Minimal Resolutions via Algebraic Morse Theory

Samuel Mimram

November 15, 2012

SO, I TRIED TO READ...

Minimal Resolutions via Algebraic Discrete Morse Theory

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THE IDEA

In order to construct a (small) resolution, we start from the (big) Bar resolution and reduce it to a decent size (sometimes minimal) by smashing triangles.

In other words, this is another point of view on homotopy reduction!

A CHAIN COMPLEX

We start from a commutative ring R and

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Define a weighted DAG $G(C_\bullet)$ with vertices $X = \cup_{i \geq 0} X_i$ and edges

$$X_i \ni c \xrightarrow{[c:c']} c' \in X_{i-1}$$

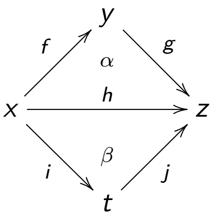
whenever $[c : c'] \neq 0$.

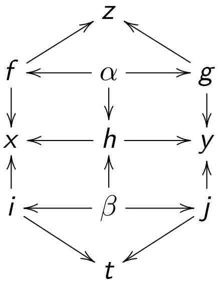
ACYCLIC MATCHINGS

A set $\mathcal{M} \subseteq E$ of $G(C_\bullet) = (X, E)$ is an **acyclic matching** when

1. For each $c \xrightarrow{[c:c']} c'$ in $\mathcal{M}$, $[c : c']$ in the center, invertible
2. Each vertex lies in a most one edge of $\mathcal{M}$
3. The graph $G_{\mathcal{M}} = (X, E_{\mathcal{M}})$ has no directed cycle with

$$E_{\mathcal{M}} = (E \setminus \mathcal{M}) \cup \left\{ c' \xrightarrow{-1/[c:c']} c \mid c \to c' \in \mathcal{M} \right\}$$





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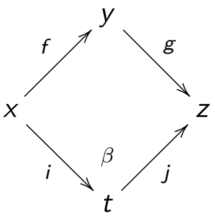

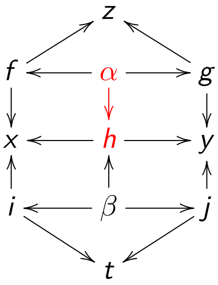

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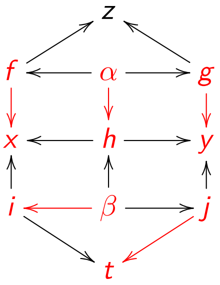

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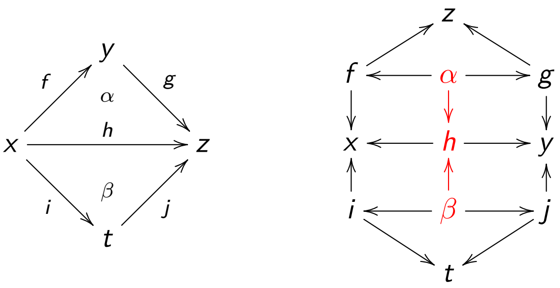

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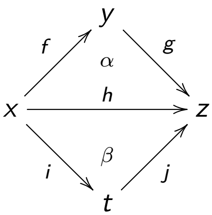

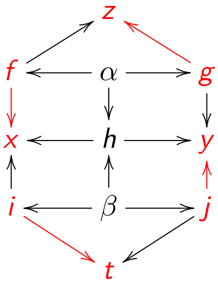

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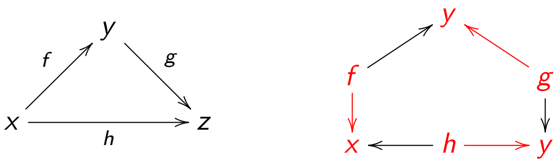

ACYCLIC MATCHINGS

Consider $G(C_\bullet)$ together with an acyclic matching $\mathcal{M}$.

