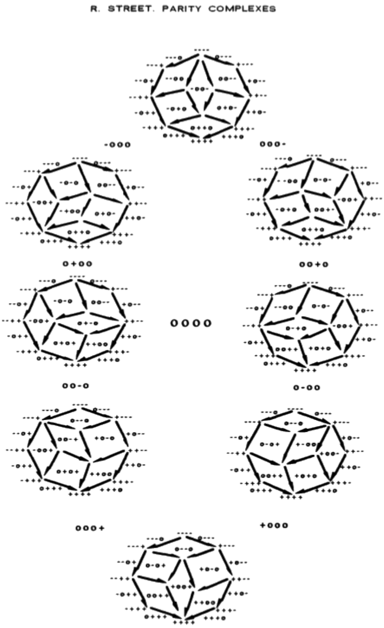
The n**-cube** is $I^{\otimes n}$ where I is the standard interval (the 1-globe or the 1-simplex):

$$I \quad = \quad - \xrightarrow{0} +$$

For instance, the 3-cube is



Cubes


R. STREET. PARITY COMPLEXES
-000
000-
0+00
00+0
0000
00-0
0-00
000+
+000

Cubes

```
# cube 3
# sigcat
...

-00 : --0 *₀ -0+ -> -0- *₀ -+0
```

$\texttt{-00 : --0} *_0 \texttt{-0+ -> -0-} *_0 \texttt{-+0}$

$\texttt{+00 : +-0} *_0 \texttt{+0+ -> +0-} *_0 \texttt{++0}$

$\texttt{0-0 : --0} *_0 \texttt{0-+ -> 0--} *_0 \texttt{+-0}$

$\texttt{0+0 : -+0} *_0 \texttt{0++ -> 0+-} *_0 \texttt{++0}$

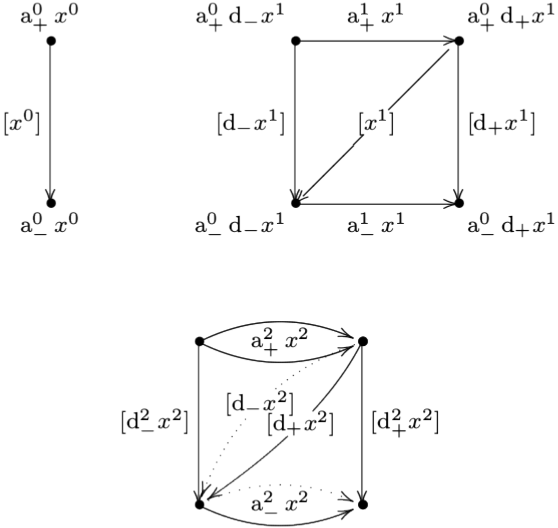
$\texttt{00- : -0-} *_0 \texttt{0+- -> 0--} *_0 \texttt{+0-}$

$\texttt{00+ : -0+} *_0 \texttt{0++ -> 0-+} *_0 \texttt{+0+}$

$\texttt{000 : (-00} *_0 \texttt{0++)} *_1 \texttt{(-0-} *_0 \texttt{0+0)} *_1 \texttt{(00-} *_0 \texttt{++0) -> (--0}$

Cylinders

The n-**cylinder** is $I \otimes G_n$. We recover François' formulas:



Cylinders

```
# globe 1 2
# sigcat
...
x0⁻|y1⁻ : x0⁻|y0⁻ -> x0⁻|y0⁺
x0⁻|y1⁺ : x0⁻|y0⁻ -> x0⁻|y0⁺
x0⁺|y1⁻ : x0⁺|y0⁻ -> x0⁺|y0⁺
x0⁺|y1⁺ : x0⁺|y0⁻ -> x0⁺|y0⁺
x|y0⁻ : x0⁻|y0⁻ -> x0⁺|y0⁻
x|y0⁺ : x0⁻|y0⁺ -> x0⁺|y0⁺

x0⁻|y : x0⁻|y1⁻ -> x0⁻|y1⁺
x0⁺|y : x0⁺|y1⁻ -> x0⁺|y1⁺
x|y1⁻ : x0⁻|y1⁻ *0 x|y0⁺ -> x|y0⁻ *0 x0⁺|y1⁻
x|y1⁺ : x0⁻|y1⁺ *0 x|y0⁺ -> x|y0⁻ *0 x0⁺|y1⁺

x|y : (x0⁻|y *0 x|y0⁺) *1 x|y1⁺
    -> x|y1⁻ *1 (x|y0⁻ *0 x0⁺|y)
```

Tensor products of globes

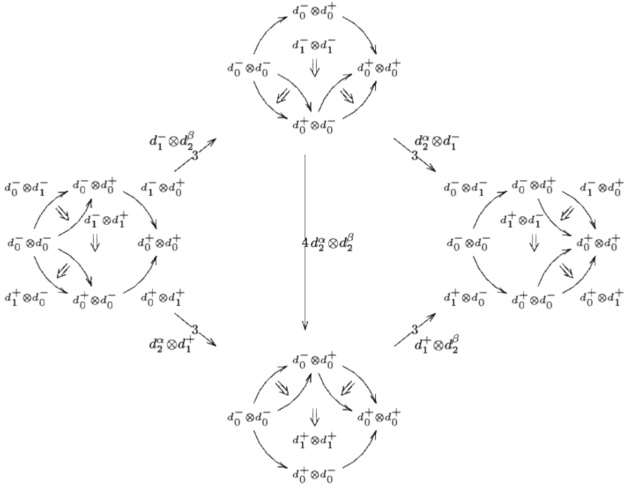
```
# globe 2 2
# sigcat
...

x1^-|y : (x0^-|y *_0 x1^-|y0^+) *_1 x1^-|y1^+ -> x1^-|y1^- *_1 (x1^-|y0^-
x1^+|y : (x0^-|y *_0 x1^+|y0^+) *_1 x1^+|y1^+ -> x1^+|y1^- *_1 (x1^+|y0^-
x|y1^- : x1^-|y1^- *_1 (x|y0^- *_0 x0^+|y1^-) -> (x0^-|y1^- *_0 x|y0^+) *
x|y1^+ : x1^-|y1^+ *_1 (x|y0^- *_0 x0^+|y1^+) -> (x0^-|y1^+ *_0 x|y0^+) *

x|y :     (x1^-|y *_1 (x|y0^- *_0 x0^+|y1^+))
      *_2 (x|y1^- *_1 (x1^+|y0^- *_0 x0^+|y))
      -> ((x0^-|y *_0 x1^-|y0^+) *_1 x|y1^+)
      *_2 ((x0^-|y1^- *_0 x|y0^+) *_1 x1^+|y)
```

Tensor products of globes

Crans in *Pasting schemes for the monoidal biclosed structure on ω-Cat* also manages to extract the formulas for the tensor product of globes from those of (degenerated) cubes:



Join

The **join** is defined by

$$(S \bullet T)_n \quad = \quad S_n + \sum_{i+j+1=n} S_i \times T_j + T_n$$

with

- if i odd

$$(xy)^- = x^- y \cup xy^- \qquad (xy)^+ = x^+ y \cup xy^+$$

- if i even

$$(xy)^- = x^- y \cup xy^+ \qquad (xy)^+ = x^+ y \cup xy^-$$

DES SIGNES DE BON GOÛT

DES SIGNES DE BON GOÛT

$$\partial(f) \quad = \quad \sum_i \pm\partial_i(f)$$

Desuspension

Consider the simplicial category Δ as a 2-category with $\star$ as 0-cell. We can present it

- as a 2-category:
  $\mu : 2 \to 1$, $\eta : 0 \to 1$, ...
- or as the category $\Delta(\star, \star)$:
  $\mu_i^n : n+1 \to n$, $\eta_i^n : n \to n+1$, ...
  (in particular, to have a convergent presentation, we now have to "orient exchange rules")

We will play the same game with our simple examples.

