DELOOPING CYCLIC GROUPS WITH LENS SPACES IN HOMOTOPY TYPE THEORY

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Homotopy type theory

There are various levels of interpretation of logic:

-1. types are booleans

$$A \vee (B \wedge C)$$

0. types are sets

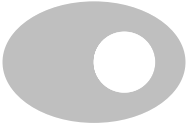
$$\mathbb{N} \to (\mathbb{N} \times \mathbb{Z})$$

∞. types are spaces

$$\Omega(\Sigma A * \Sigma B)$$

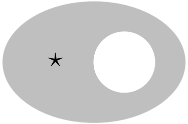
Loop spaces

Suppose given a **space** A, i.e. a type.



Loop spaces

Suppose given a space A which is **pointed** by $\star : A$.



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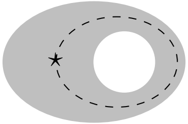

Its **loop space** is

$$\Omega A \quad := \quad \star = \star$$

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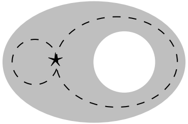

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it looks like a **group**

- we can concatenate paths,
- we can take path backwards,
- etc.

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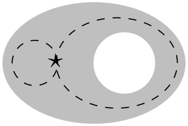

Its **loop space** is

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it looks like a **group**

- we can concatenate paths,
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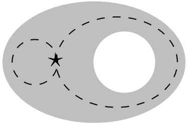
excepting that it might not be a set!

Loop spaces

Suppose given a space A which is pointed by $\star : A$ and a **groupoid**.



Its **loop space** is

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it is a **group**

- we can concatenate paths,
- we can take path backwards,
- etc.

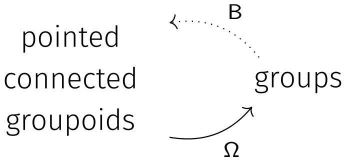
Delooping of groups

We have a map

$$\begin{array}{c}\text{pointed}\\ \text{connected}\\ \text{groupoids}\end{array} \xrightarrow{\ \Omega\ } \text{groups}$$

Delooping of groups

Do we have a map



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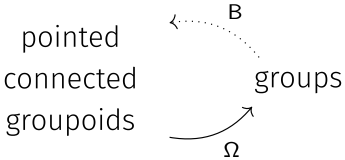

Given a group G, a **delooping** is a space $\mathrm{B}\,G$ such that

$$\Omega\,\mathrm{B}\,G = G$$

Those are very useful:

- in order to perform group theory internally,
- to compute invariants (homology, cohomology, ...)
- etc.

Delooping of groups

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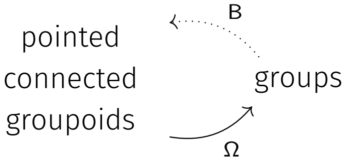

Given a group G, a **delooping** is a space $\mathrm{B}\,G$ such that

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In fact, B exists and the above is an equivalence!

Delooping of groups

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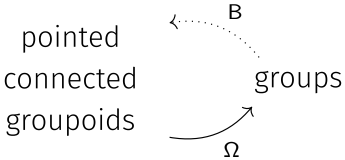

Given a group G, a **delooping** is a space $\mathrm{B}\,G$ such that

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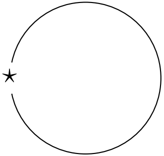
For instance, $\mathrm{B}\,\mathbb{Z} = \mathsf{S}^1$:



Constructing deloopings

Suppose that we want to construct a delooping of $\mathbb{Z}_2$.

Let's try with a higher inductive type A_0 generated by

- $\star : A_0$

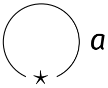
Its loop space ΩA_0 is

$$0$$

Constructing deloopings

Suppose that we want to construct a delooping of $\mathbb{Z}_2$.