- When $e \to f \in \mathcal{M}$, e is **collapsible** and f is **redundant**.
- A vertex $c \in X$ is **critical** when it lies in no edge of $\mathcal{M}$.
- We write $X_i^{\mathcal{M}} \subseteq X_i$ for the critical vertices.
- The **weight** of a path is

$$w(c_1 \to c_2 \to \ldots \to c_r) = \prod_{i=1}^{r-1} w(c_i \to c_{i+1})$$

  with $w(c \xrightarrow{\ell} c') = \ell$.
- We write

$$\Gamma(c, c') = \sum_{p \in \mathrm{path}(c,c')} w(p)$$

THE MORSE COMPLEX

The **Morse complex** $C_\bullet^{\mathcal{M}} = (C_i^{\mathcal{M}}, \partial_i^{\mathcal{M}})$ is defined by $C_i^{\mathcal{M}} = RX_i^{\mathcal{M}}$ and $\partial_i^{\mathcal{M}} : C_i^{\mathcal{M}} \to C_{i-1}^{\mathcal{M}}$ by

$$\partial_i^{\mathcal{M}}(c) \quad = \quad \sum_{c' \in X_{i-1}^{\mathcal{M}}} \Gamma(c, c')c'$$

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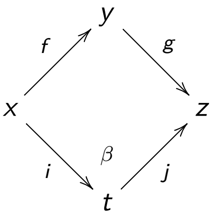

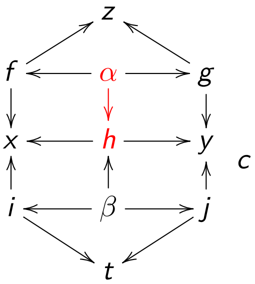

$$\partial_2^{\mathcal{M}}(\beta) \quad = \quad i + j - f - g$$

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Theorem

The complex $C_\bullet^{\mathcal{M}}$ of free R-modules is homotopy equivalent to $C_\bullet$. The maps $f : C_\bullet \to C_\bullet^{\mathcal{M}}$ and $g : C_\bullet^{\mathcal{M}} \to C_\bullet$ give a chain homotopy (and thus a quasi-iso) between $C_\bullet$ and $C_\bullet^{\mathcal{M}}$:

$$f_i(c) = \sum_{c' \in X_i^{\mathcal{M}}} \Gamma(c, c')c' \qquad\qquad g_i(c) = \sum_{c' \in X_i} \Gamma(c, c')c'$$

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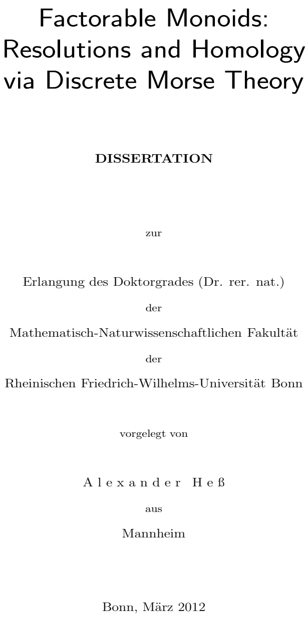
Proposition

If $\mathcal{M}$ is a set of edges with different source and targets, then $C_\bullet^{\mathcal{M}} \cong C_\bullet$ iff $\mathcal{M}$ is an acyclic matching.

SOME MORE DETAILS CAN BE FOUND IN



GAUß ELIMINATION

- Fix a free chain complex

$$0 \to RX_k \xrightarrow{\partial} RX_{k-1} \to 0 \tag{1}$$

with $X_k = \{x_1, \ldots, x_m\}$ and $X_{k-1} = \{y_1, \ldots, y_n\}$.

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- We define a matrix $A \in R^{n \times m}$ with

$$a_{j,i} \quad = \quad [\partial x_i : y_j]$$

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- By Gauß elimination A is similar to

$$N^{-1}AM \quad = \quad \begin{pmatrix} 1 & 0 \\ 0 & A' \end{pmatrix}$$

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- Then (1) has the same homology as

$$0 \to RX'_k \xrightarrow{A'} RX'_{k-1} \to 0$$

  with $X'_k = X_k \setminus \{x_i\}$ and $X'_{k-1} = X_{k-1} \setminus \{y_j\}$.