The associahedron

- The associahedron is the polytope generated by the critical n-uple of the rewriting system $m(m(x,y),z) \Rightarrow m(x,m(y,z))$. Can we come up with a direct computation of the faces?

The associahedron

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- It can also be seen as the hom-n-category

$$\Delta^{\mathrm{op}}_{n+2}((0<1) *_0 (1<2) *_0 \ldots *_0 (n-1<n), (0<n))$$

of the $(n+2)$-simplex.

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  of the $(n+2)$-simplex.
- The operad Ass is the terminal non-unital operad: it is the theory of an associative binary operation. An A_∞ algebra is an algebra over a resolution of this operad, and Stasheff seems to have computed it all for us!

The associahedron

Definition

An A_∞-algebra consists of a graded vector space A together with n-ary operations $m_n : A^{\otimes n} \to A$ of degree $n-2$ satsifying

$$\sum_{p+q+r=n} (-1)^{p+qr} m_{p+1+r} \circ (\mathrm{id}_p \otimes m_q \otimes \mathrm{id}_r)$$

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In particular $d = -m_1$ is a differential and the induced derivative is

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and explicitly

$$\partial(m_n) \quad = \quad \sum_{\substack{p+q+r=n \\ p+1+r \geq 2 \\ q \geq 2}} (-1)^{p+qr} m_{p+1+r} \circ (\mathrm{id}_p \otimes m_q \otimes \mathrm{id}_r)$$

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e.g. $\qquad \partial(m_3) \quad = \quad m_2 \circ (m_2 \otimes \mathrm{id}) - m_2 \circ (\mathrm{id} \otimes m_2)$

The associahedron

We expect the n-associahedron to be defined as the complex whose

- elements are trees with $n + 2$ leaves and nodes of arity ≥ 2
- the faces of a tree are obtained by splitting a node into two
- the degree of a tree is the sum of arities - 2 of nodes
- the signs are given by the above formula

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- the degree of a tree is the sum of arities - 2 of nodes
- the signs are given by the above formula

. . . excepting that it does not work . . .

The associahedron

```
# associahedron 2
# sigcat
(((..).).)
((.(..)).)
(.((..).))
(.(.(..)))
((..)(..))

((...).) : ((.(..)).) -> (((..).).)
(.(...)) : (.(.(..))) -> (.((..).))
((..)..) : ((..)(..)) -> (((..).).)
(.(..).) : (.((..).)) -> ((.(..)).)
(..(..)) : (.(.(..))) -> ((..)(..))
```

(....) : (.(...)) $*_0$ (.(..).) $*_0$ ((...).) -> (..(..)) $*_0$ ((

The associahedron

```
# associahedron 3
((((..).).).) , (((.(..)).).) , ((.((..).)).) , ((.(.(..)))
(((...).).) : (((.(..)).).) -> ((((..).).).) , ((.(...)).)
((....).) : (((...).).),((.(...)).),((.(..).).) -> (((..).
(.....) : (.(....)),(.(..)..),(...(..)) -> ((....).),((...)
# check
Error: Invalid signature (cylic).
```

The associahedron

Remark

Can we also describe the following categories?

- $\Delta_n(0, n)$
- $\Delta_n(\text{left comb}, \text{right comb})$
- etc.

Answer to the first question

Kapranov and Voevodsky note

$$\Delta_n(0, n) \quad = \quad I^{\otimes(n-1)}/\sim$$

(some squares corresponding to exchanges become equalities)

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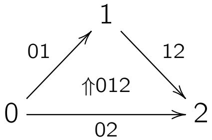
For instance:



$$02 \xrightarrow{012} 01 *_0 12$$

Answer to the first question

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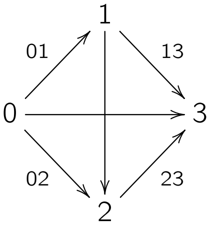
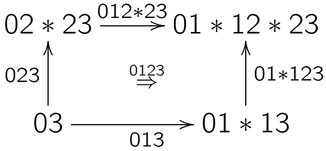
For instance:





Answer to the first question

Kapranov and Voevodsky note

$$\Delta_n(0, n) \quad = \quad I^{\otimes(n-1)}/\sim$$

(some squares corresponding to exchanges become equalities)

For an answer to the other questions, see later on.

The same problem occurs with
the *permutohedron*,
let's study this in more details...

THE PERMUTOHEDRON

Categorical definition

The n-**permutohedron** is the hom-n-category of the $(n+1)$-cube:

$$\Pi_n \quad = \quad I^{\otimes(n+1)}(-^{n+1}, +^{n+1})$$

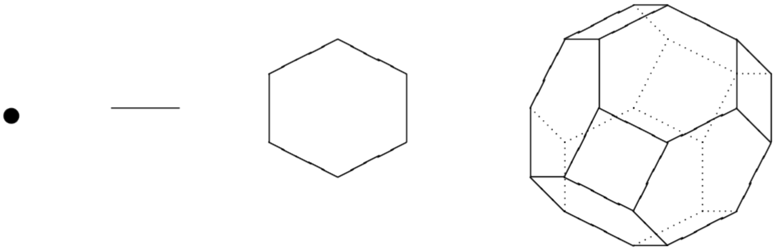
Categorical definition

The n-**permutohedron** is the hom-n-category of the $(n+1)$-cube:

$$\Pi_n \quad = \quad I^{\otimes(n+1)}(-^{n+1}, +^{n+1})$$

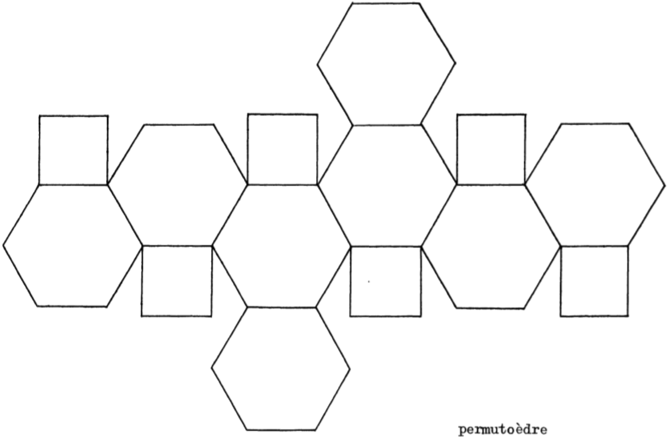
Note that because of this we have an orientation for most generators, excepting exchange.