Let's try with a higher inductive type A_1 generated by

- $\star : A_1$
- $a : \star = \star$, i.e. $A_1 =$ 

Its loop space ΩA_1 is

$$\cdots \qquad a^{-1} \qquad a^0 \qquad a^1 \qquad a^2 \qquad a^3 \qquad \cdots$$

Constructing deloopings

Suppose that we want to construct a delooping of $\mathbb{Z}_2$.

Let's try with a higher inductive type A_2 generated by

- $\star : A_2$
- $a : \star = \star$
- $p : aa = 1$

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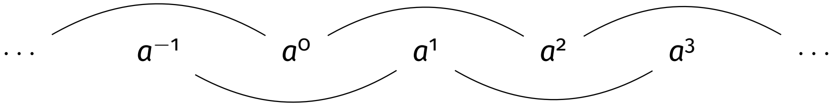

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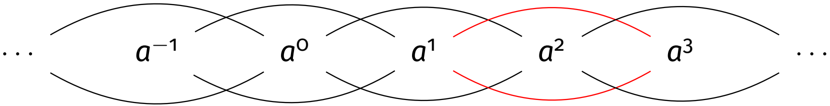

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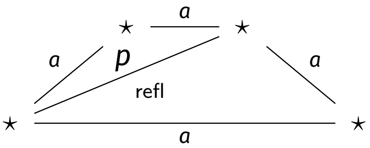

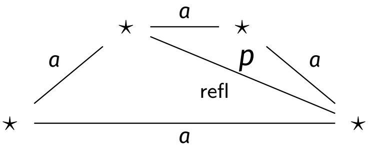

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Suppose that we want to construct a delooping of $\mathbb{Z}_2$.

Let's try with a higher inductive type A_3 generated by

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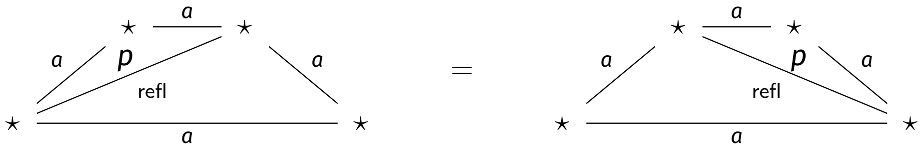
- $p : aa = 1$
- $pa = ap$

Its loop space ΩA_3 is



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Its loop space ΩA_3 is

We should keep on adding identities between identities forever… But which?

If we stop after n steps, we obtain an "approximation" A_n of $\mathrm{B}\,G$ up to dimension n.

A_{n+1} is obtained from A_n by making the canonical map $A_n \to \mathbb{Z}_2$ "more injective".

Constructing deloopings

One way to handle this is to use **truncation** (Finster-Licata, LICS'14):

Theorem

The following type A is a delooping of $\mathbb{Z}_2$:

- $\star : A$
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- $\mathrm{isGroupoid}(A)$

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- $\mathrm{isGroupoid}(A)$

Problem: because truncation is formal, it's very difficult to use in practice.

Delooping $\mathbb{Z}_2$ using real projective spaces

The **real projective spaces**

$$\mathbb{R}\mathrm{P}^n \quad = \quad \{\text{lines in } \mathbb{R}^n\}$$

are the "topological analogous" of the n-approximation and we can define

$$\mathrm{B}\,\mathbb{Z}_2 \quad = \quad \mathbb{R}\mathrm{P}^\infty$$

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Those were defined in homotopy type theory (Buchholtz-Rijke, LICS'17).

Delooping $\mathbb{Z}_m$ using real projective spaces

Here, we define **lens spaces**

$$\mathsf{L}_m^n \quad = \quad \text{quotient of } \mathsf{S}^{2n-1} \subseteq \mathbb{C}^n \text{ under some rotations}$$

which are such that

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The general approach is the same as for projective spaces
although generalization is not straightforward.