GAUß ELIMINATION

For instance

$$0 \to \mathbb{Z}^3 \xrightarrow{A} \mathbb{Z}^2 \to 0$$

with

$$A = \begin{pmatrix} 3 & 2 & -1 \\ 0 & 1 & 4 \end{pmatrix}$$

Taking $a_{2,2}$ as pivoting element,

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with $a_{j,i}$ as pivoting element where

$$\begin{aligned} M \quad &= \quad \left(x_i \mid x_1 - \frac{a_{j,1}}{a_{j,i}} x_i \mid \ldots \mid \hat{0} \mid \ldots \mid x_m - \frac{a_{j,m}}{a_{j,i}} x_i \right) \\ N \quad &= \quad (Ax_i \mid y_1 \mid \ldots \mid \hat{y}_j \mid \ldots \mid y_n) \end{aligned}$$

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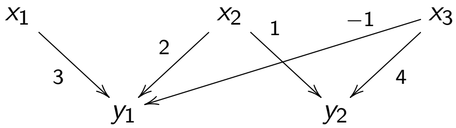

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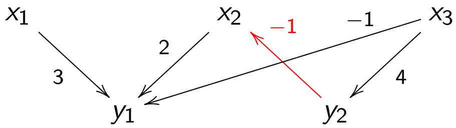

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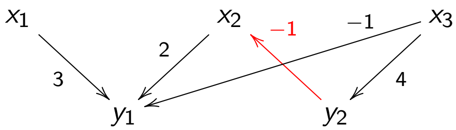

We have

$$A \quad \approx \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -9 \end{pmatrix}$$

and the flow from x_3 to y_1 is $-1 + 4 \times (-1) \times 2 = -9$, etc.

Let's use those ideas to reduce bar

ALGEBRAS

We consider the quotient (non-commutative) algebra $A = S/\mathfrak{a}$ with $S = \mathbb{K}\langle x_1, \ldots, x_n \rangle$ and $\mathfrak{a}$ an ideal of S.

From now on, we suppose fixed an order $x_1 \prec x_2 \prec \ldots \prec x_n$ on letters and extend it by deglex on monomials in S.

We also suppose that $\mathfrak{a} = \langle f_1, \ldots, f_s \rangle$ such that $\{f_1, \ldots, f_s\}$ is a minimal reduced Gröbner basis of S.

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$\mathrm{MinGen}(\mathrm{in}_\prec(I))$: a minimal (wrt $\prec$) generating system of $\mathrm{in}_\prec(I)$.

G is **reduced** when

1. for each $g \in G$ the coefficient of $\mathrm{in}_\prec(g)$ in g is 1
2. the set $\{\mathrm{in}_\prec(g) \mid g \in G\}$ minimally generates $\mathrm{in}_\prec(I)$
3. no trailing term of any $g \in G$ lies in $\mathrm{in}_\prec(I)$

A PBW BASIS

The set $\mathcal{G}$ of standard monomials of degree ≥ 1 is such that that $\mathcal{G} \cup \{1\}$ is a basis of the $\mathbb{K}$-vector space A: every $w \in A$ has a unique representation

$$w \quad = \quad a_1 + \sum_{v \in \mathcal{G}} a_v v$$

And it satisfies $a_v = 0$ when $|v| > |w|$.

THE NORMALIZED BAR RESOLUTION

The **normalized Bar resolution** $\mathrm{NB}_\bullet^A = (B_i, \partial_i)$ is

$$B_i = \bigoplus_{w_1,\dots,w_i \in \mathcal{G}} A[w_1|\dots|w_i]$$

with differential

$$\begin{aligned}\partial_i([w_1|\dots|w_i]) = {} & w_1[w_2|\dots|w_i] \\ & + \textstyle\sum_{j=1}^{i-1}(-1)^j \sum_{v\in\mathcal{G}} a_{jv}[w_1|\dots|w_{j-1}|v|w_{j+2}|\dots|w_i] \\ & +(-1)^i[w_1|\dots|w_{j-1}|v|w_{j+2}|\dots|w_{i-1}]\end{aligned}$$

with $w_j w_{j+1} = a_{j1} + \sum_{v\in\mathcal{G}} a_{jv} v$.

ANICK'S RESOLUTION

We define

- $C_0 = \{1\}$
- $C_1 = \{(1, x_1), \dots, (1, x_n)\}$
- C_{i+1} contains (ut, t') such that $(u, t) \in C_i$, $t' \in \mathcal{G}$ and tt' has exactly one occurrence of a $\mathrm{MinGen}(\mathrm{in}_{\prec}(\mathfrak{a}))$, which is a suffix of tt'.