Some pictures



0 1 2 3

Some pictures



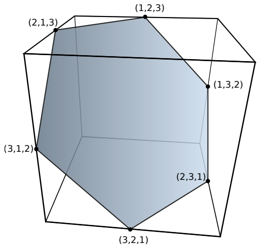
Le Conte de Poly-Barbut, *Le diagramme du treillis permutoèdre est intersection des diagrammes de deux produits directs d'ordres totaux*, 1990.

Geometric realization

The permutohedron can be defined as the convex hull of points

$$(\sigma(0), \sigma(1), \ldots, \sigma(n)) \in \mathbb{R}^{n+1}$$

where σ runs over $\mathfrak{S}_{n+1}$.

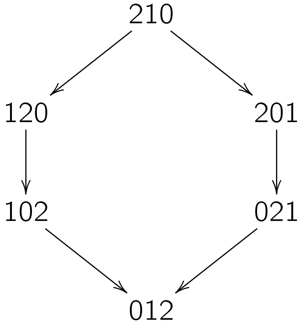


It lies in the hyperplane

$$\left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \;\middle|\; \sum_i x_i = n(n-1)/2 \right\}$$

Edges

Edges correspond to transpositions:



Faces

More generally, cells are indexed by surjections (Loday-Ronco, *Permutads*):

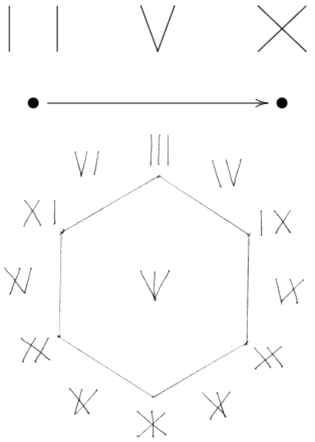
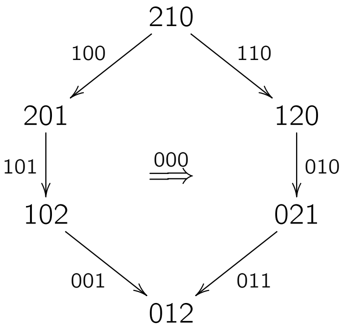


Figure 1: The permutad clock

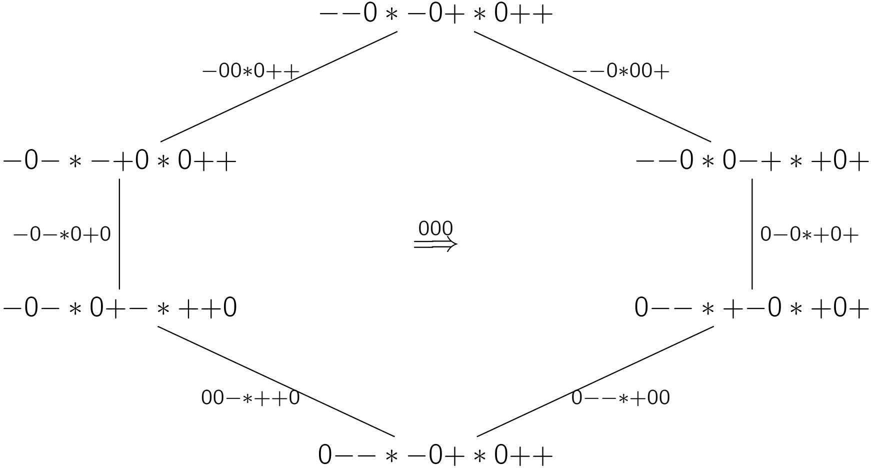
Faces

More generally, cells are indexed by surjections (same as previous figure up to symmetry):



Links with $I^{\otimes n}$

The correspondence between surjections and cells of the cube is as follows:



A cochain complex

Chapoton, in *Opérades différentielles graduées sur les simplexes et les permutoèdres*, defines a cochain complex $\Pi(I) \subseteq T(\Lambda X)$:

- elements are of the form

$$\pi = \pi_1 \otimes \pi_2 \otimes \ldots \otimes \pi_n$$

  with

$$\pi_j = i_{j,1} \wedge i_{j,2} \wedge \ldots \wedge i_{j,p_j}$$

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- the differential (of degree +1) is given by

$$d(\pi) \quad = \quad \sum_{j=1}^{p-1} (-1)^{\sum_{k=1}^{j} \dim(\pi_k)} \pi_1 \otimes \ldots \otimes \pi_j \wedge \pi_{j+1} \otimes \ldots \otimes \pi_n$$

A cochain complex

It can be checked that this is a codifferential:

- we have $\dim(\pi_i \wedge \pi_j) = \dim(\pi_i) + \dim(\pi_j) - 1$,

A cochain complex

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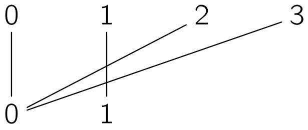
- we have $\dim(\pi_i \wedge \pi_j) = \dim(\pi_i) + \dim(\pi_j) - 1$,
- with $n = 3$, we have

$$
\begin{aligned}
d(\pi_1 \otimes \pi_2 \otimes \pi_3) &= (-1)^{\dim(\pi_1)} \pi_1 \wedge \pi_2 \otimes \pi_3 \\
&\quad + (-1)^{\dim(\pi_1)+\dim(\pi_2)} \pi_1 \otimes \pi_2 \wedge \pi_3 \\
d^2(\pi_1 \otimes \pi_2 \otimes \pi_3) &= (-1)^{2\dim(\pi_1)+\dim(\pi_2)-1} \pi_1 \wedge \pi_2 \wedge \pi_3 \\
&\quad + (-1)^{2\dim(\pi_1)+\dim(\pi_2)} \pi_1 \wedge \pi_2 \wedge \pi_3 \\
&= 0
\end{aligned}
$$

and the general case is essentially similar.

A point on notations

A surjection

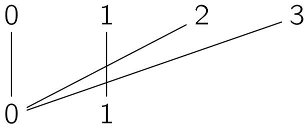


can be denoted

- by images: 0100

A point on notations

A surjection

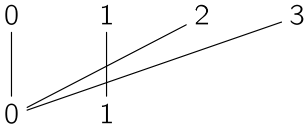


can be denoted

- by images: 0100
- by preimages: $(0 \wedge 2 \wedge 3) \otimes 1$
  (= ordered partitions of the source)

A point on notations

A surjection



can be denoted

- by images: 0100
- by preimages: $(0 \wedge 2 \wedge 3) \otimes 1$
  (= ordered partitions of the source)
- by categorical notation, with explicit exchange:

....