The general approach

The general approach is as follows:

- we know that a type $\mathrm{B}\,\mathbb{Z}_m$ exists
- we iteratively construct a family of types A_n together maps

$$f_n : A_n \to \mathrm{B}\,\mathbb{Z}_m$$

  which are $(n-1)$-connected:

$$\|\operatorname{fib}(f_n)\|_{n-1} = 1$$

Getting started

As first approximation to $\mathsf{B}\,\mathbb{Z}_m$ (a pointed connected groupoid), we can take

$$f_0 : \mathbf{1} \to \mathsf{B}\,\mathbb{Z}_m$$

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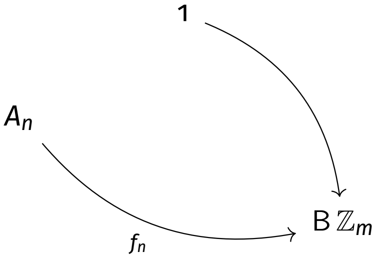
$$f_0 : \mathbf{1} \to \mathsf{B}\,\mathbb{Z}_m$$

Note: any map $X \to \mathsf{B}\,\mathbb{Z}_m$ would actually work as long as X contains a point

The inductive step

In order to compute f_{n+1}, we compute

which corresponds to



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$$\ker(f_n) = \sum(x : A_n).(f_n(x) = \star)$$

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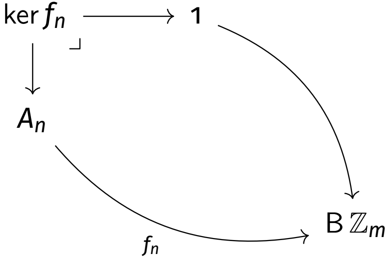

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In order to compute f_{n+1}, we compute

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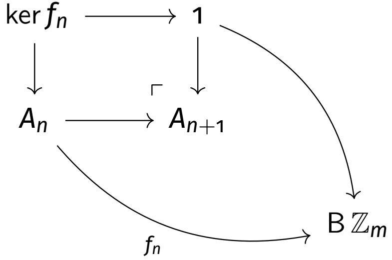

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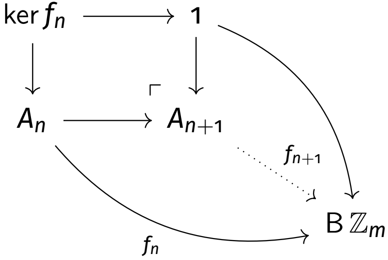


Lens spaces

Definition
If we begin with a map

$$S^1 \to B\,\mathbb{Z}_m$$

and iterate the same construction, we obtain types L_n which correspond to **lens spaces**.

Theorem

$$L_\infty = B\,\mathbb{Z}_m$$

Lens spaces

Definition

If we begin with **any** map

$$S^1 \to \mathrm{B}\,\mathbb{Z}_m$$

and iterate the same construction, we obtain types L_n which correspond to **lens spaces**.

Theorem

$$L_\infty = \mathrm{B}\,\mathbb{Z}_m$$

Results and applications

Theorem

We have a pushout

$$\begin{array}{ccc} (\mathrm{B}\,\mathbb{Z}^2)^{*_{S^1} n} & \longrightarrow & S^1 \\ \downarrow & & \downarrow \\ L_n & \longrightarrow & L_{n+1} \end{array}$$

from which we can (hope to)

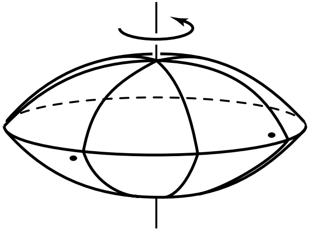
- define actions of G on higher types

$$\mathrm{B}\,G \to \mathcal{U}$$

- compute cohomology of $\mathbb{Z}_m$

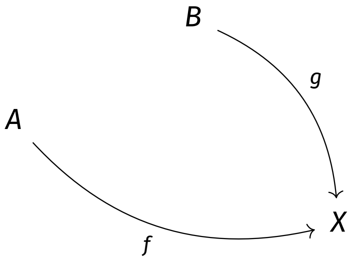
$$H^n(\mathbb{Z}_m) := \|\,\mathrm{B}\,\mathbb{Z}_m \to K(\mathbb{Z}, n)\|_0$$

Questions?