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An element of F_l is of the form

$$[x_1 | t_2 | t_3 | \ldots | t_l]$$

with

- $t_i = v_i w_i$
- $w_i t_{i+1} \in \mathrm{MinGen}(\mathrm{in}_\prec(\mathfrak{a}))$
- for each prefix u of t_{i+1}, $t_i u \in \mathcal{G}$

ANICK'S RESOLUTION

An element of F_l is of the form

$$[x_1|v_2w_2|v_3w_3|\dots|v_lw_l]$$

We define $\partial_i : C_i \otimes A \to C_{i-1} \otimes A$ and $\iota_i : C_{i-1} \otimes A \to C_i \otimes A$ by

$$\partial_1([x]) = []x \qquad \iota_1([]\widehat{xu}) = [x]\widehat{u}$$

and given ∂_i and ι_i such that

$$\partial_{i-1}\partial_i = 0 \qquad \partial_i\iota_i = \mathrm{id}_{\mathrm{Ker}\,\partial_{i-1}}$$

we define

$$\partial_{i+1}([x_1|t_2|\dots|t_{i+1}]) = [x_1|t_2|\dots|t_i]t_{i+1} - \iota_i\partial_i([x_1|t_2|\dots|t_i]t_{i+1})$$
$$\iota_{i+1}(\underbrace{[x_1|t_2|\dots|t_i]t}_{\text{maximal degree}} + \dots) = \iota_{n+1}([x_1|t_2|\dots|t_i|t']t'') + \dots'$$

with $t = t't''$ and t' smallest prefix of t admitting a reducible suffix.

AN ACYCLIC MATCHING ON NB

We define matchings $\mathcal{M}_j$ of the normalized Bar resolution inductively wrt the "j-th coordinate".

$$\mathcal{M}_1 = \left\{ \begin{array}{c} [x_i|w_1'|w_2|\ldots|w_l] \\ \downarrow \\ [w_1|w_2|\ldots|w_l] \end{array} \in G(\mathrm{NB}_\bullet^A) \mid w_1 = x_i w_1' \right\}$$

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Critical cells $\mathcal{B}_l^{\mathcal{M}_1}$ in homological degree $l \geq 1$ are

- $\mathcal{B}_1^{\mathcal{M}_1} = \{[x_i] \mid 1 \leq i \leq n\}$
- for $l > 1$, $[x_i|w_2|\dots|w_l] \in \mathcal{B}_l^{\mathcal{M}_1}$ when $x_i w_2$ is reducible

AN ACYCLIC MATCHING ON NB

Suppose that $\mathcal{M}_{j-1}$ is defined. We write

- $\mathcal{B}^{\mathcal{M}_{j-1}}$ for the set of critical cells wrt $\mathcal{M}_1 \cup \ldots \cup \mathcal{M}_{j-1}$
- $\mathcal{E}_j$: edges of $G(\mathrm{NB}_\bullet^A)$ connecting critical cells in $\mathcal{B}^{\mathcal{M}_{j-1}}$

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We define $\mathcal{M}_j$ as the set of edges

$$[x_{i_1}|w_2|\ldots|w_{j-1}|u_1|u_2|w_{j+1}|\ldots|w_l] \to [x_{i_1}|w_2|\ldots|w_{j-1}|w_j|w_{j+1}|\ldots|w_l]$$

in $\mathcal{E}_j$ (so $w_j = u_1 u_2$) such that

1. $[x_{i_1}|w_2|\ldots|w_{j-1}|u_1|u_2|w_{j+1}|\ldots|w_l] \in \mathcal{B}^{\mathcal{M}_{j-1}}$
2. $[x_{i_1}|w_2|\ldots|w_{j-1}|v_1|v_2|w_{j+1}|\ldots|w_l] \notin \mathcal{B}^{\mathcal{M}_{j-1}}$ for every prefix v_1 of u_1, $v_1 v_2 = w_j$.

AN ACYCLIC MATCHING ON NB

The critical cells $\mathcal{B}_l^{\mathcal{M}_j}$ are

- $\mathcal{B}_1^{\mathcal{M}_j} = \{[x_i] \mid 1 \leq i \leq n\}$
- $[x_i|w_2] \in \mathcal{B}_2^{\mathcal{M}_j}$ when $x_i w_2 \in \mathrm{MinGen}(\mathrm{in}_\prec(\mathfrak{a}))$
- for $l > 2$, $[x_i|w_2|\ldots|w_j|\ldots|w_l] \in \mathcal{B}_l^{\mathcal{M}_{j-1}}$ when
  - for each prefix u of w_j we have $[x_i|w_2|\ldots|w_{j-1}|u|w_{j+1}|\ldots|w_l] \notin \mathcal{B}_l^{\mathcal{M}_{j-1}}$
  - and $w_j w_{j+1}$ is reducible.