A point on notations

There are two possible actions. In the representation by images, with $\tau = 12$,

- on the left:

$$\tau \cdot 2031 \quad = \quad 2301$$

- on the right:

$$2031 \cdot \tau \quad = \quad 1032$$

A point on notations

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- on the left:

$$\tau \cdot 2031 \quad = \quad 2301$$

- on the right:

$$2031 \cdot \tau \quad = \quad 1032$$

The one on the right is more natural wrt to surjections:

$$2031 \cdot \mu_{12} \quad = \quad 1021$$

instead of

$$\mu_{12} \cdot 2031 \quad = \quad 30412$$

Test in dimension 2

```
# permutohedron 2
# sigcat
012,102,021,120,201,210

001 : 102 -> 012
010 : 120 -> 021
011 : 021 -> 012
100 : 210 -> 201
101 : 201 -> 102
110 : 210 -> 120
```

000 : 100 $*_0$ 101 $*_0$ 001 -> 110 $*_0$ 010 $*_0$ 011

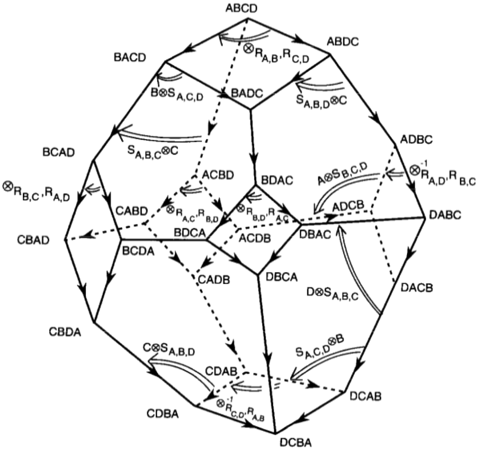
Test in dimension 3

```
# permutohedron 3
0123 , 1023 , 0213 , 1203 , 2013 , 2103 , 0132 , 1032 , 023
0012 : 1023 -> 0123 , 0102 : 1203 -> 0213 , 0112 : 0213 ->
0001 : 0012,1002,1012 -> 0102,0112,1102 , 0010 : 0021,1020,
0000 : 0010,0011,0110,1000,1001,1011,1100,1110 -> 0001,010
# check
Error: Invalid signature (cylic).
```

The signs are not right!

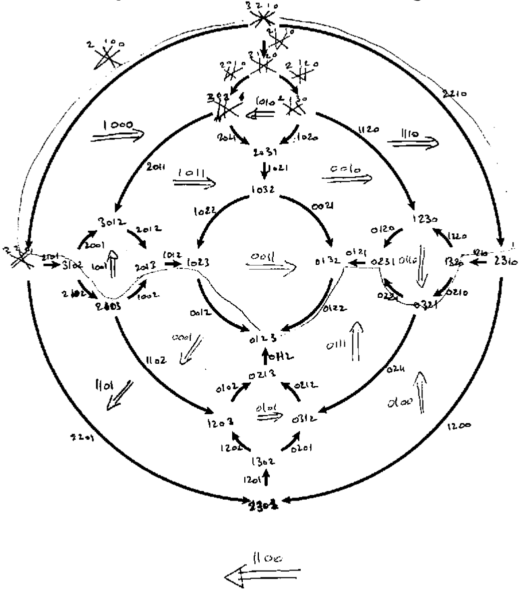
Categorical permutohedra in nature

However, it should exist, e.g. Kapranov and Voevodsky, *Braided monoidal 2-categories and Manin-Schechtman higher braid groups*:

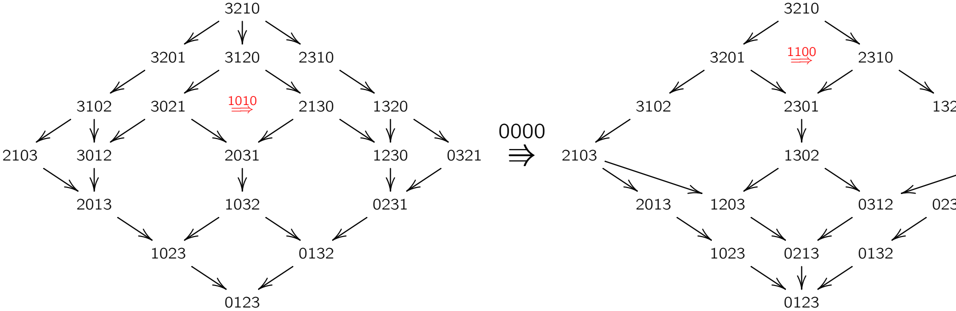


The permutohedron

The signs obtained by this method are not right:



Our orientation



For instance 2010 : 3120 → 3021 corresponds to $1 \wedge 3 \otimes 2 \otimes 0$ and therefore occurs in the target (instead of the source) of 1010 which corresponds to $1 \wedge 3 \otimes 0 \wedge 2$.

Our orientation

The signs obtained by this method are not right:

- 1100 is badly oriented
- 1010 is badly oriented
- the type of 0000 is from

  $$0010, 0011, 0110, 1000, 1001, 1011, 1100, 1110$$

  to

  $$0001, 0100, 0101, 0111, 1010, 1101$$

  instead of from

  $$0010, 0011, 0110, 1000, 1001, 1011, 1010, 1110$$

  to

  $$0001, 0100, 0101, 0111, 1100, 1101$$

Reorienting

```
# permutohedron 3
# check
Error: Invalid signature (cylic).
# reorient 1010
# reorient 1100
# remove 0000
# gen 0000
      1000,1001,1011,1010,0011,0010,1110,0110
      0001,1101,1100,0101,0111,0100
# sigcat
...
```

1010 : 2010 $*_0$ 2021 -> 2120 $*_0$ 1020
1011 : 2011 $*_0$ 2012 $*_0$ 1012 -> 2021 $*_0$ 1021 $*_0$ 1022
1100 : 2100 $*_0$ 2201 -> 2210 $*_0$ 1200
1101 : 2101 $*_0$ 2102 $*_0$ 1102 -> 2201 $*_0$ 1201 $*_0$ 1202
1110 : 2110 $*_0$ 2120 $*_0$ 1120 -> 2210 $*_0$ 1210 $*_0$ 1220

0000 : (2100 $*_0$ 2101 $*_0$ 1001 $*_0$ 1012 $*_0$ 0012) $*_1$ (1000 $*_0$ 20

Coxeter groups

Question

Can we find categorical polytopes associated to finite Coxeter groups?

