Bisimulations as weak equivalences?

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In [4] (see also [2, 1]), Joyal, Nielsen and Winskel proposed an abstract definition of bisimulations of labeled transition systems, by using lifting properties. It is then very natural to wonder whether there is a model structure (see [3]) on the category of labeled transition systems for which bisimulations are weak equivalences. In this short note, we show that the direct attempts at constructing it fail.

Labeled transition systems. We write L for the sets of labels. A *graph* is a diagram

$$G_0 \underset{t}{\overset{s}{\leftleftarrows}} G_1$$

in **Set**, where the elements of G_0 and G_1 are respectively called *vertices* and *edges*. A *pointed* graph is a graph together with a distinguished vertex, sometimes called the *initial vertex* and noted $\odot$. A *labeling* ℓ of a graph G in a set L is a morphism from G to the graph

$$1 \leftleftarrows L$$

A *labeled transition system*, or LTS, is a pointed graph together with a labeling. We write **LTS** for the category of labeled transition systems, where morphisms are graph morphisms preserving the distinguished vertex and the labels.

Path extensions. A *path extension* is a morphism exhibiting the inclusion from a linear LTS of length n

$$\odot \xrightarrow{a_1} \cdot \xrightarrow{a_2} \cdot \xrightarrow{a_3} \cdots \xrightarrow{a_n} \cdot$$

to a linear LTS of length $n+1$

$$\odot \xrightarrow{a_1} \cdot \xrightarrow{a_2} \cdot \xrightarrow{a_3} \cdots \xrightarrow{a_n} \cdot \xrightarrow{a_{n+1}} \cdot$$

We write $\mathcal{J}$ for the set of path extensions.

Simuations. A morphism $p : X \to Y$ is a *simulation* (or is *open*) when it has the right lifting property with respect to every morphism $i : A \to B$ in $\mathcal{J}$. This

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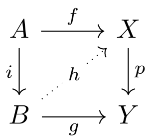
means that for every commuting square



with $i \in \mathcal{J}$, there exists a morphism $h : B \to X$ making the two triangles commute. We write $\mathcal{J}^{\boxslash}$ for the class of simulations.

Bisimuations. A *bisimulation* is a span

$$A \longleftarrow X \longrightarrow B$$

of simulations.

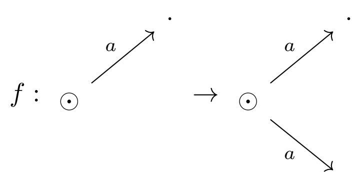
Cofibrations. A *cofibration* is a morphism which has the left lifting property with every simulation. We write $\mathcal{C} = {}^{\boxslash}(\mathcal{J}^{\boxslash})$ for the class of cofibrations. By general properties about lifting classes, we have $\mathcal{J} \subseteq \mathcal{C}$ and $\mathcal{C}$ is closed under coproducts, pushouts (along any morphism), countable compositions and retracts [5, Lemma 11.1.4].

Toward a model structure. It is natural to expect to have a model structure with

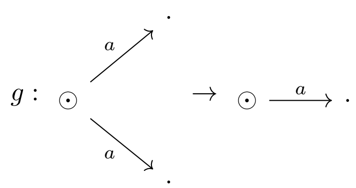
- path extensions as generating cofibrations,
- simulations as trivial fibrations,
- cofibrations as cofibrations,
- bisimulations as weak equivalences.

This determines all the classes (if there is such a model structure) since trivial cofibrations are cofibrations which are weak equivalences, and fibrations are the morphisms with the right lifting property with respect to those.

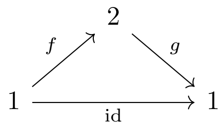
A problem. This does not quite work, as we now illustrate. Consider the morphisms



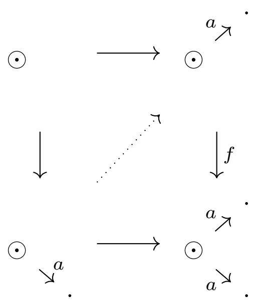
and



Clearly g is a bisimulation and thus a weak equivalence. Moreover, we have $g \circ f = \mathrm{id}$, i.e. the following triangle commutes:

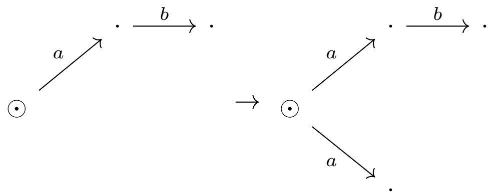


By the 2-out-of-3 property, f must be a weak equivalence since both g and id are. However, f is not a bisimulation: we cannot fill in the square



with a dotted arrow.

We could think of having a class of weak equivalences which is a bit larger than bisimulations and would include morphisms such as f, but things get worse. Since trivial cofibrations should be closed under pushouts (as a class of left liftings), this means that the morphism



obtained from f by pushout should also be a weak equivalence, and this is really bad because the two labeled transition systems are "clearly not bisimilar".

Culprits. There are two properties of model structures that we can question here:

- the 2-out-of-3 property which has allowed us to conclude that f is a weak equivalence,
- the closure of trivial cofibrations under pushouts.

However, removing either of them gets us very far from traditional theory.

References

[1] Gian Luca Cattani and Glynn Winskel. Profunctors, open maps and bisimulation. *Mathematical Structures in Computer Science*, 15(3):553–614, 2005. `doi:10.1017/S0960129505004718`.

[2] Marcelo Fiore, Gian Luca Cattani, and Glynn Winskel. Weak bisimulation and open maps. In *Proceedings of the 14th Symposium on Logic in Computer Science*, pages 67–76. IEEE, 1999. doi:10.1109/LICS.1999.782590.

[3] Mark Hovey. *Model categories*. Number 63 in Mathematical Surveys and Monographs. American Mathematical Society, 2007. doi:10.1090/surv/063.

[4] André Joyal, Mogens Nielsen, and Glynn Winskel. Bisimulation from open maps. *Information and Computation*, 127(2):164–185, 1996. doi:10.1006/inco.1996.0057.

[5] Emily Riehl. *Categorical homotopy theory*, volume 24 of *New Mathematical Monographs*. Cambridge University Press, 2014. doi:10.1017/CBO9781107261457.