AN ACYCLIC MATCHING ON NB

Lemma

$\mathcal{M} = \cup_{j \geq 1} \mathcal{M}_j$ *is an acyclic matching.*

We write $\mathcal{B}^{\mathcal{M}}$ for the set of critical cells wrt $\mathcal{M}$.

FULLY ATTACHED CELLS

Definition

Let $m_{i_1}, \ldots, m_{i_{l-1}} \in \mathrm{MinGen}(\mathrm{in}_{\prec}(\mathfrak{a}))$ be monomials such that for every j, $m_{i_j} = u_{i_j} v_{i_j} w_{i_j}$ with $u_{i_{j+1}} = w_{i_j}$ and $|u_{i_1}| = 1$. Then

$$[u_{i_1} | v_{i_2} w_{i_2} | v_{i_3} w_{i_3} | \ldots | v_{i_l} w_{i_l}]$$

is **fully attached** if for all j and each prefix u of $v_{i_{j+1}} w_{i_{j+1}}$ the monomial $v_{i_j} w_{i_j} u$ is in $\mathcal{G}$.

We write $\mathcal{B}_j$ for the set of fully attached j-uples.

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We write $\mathcal{B}_j$ for the set of fully attached j-uples.

Remark

Critical cells are exactly the fully attached ones: $\mathcal{B}_j^{\mathcal{M}} = \mathcal{B}_j$.

ANOTHER DESCRIPTION

The *height* of a cell $[w_1|\dots|w_n]$ is the maximal h such that $[w_1|\dots|w_h]$ is f.a.

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We want f.a. to be critical, so

$$e = [w_1|\dots|w_n] \to f$$

with

- $f = [w_1|\dots|w_h w_{h+1}|\dots|w_n]$ if collapsible of height h
- $f = [x_1|w_1'|w_2|\dots|w_n]$ if redundant of height $h = 0$ with $w_1 = w_1 w_1'$
- $f = [w_1|\dots|w_h|u_h|v_h|w_{h+2}|\dots|w_n]$ if redundant of height $0 < h < n$ with u_h minimal such that $u_h v_h = w_h$ and $w_h u_h$ reducible

BIBLIOGRAPHICAL REMARKS

- The first proof of this kind is: **K. S. Brown**. The geometry of rewriting systems: a proof of the Anick-Groves- Squier theorem. In *Algorithms and classification in combinatorial group theory* (Berkeley, CA, 1989), volume 23 of Math. Sci. Res. Inst. Publ., pages 137–163. Springer, New York, 1992.
- It was topological and algebraized in: **D. E. Cohen**. String rewriting and homology of monoids. *Math. Structures Comput. Sci.*, 7(3):207–240, 1997.
  (which simplifies an article of 1993)

Let's describe
the Morse differential

REDUCTION RULES

We write $\mathcal{R}$ for the reduction rules associated to the Gröbner basis. These are of the form

$$v_1 v_2 \xrightarrow{a_w} w$$

with

- $v_1, v_2 \in \mathcal{G}$,
- $v_1 v_2 \notin \mathcal{G}$,
- $v_1 \cdot v_2 = a_0 + \sum_{w \in \mathcal{G}} a_w w$.

TYPES OF REDUCTIONS

Suppose given a tuple of standard monomials

$$e_1 \quad = \quad [w_1|\dots|w_l]$$

We define the following three types of reductions from e_1 to e_2:

I. $e_1 \xrightarrow{-a}_I e_2$
   - e_1 fully attached
   - $e_2 = [w_1|\dots|w_{i-1}|v_i|v_{i+1}|w_{i+2}|\dots|w_l]$
   - $[w_1|\dots|w_{i-1}|v_i]$ fully attached
   - $v_i v_{i+1} \in \mathcal{G}$
   - $w_i w_{i+1} \xrightarrow{a} v_i v_{i+1} \in \mathcal{R}$ with $a \neq 0$