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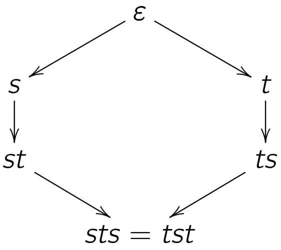
For instance, the permutohedron is the Hasse diagram of the *weak Bruhat order* (= "prefix order") for A_n:



Higher Bruhat orders

In fact, higher analogues of Bruhat orders have been defined by Manin and Schechtman in

- *Arrangements of real hyperplanes and Zamolodchikov equations*, 1986
- *Higher Bruhat orders, related to the symmetric group*, 1986
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and reworked by Voevosky and Kapranov with exactly the same motivations as us...

- *Free n-categories generated by a cube, oriented matroids, and higher Bruhat orders*, 1990
- *Combinatorial-geometric aspects of polycategory theory: pasting schemes and higher Bruhat orders (list of results)*, 1991

Desuspension

Given a $(n+1)$-category $\mathcal{C}$ with initial object A_- and terminal object A_+, we write $\Omega\mathcal{C} = \mathcal{C}(A_-, A_+)$.

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In particular, we have seen

- $\Omega\Delta_{n+1} = I^{\otimes n}$
- $\Omega^2\Delta_{n+2} =$ n-associahedron
- $\Omega I^{\otimes(n+1)} =$ n-permutohedron
- $\Omega^k I^{\otimes n} =?$

Higher Bruhat orders

M-S define posets $B(n,k)$ so that $B(n,1)$ is S_n with the weak Bruhat order.

- $C(n,k)$: k-elements subsets of $\{0,\ldots,n\}$,
- we write $x = x_0 < \ldots < x_{k-1}$ for an element and $\partial_i x$ for x with i-th element removed,
- $A(n,k)$: total orders on $C(n,k)$ such that for each $x \in C(n,k+1)$, either

$$\partial_0 x < \partial_1 x < \ldots < \partial_k x \qquad \text{or} \qquad \partial_0 x > \partial_1 x > \ldots > \partial_k x$$

- we write $a = a_0 < \ldots < a_N$, with $N = \binom{n}{k}$ for an element of $A(n,k)$,
- for $a, a' \in A(n,k)$, we write $a \sim a'$ when a' is obtained from a by permuting a_i and a_{i+1} such that $|a_i \cap a_{i+1}| < k-1$,
- $B(n,k) = A(n,k)/\sim$,
- a partial order can be defined on $B(n,k)$.

Higher Bruhat orders

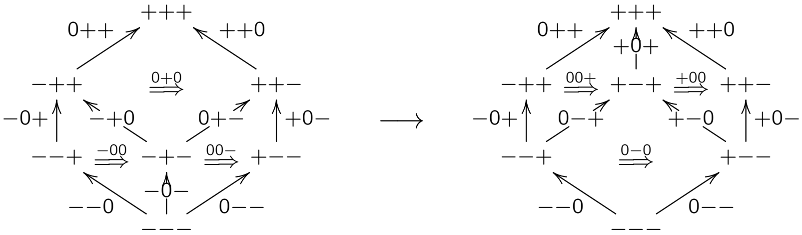
K-V have "shown":

Theorem
$B(n,k) \cong \operatorname{Ob}\Omega^k I^{\otimes n}$.

They thus correspond to maximal cells (up to permutations).

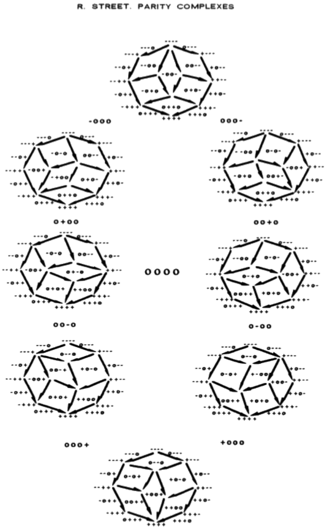
Higher Bruhat orders

- $B(3,2)$ is an edge:



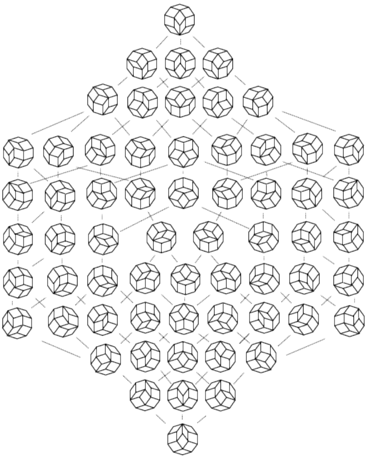
Higher Bruhat orders

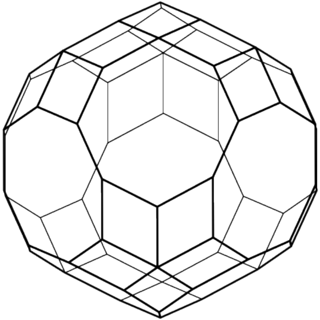
- $B(3,2)$ is an edge:
- $B(4,2)$ is an 8-gon:



Higher Bruhat orders

From Felsner and Ziegler, *Zonotopes Associated with Higher Bruhat Orders*, $B(5,2) = \Omega^2 I^{\otimes 5}$ is



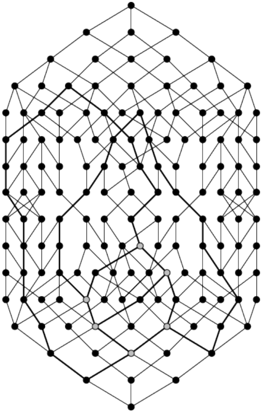


Higher Bruhat orders

F-Z observe that the graph of $B(6, 3)$ is not polytopal:

Proof.

It has vertices of degree 3, thus it is not the graph of a polytope of dimension ≥ 4. Moreover, it contains a $K_{3,3}$ and is thus not planar. □



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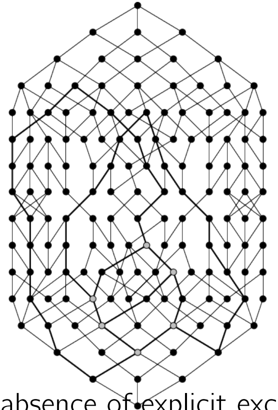


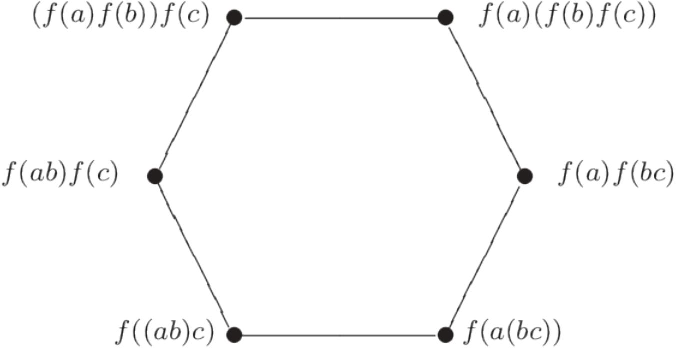
Is this because of the absence of explicit exchange? Or what?