Join of maps

Given two maps



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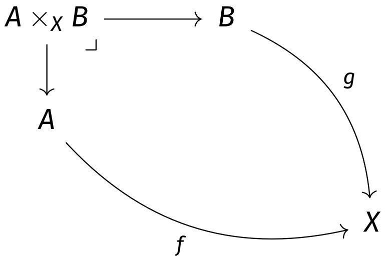

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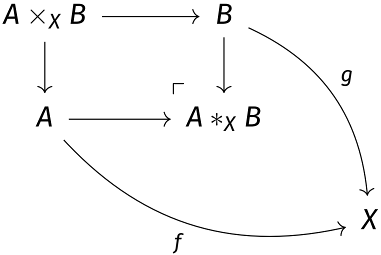

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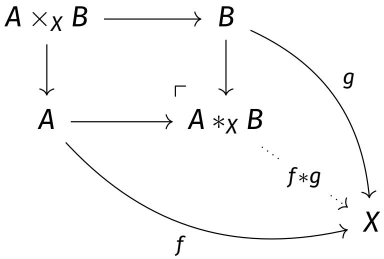

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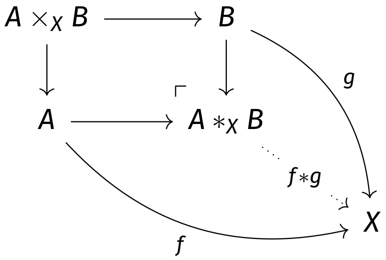

Lemma

If f is m-connected and g is n-connected then $f * g$ is $(m+n)$-connected.

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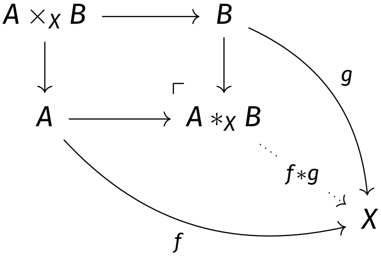

Lemma

If f is m-connected and g is n-connected then $f * g$ is $(m+n)$-connected.

Lemma

Given $f : A \to B$ where A has a point, f^{*n} converges toward an equivalence.

Future application: defining actions on higher types

An **action** of a group G on a set X is a map

$$G \times X \to X(a, x) \qquad \mapsto a \cdot x$$

satisfying

$$a \cdot (b \cdot x) = (a \times b) \cdot x \qquad 1 \cdot x = x$$

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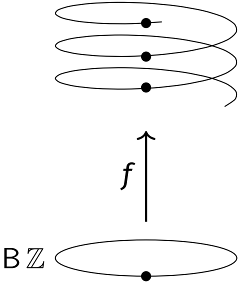

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An **action** of a group G on a type X is a map

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with $X := f(\star)$.

With the definition of $\mathrm{B}\,G$ as a HIT, we have $\mathsf{isGroupoid}(\mathrm{B}\,G)$ and we can only eliminate to a groupoid, e.g. define

$$f : \mathrm{B}\,G \to \mathbf{Set}$$

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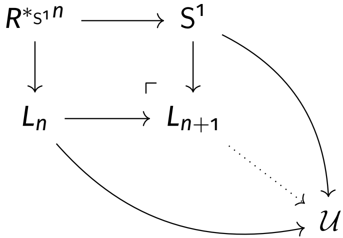

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with $R = \mathrm{B}\, \mathbb{Z}^2$ *and a map* $\mathrm{B}\, \mathbb{Z}_m \to \mathcal{U}$ *is the limit of maps* $L_n \to \mathcal{U}$.

Future application: computing cohomology groups

The n-th **cohomology group** of $\mathbb{Z}_m$ is

$$H^n(\mathbb{Z}_m) :\equiv \| \mathsf{B}\,\mathbb{Z}_m \to K(\mathbb{Z}, n) \|_0$$