II. $e_1 \xrightarrow{(-1)^i a}_{II} e_2$
   - $e_2 = [w_1|\dots|w_{i-1}|v|w_{i+2}|\dots|w_l]$ fully attached
   - $w_i w_{i+1} \xrightarrow{a} v \in \mathcal{R}$ with $a \neq 0$

III. $e_1 \xrightarrow{w_1}_{III} e_2$
   - $e_2 = [x_{i_2}|w_2'|w_3|\dots|w_l]$
   - $w_2 = x_{i_2} w_2'$

TYPES OF REDUCTIONS

Given $e = [w_1|\dots|w_l]$ and $f = [v_1|\dots|v_{l-1}]$ fully attached,

$$e \xrightarrow{c} f$$

whenever either

1. $e = e_0 \xrightarrow{-a_1} e_1 \xrightarrow{-a_2} e_2 \xrightarrow{-a_3} \dots \xrightarrow{-a_r} e_r = [u|f]$
   with reductions of type I and III,
   and $c = ((-1)^r \prod_{i=1}^r a_i)u$
2. $e = e_0 \xrightarrow{-a_1} e_1 \xrightarrow{-a_2} e_2 \xrightarrow{-a_3} \dots \xrightarrow{-a_r} e_r \xrightarrow{(-1)^j b} f$
   with reductions of type I and III, excepting the last of type II,
   and $c = (-1)^{r+j} b \prod_{i=1}^r a_i$

There may be multiple paths from e to f with different coefficients, in this case the coefficient $[e : f]$ is the sum over paths (and 0 if there is no path).

THE MORSE RESOLUTION

The Morse complex $F_\bullet$ is then

$$F_j = \bigoplus_{e \in \mathcal{B}_j} Ae$$

and $\partial_i : F_i \to F_{i-1}$ by

$$\partial_i(e) = \sum_{f \in \mathcal{B}_{i-1}} [e : f] f$$

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Remark

This is Anick's resolution!

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Theorem

$F_\bullet$ is an A-free resolution of the field $\mathbb{K}$. It is minimal iff no type II reduction is possible.

MINIMAL RESOLUTIONS

Theorem

$F_\bullet$ is an A-free resolution of the field $\mathbb{K}$. It is minimal iff no type II reduction is possible.

Proof.

Fully attached tuples are exactly the critical cells ($\mathcal{B}_j^{\mathcal{M}} = \mathcal{B}_j$).
Reduction rules describe the Morse differential:

$$\partial^{\mathcal{M}}([w_1|\dots|w_l]) = w_1[w_2|\dots|w_l] + \sum_{i=1}^{l-1} (-1)^i [w_1|\dots|w_i w_{i+1}|\dots|w_l]$$

If $e = [w_2|\dots|w_l] \notin \mathcal{B}$, we have $e = \partial([x_{i_2}|w_2'|w_3|\dots|w_l])$ which is described by type III reductions.
For $[w_1|\dots|w_i w_{i+1}|\dots|w_l]$, we distinguish three cases... □

MINIMAL RESOLUTIONS

Proof.

For $[w_1|\dots|w_i w_{i+1}|\dots|w_l]$, we distinguish three cases:

1. $[w_1|\dots|v_{ij}|\dots|w_l]$ is critical. $w_{i-1}v_{ij}$ and $v_{ij}w_{i+2}$ reducible, $w_{i-1}u_1 \in \mathcal{G}$ and $v_{ij}u_2 \in \mathcal{G}$ for every prefix u_1 of v_{ij} and u_2 of w_{i+2}. Situation described by reductions of type II.
2. $[w_1|\dots|v_{ij}|\dots|w_l]$ is matched by a higher-degree cell. $w_{i-1}u_1$ reducible for $v_{ij} = u_1u_2$, and $W_{i-1}u' \in \mathcal{G}$ for prefixes u' of u_1. Then $[w_1|\dots|v_{ij}|\dots|w_l] = (-1)^{i+1}[w_1|\dots|u_1|u_2|\dots|w_l]$ which is a reduction of type I.
3. $[w_1|\dots|v_{ij}|\dots|w_l]$ is matched by a lower-degree cell. We have $[w_1|\dots|v_{ij}|\dots|w_l] = 0$.