Back to the associahedron

- The analoguous of weak Bruhat order in the case of the associahedron is the *Tamari lattice*, ordering planar trees.
- Higher-dimensional generalizations exist and correspond to $\Omega^k \Delta_n$.

The multiplihedron

There are other interesting polytopes such as the **multiplihedron**



generated by $(ab)c \to a(bc)$ and $f(a)f(b) \to f(ab)$.

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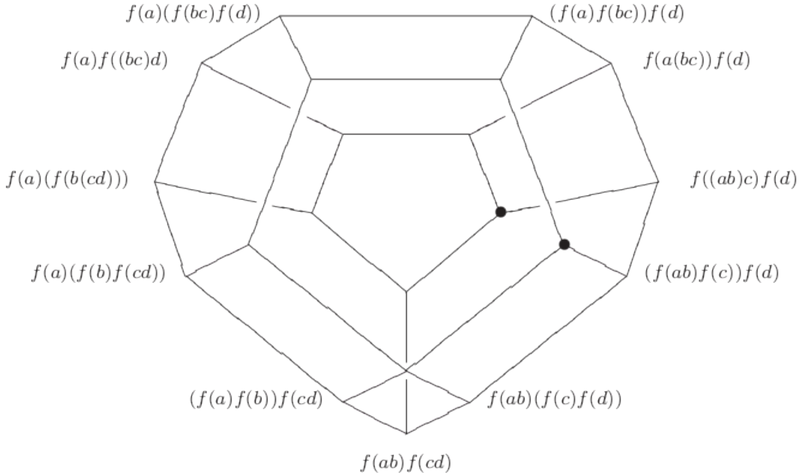

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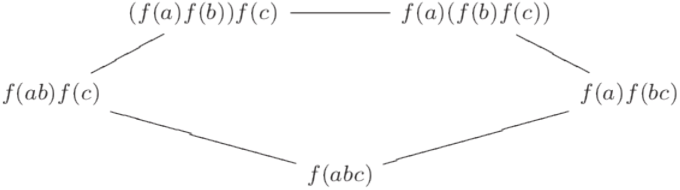
The composihedron

The **composihedron** is obtained from the multiplihedron by quotienting under associativity $(ab)c = a(bc)$:



From Forcey, *Quotients of the multiplihedron as categorified associahedra*.

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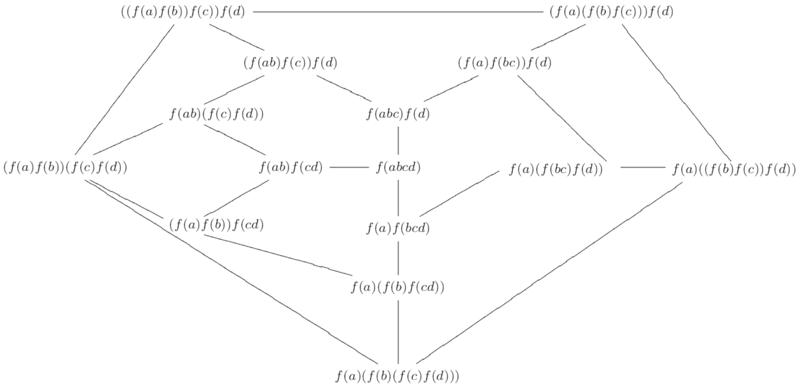

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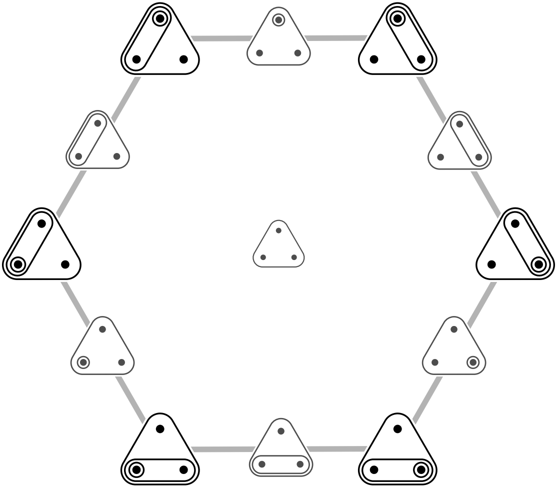
Graph composihedra

Composihedra associated to a graph (= **graph composihedra**) have also been investigated, it would be interesting to look at composihedra generated by 2-dimensional (or higher) pasting schemes!

The cyclohedron

The **cyclohedron** is close to the associahedron excepting that we bracket a cycle instead of a word:



Coxeteredra

Generalizing the permutohedron associated to the symmetric group A_n, we have **Coxeterhedra** associated to other Coxeter groups. From Reiner, Ziegler, *Coxeter-associahedra*:

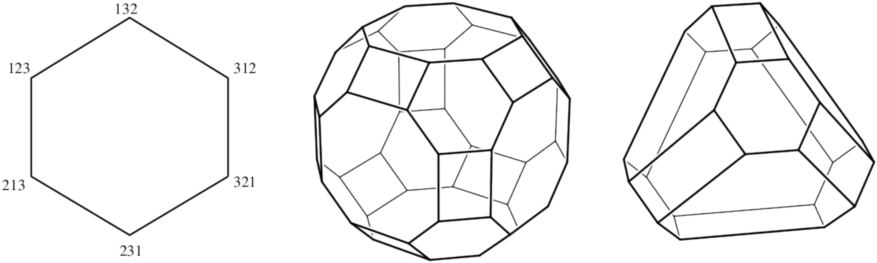


Figure 1: The Coxeterhedra PA_2, PB_3, PD_3

Biassociahedra

There are also **biassociahedra**, resolving bialgebra laws:

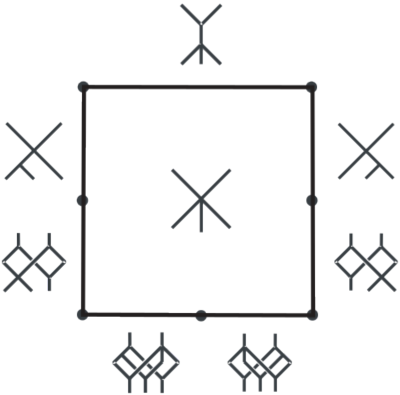


Figure 2. The biassociahedron $KK_{2,3}$.

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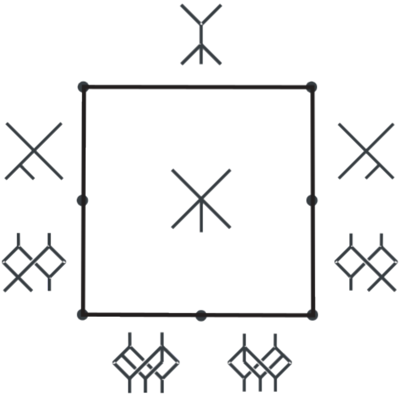


Figure 2. The biassociahedron $KK_{2,3}$.

and as you have guessed there are also **bipermutohedra**.