The Morse differential is thus obtained by the $e \xrightarrow{c} f$ reductions: I/III reductions, ended by either a reduction of type II or $[v_1|\dots|v_l] \xrightarrow{v_1} [v_2|\dots|v_l]$. □

COROLLARY

Proposition

The following are equivalent:

1. $(F_\bullet, \partial)$ *is not minimal*
2. *There is a reduction of type II*
3. *There exist standard monomials* $w_1, \ldots, w_4 \in \mathcal{G}$ *and* $m_1, m_2, m_3 \in \mathrm{MinGen}(\mathrm{in}_\prec(\mathfrak{a}))$ *such that*
   - $w_1 w_2 = u_1 m_1$
   - $w_2 w_3 = u_2 m_2$
   - $w_1 w_4 = u_1' m_3$

   with
   - u_1, u_1' *suffixes of* w_1
   - u_2 *suffix of* w_2
   - $w_2 w_3 \to w_4 \in \mathcal{R}$

AN EXAMPLE OF NON-MINIMAL ANICK

Consider the convergent rewriting system

$$\langle a, b, c, d, e \mid abb \xrightarrow{A} ee, abdd \xrightarrow{B} eec, bc \xrightarrow{C} dd \rangle$$

The words

$$w_1 = aa \qquad w_2 = bb \qquad w_3 = c \qquad w_4 = bdd$$

satisfy the hypothesis of previous proposition:

$$w_1 w_2 = a|abb \qquad w_2 w_3 = b|bc$$
$$w_1 w_4 = a|abdd \qquad w_2 w_3 = bbc \to bdd = w_4 \in \mathcal{R}$$

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$$w_1 w_4 = a|abdd \qquad w_2 w_3 = bbc \to bdd = w_4 \in \mathcal{R}$$

We have

- $F_1 = \{[a], [b], [c], [d], [e]\}$
- $F_2 = \{[a|bb], [a|bd], [b|c]\}$
- $F_3 = \{[a|bb|c]\}$
- $F_{i>3} = \emptyset$

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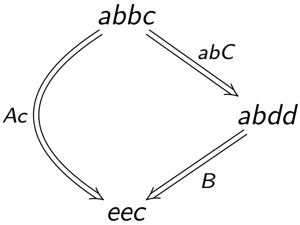
The critical pair:



Therefore B is not necessary!

AN EXAMPLE OF NON-MINIMAL ANICK

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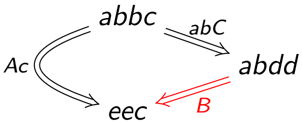
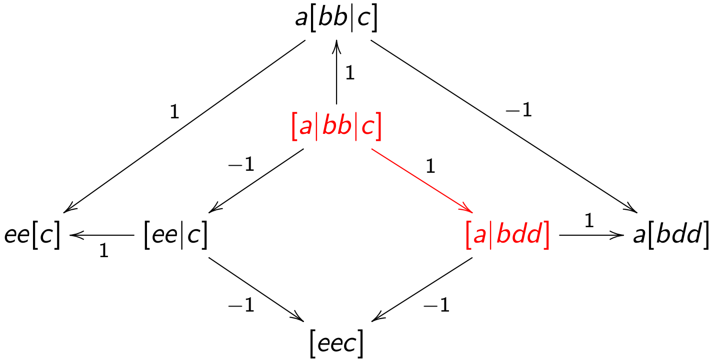


induces in the matching graph



COROLLARY

Proposition

The resolution $F_\bullet$ is minimal in the following cases

1. $\mathfrak{a}$ *admits a monomial Gröbner basis*
2. *The Gröbner basis of* $\mathfrak{a}$ *consists of homogeneous polynomials all of the same degree*

Proof.

1. Obvious.
2. We write l for the degree of the Gröbner basis. We have $w_1 w_4 = u_1' m_{i_3}$ with $w_1, w_4 \in \mathcal{G}$ and $m_{i_3} \in \text{MinGen}(\text{in}_\prec(\mathfrak{a}))$, therefore $|w_4| < l$. However, $w_2 w_3 \to w_4 \in \mathcal{R}$ implies $|w_4| = l$. Contradiction. □

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Remark

In the case of a quadratic algebra, we recover the Koszul complex!

MINIMAL VS KOSZUL (?)