An online encyclopedia

Forcey has compiled an interesting list of polyhedra:

http://www.math.uakron.edu/~sf34/hedra.htm

The erasohedron

We can even come up with "new" ones. For instance, consider the free monoidal category $\mathcal{C}$ on $\varepsilon : 1 \to 0$. I call $\mathcal{C}(n, 0)$ the n-**erasohedron** E_n.

In low dimensions:

- E_1:

$$1 \xrightarrow{\;*\;} 0$$

- E_2:

$$\begin{array}{ccc} 2 & \xrightarrow{\;0*\;} & 1 \\ {\scriptstyle *0}\big\downarrow & \overset{**}{\Longrightarrow} & \big\downarrow{\scriptstyle *} \\ 1 & \xrightarrow[\;*\;]{} & 0 \end{array}$$

The erasohedron

We can even come up with “new” ones. For instance, consider the free monoidal category $\mathcal{C}$ on $\varepsilon : 1 \to 0$. I call $\mathcal{C}(n, 0)$ the n-**erasohedron** E_n.

Conjecture

The cells of E_n are in bijection with injections $m \to n$ with $m \leq n$.

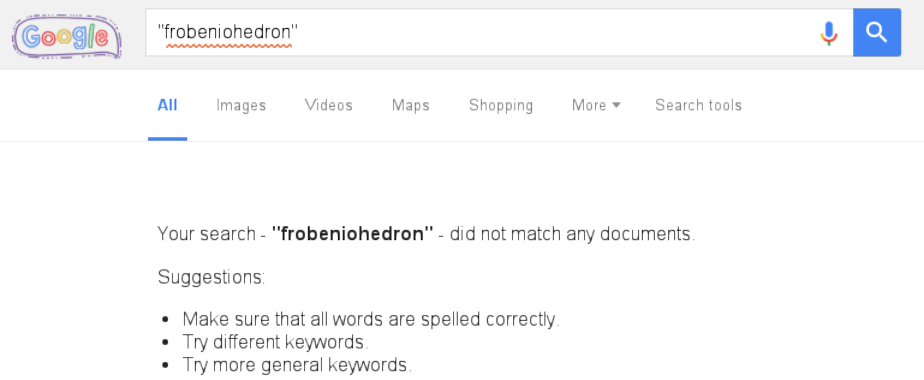
Conjecture

The erasohedron is isomorphic to $I^{\otimes n}$.

New examples

`Anyone with interesting new examples?`

The Frobeniohedron



A good question

Question

How do we generate signs in a general way?

COMPARING WITH STREET'S DEFINITION


MEAN
STREETS

Parity complexes

Definition

A **parity complex** is a graded set S such that

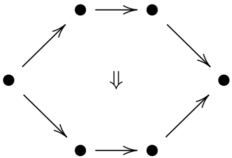
1. $x^- \neq \emptyset \neq x^+$ and $x^- \cap x^+ = \emptyset$
2. $x^{--} \cup x^{++} = x^{-+} \cup x^{+-}$
3. x^- and x^+ are *well-formed*:
   - they contain at most one 0-generator
   - for every $y \neq z$, we have $(y^- \cap z^-) \cup (y^+ \cap z^+) = \emptyset$
4. $\triangleleft$ is acyclic
5. $x \triangleleft y$, $x \in z^\varepsilon$, $y \in z^\eta$ imply $\varepsilon = \eta$

About the second axiom

$$x^{--} \cup x^{++} \quad = \quad x^{-+} \cup x^{+-}$$

Note that x^{--} is not exactly what you think of first:

$x \quad =$



Therefore it is the closest we can do to globular identities
(this still ensures that we do have a chain complex for instance).

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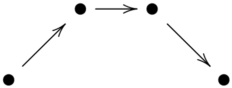
$x^{-} \quad =$ 

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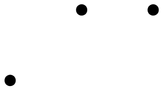


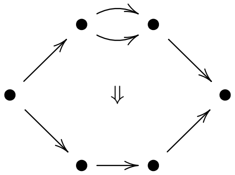
Therefore it is the closest we can do to globular identities
(this still ensures that we do have a chain complex for instance).

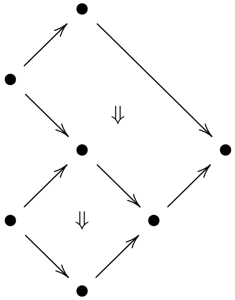
About the third axiom

For $y, z \in x^{\epsilon}$,

$$(y^- \cap z^-) \cup (y^+ \cap z^+) \quad = \quad \emptyset$$

This typically forbids

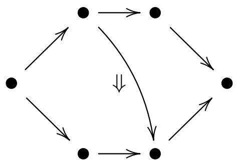




About the last axiom

$$x \lhd y,\ x \in z^{\varepsilon},\ y \in z^{\eta} \text{ imply } \varepsilon = \eta$$

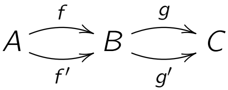
Typically, the following is forbidden



About the last axiom

$$x \triangleleft y,\ x \in z^{\varepsilon},\ y \in z^{\eta} \text{ imply } \varepsilon = \eta$$

Note that it also forbids



with

$$\alpha : f *_0 g \Rightarrow f' *_0 g' \qquad \alpha^{-} = \{f, g\} \qquad \alpha^{+} = \left\{f', g'\right\}$$

because $f \triangleleft g'$.

Note that this excludes Power's counter-example!

Movement

Given subsets P, F, Q of S, we say that F **moves** P to Q and write $F : P \longrightarrow Q$ when

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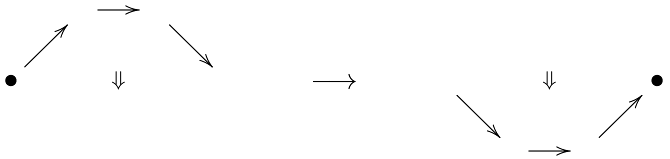
Example

The complex on the left moves the complex on the left to the one on the right:



Typical movement

Lemma

Writing $F^{\mp} = F^{-} \setminus F^{+}$*, given* F *and* P*, there exists* Q *such that* $F : P \longrightarrow Q$ *if and only if*

$$F^{\mp} \subseteq P \qquad \text{and} \qquad P \cap F^{+} = \emptyset$$

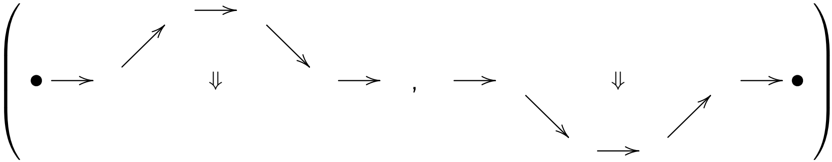
The ω-category of a parity complex

Definition

A cell $C = (P, Q)$ is a pair of non-empty well-formed finite subsets such that

$$P : P \longrightarrow Q \qquad \text{and} \qquad Q : P \longrightarrow Q$$

Typically,



(or with multiple top-dimensional generators).

The ω-category of a parity complex

Source and target are defined "as expected":

$$s_n(P, Q) \quad = \quad (P^{(n)}, P_n \cup Q^{(n-1)})$$

where $P^{(n)}$ is the n-truncation (we empty sets P_i with $i > n$).

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The "only" difficult thing is to check that the composite of two cells is a cell (which takes up a few pages).

Freeness

To each n-generator x one can easily associate a cell $\langle x \rangle$ with x as only n-generator: we take

$$P_n = \{x\} \qquad\qquad P_i = P_{i+1}^{\mp}$$

and similarly for Q.

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Theorem (Street)

The ω-category is freely generated by cells of this form.

Excision of extremals

Consider an n-cell (P, Q) containing $u \in P_n \cap Q_n$ and different from $\langle u \rangle$.

1. Find the largest $m < n$ such that $(P_{m+1}, Q_{m+1}) \neq \langle u \rangle_{m+1}$ and pick $w \in P_{m+1} \cap Q_{m+1}$.

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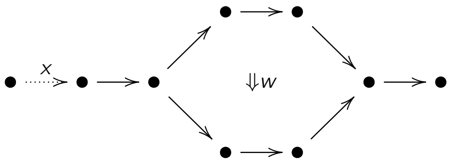
2. In M_{n+1}, pick $x \lhd w$ minimal and $y \rhd w$ maximal. One of them belongs to $P_{m+1} \cap Q_{m+1}$, suppose x.
3. We get a decomposition $(P, Q) = (P', Q') *_m (P'', Q'')$ with

$$P' = P^{(m)} \cup \{x\} \quad Q' = P^{(m-1)} \cup \left(\left(M_m \cup x^+\right) \setminus x^-\right) \cup \{x\}$$
$$Q'' = P \setminus \{x\} \qquad P'' = \left((P \setminus \{x\}) \cup x^+\right) \setminus x^-$$



COMPARING WITH PASTING SCHEMES

Pasting schemes

Those are defined in Johnson, *The Combinatorics of n-Categorical Pasting*.

Pasting schemes

A **pasting scheme** consists of

- a graded set (A_i) of generators
- relations $B^i_j, E^i_j : A_i \to A_j$, for $j \leq i$, expressing whether a j-generators occurs in the beginning (resp. end) of a generator

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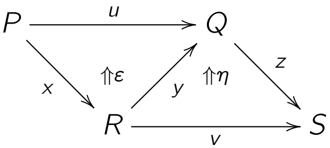
and dually (replace E with B and vice versa).

Pasting schemes

Example



has

$$E_2^2 = \{(\varepsilon, \varepsilon), (\eta, \eta)\} \qquad B_2^2 = \{(\varepsilon, \varepsilon), (\eta, \eta)\}$$

$$E_1^2 = \{(\varepsilon, u), (\eta, y), (\eta, z)\} \qquad B_1^2 = \{(\varepsilon, x), (\varepsilon, y), (\eta, v)\}$$

$$E_0^2 = \{(\eta, Q)\} \qquad B_0^2 = \{(\varepsilon, R)\}$$

$$\vdots \qquad \vdots$$

Directed loops

We write $\triangleleft$ for the preorder such that, for $x, y \in A_{n+1}$, $x \triangleleft y$ whenever $E_n(x) \cap B_n(y) \neq \emptyset$.

Directed loops

We write $\lhd$ for the preorder such that, for $x, y \in A_{n+1}$, $x \lhd y$ whenever $E_n(x) \cap B_n(y) \neq \emptyset$.

A has **no directed loops** when for $x, y \in A_n$,

- $B(x) \cap E(x) = \{x\}$, and
- $x \lhd y$ then $B(x) \cap E(y) = \emptyset$.

Domain and codomain

Given a graded subset $X \subseteq A$, we define

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Theorem

If A has no directed loops then the globular identities are satisfied.

Well-formedness

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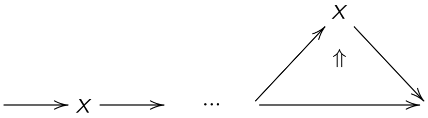
3. for all k, $\mathrm{dom}^k(A)$ and $\mathrm{codom}^k(A)$ are compatible subpasting schemes of A.

Loop-freeness



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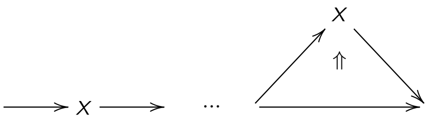

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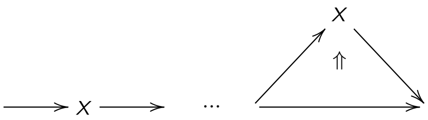

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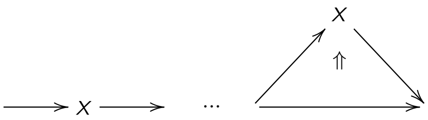

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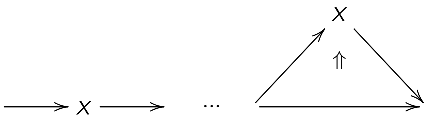

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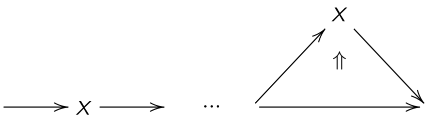

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and dually.

A free construction

Starting from a loop-free pasting scheme, Johnson defines an ω-category (roughly as we do), with

- well-formed (not necessarily loop-free) subschemes as cells,
- source and target given by dom and codom,
- composition given by union.

Theorem

It is the free ω-category generated by the cells $R(x)$.

Remark

Power's counter-examples explains why we have to include non-loop-free schemes, since loop-free are not closed under composition.

The pasting theorem

Theorem

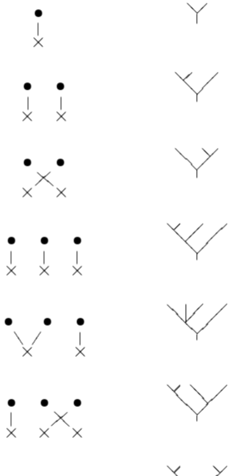
The realization of a well-formed loop-free pasting scheme in a category gives rise to a unique composite cell.

SIDE NOTES

The permutohedron

Surjections are in bijection with leveled planar trees:

surjection	leveled planar tree
$t : \underline{n} \to \underline{k}$	$n+1$ leaves, k levels



From leveled trees to surjections:

- label leaves from left to right by $0, 1, \ldots, n$
- label levels downward from 1 to k
- $f(i)$ is the level attained by a ball dropped between i and $i+1$

So, the permutohedron is a "leveled" variant of the associahedron.