The following example shows that even in the case where the Gröbner basis is not finite one can apply our theory:

Example 4.12. Consider the two-side ideal $\mathfrak{a} = \langle x^2 - xy \rangle$. By [**19**] there does not exist a finite Gröbner basis with respect to degree-lex for $\mathfrak{a}$. One can show that $\mathfrak{a} = \langle xy^n x - xy^{n+1} \mid n \in \mathbb{N} \rangle$ and that $\{xy^n x - xy^{n+1} \mid n \in \mathbb{N}\}$ is an infinite Gröbner basis with respect to degree-lex.

If one applies our matching from Lemma 4.2, it is easy to see that the critical cells are given by tuples of the form

$$[x|y^{n_1}|x|y^{n_2}|x|\ldots|x|y^{n_l}|x] \text{ and } [x|y^{n_1}|x|y^{n_2}|x|\ldots|x|y^{n_l}]$$

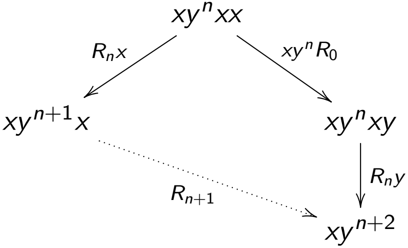
with $n_1, \ldots, n_l \in \mathbb{N}$.
A degree argument implies that the Morse complex is even a minimal resolution. Therefore, we get a minimal resolution $F_\bullet$ of k over $A = k\langle x_1, \ldots x_n \rangle / \mathfrak{a}$.

In this case, this proves that k does not admit a linear resolution and hence A is not Koszul.

MINIMAL VS KOSZUL (?)

We start from $R_0 : xx \to xy$ with $x > y$.

If we have $R_n : xy^n x \to xy^{n+1}$, we have by Knuth-Bendix



So we have all the $R_n : xy^n x \to xy^{n+1}$ for $n \in \mathbb{N}$.

MINIMAL VS KOSZUL (?)

More generally, the form for critical pairs is

$$
\begin{array}{ccccc}
 & & xy^n xy^m x & & \\
 & \swarrow^{R_n y^m x} & & \searrow^{xy^n R_m} & \\
xy^{n+m+1}x & & & & xy^n xy^{m+1} \\
 & \searrow_{R_{n+m+1}} & & \swarrow_{R_n y^{m+1}} & \\
 & & xy^{n+m+2} & &
\end{array}
$$

MINIMAL VS KOSZUL (?)

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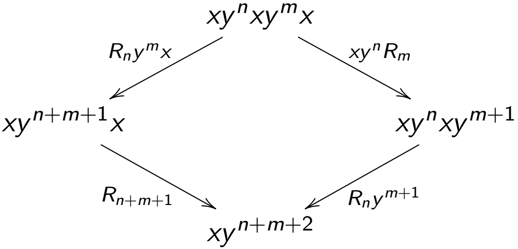

By Morse reduction, this cell is collapsible with R_{n+m+1} redundant. Moreover R_{n+m+1} can be expressed in terms of R_i with $i < n+m+1$; so we can remove any (every?) R_n with $n > 0$.

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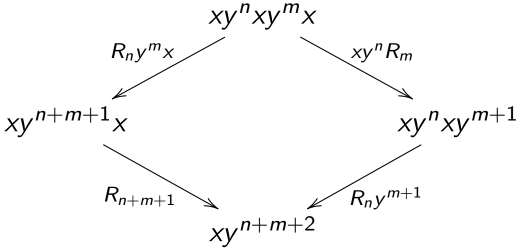

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Actually, Knuth-Bendix always produces collapsible cells...

MINIMAL VS KOSZUL (?)

If we choose $x < y$, we have

$$xy \to xx$$

There is no critical pair and the Anick resolution is

- $F_1 = \{[x], [y]\}$
- $F_2 = \{[x|y]\}$
- $F_3 = \emptyset$
- ...

Which is obviously minimal...

ANOTHER PROBLEM

(certainly not fundamental but. . .)

If we consider $A = \langle x, y \mid x \to y \rangle$, the Anick resolution is not minimal:

- $F_1 = \{[x], [y]\}$
- $F_2 = \emptyset$
- $\ldots$

and $A = \langle x \mid \rangle$.

EXTENSIONS

This can be extended to

- commutative algebras
- Hochschild resolution of A as an $A \otimes A^{\mathrm{op}}$-module
- ...