THE ALGEBRAIC STRUCTURE OF FIRST ORDER CAUSALITY

SAMUEL MIMRAM

LOGIC AND INTERACTIONS

ALGEBRA AND COMPUTATION

28 FEBRUARY 2012

IN THIS TALK

- Uses of higher-dimensional rewriting techniques to study models of logics and computation.
- Techniques to study rewriting in monoidal categories (PROs).
- We don't necessarily need termination to have normal forms.

GAME
SEMANTICS

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- morphisms are programs of type $A \to B$, up to reduction

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For instance, one can try to interpret the language in **Set**:

$$[\![\texttt{int}]\!] = \mathbb{N} \qquad [\![\texttt{A} * \texttt{B}]\!] = [\![\texttt{A}]\!] \times [\![\texttt{B}]\!]$$

$$[\![\texttt{+:int * int -> int}]\!] \quad = \quad + : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

TOWARDS GAME SEMANTICS

This naive interpretation in **Set** will fail in general.

- Most programming languages require more structure than we have in **Set**:

  ```
  x = 0; while (x < 5) { x = x + 1 }; return x
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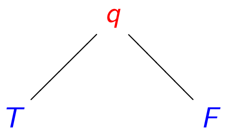
We have to interpret our programming language in categories $\mathcal{C}$ which are "richer" than **Set**. Here, I will be interested in $\mathcal{C} = \mathbf{Games}$ whose objects are *games* and morphisms are *strategies*.

GAME SEMANTICS

A **game** $G = (M, \leq, \lambda)$ is

- a set M of *moves*
- a partial order $\leq$ on M called *causality*
- a *polarization* function $\lambda : M \to \{O, P\}$

For instance, the game $\mathbb{B}$ corresponding to booleans is



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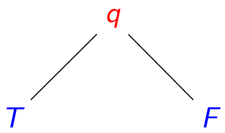
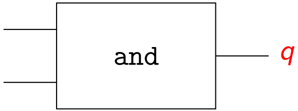


A **strategy** on a game is a set of sequences of moves (*plays*) respecting the order of the game and closed under prefix.

For instance, the strategy corresponding to *true* on $\mathbb{B}$ is $\{\varepsilon, q, qT\}$.

STRATEGIES

The general idea is to see a program as a black-box and see how it reacts to its environment:



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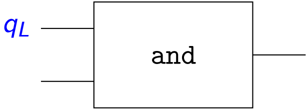

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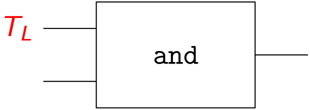

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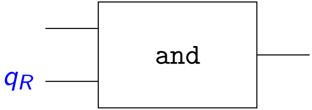

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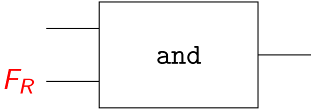

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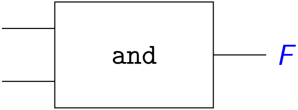

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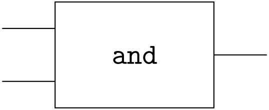

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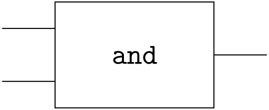


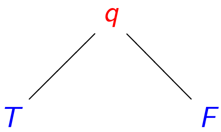
So, the strategy corresponding to `and` will contain

$$q q_L T_L q_R F_R F$$
$$q q_L T_L q_R T_R T$$
$$\ldots$$

A CATEGORY OF GAMES

It was discovered by Joyal that Conway games can be made into a category: given a game $G = (M, \leq, \lambda)$ we define

$\mathbb{B} \quad =$



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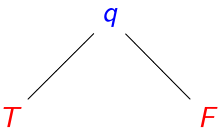

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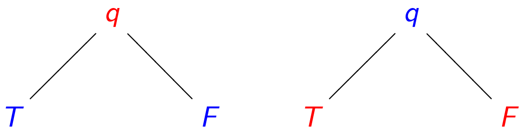

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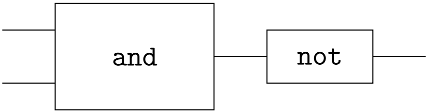

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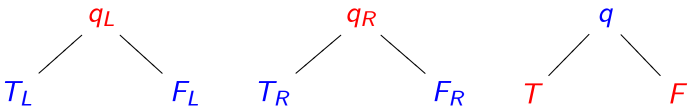

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For instance, $[\![\texttt{and}]\!]$ is a strategy on $\mathbb{B}^* \| \mathbb{B}^* \| \mathbb{B}$.

PRESENTING A GAME SEMANTICS FOR FIRST-ORDER LOGIC

- A lot can be done in this framework, in particular **definable** strategies (in the image of the semantics) can be characterized as *innocent strategies*.

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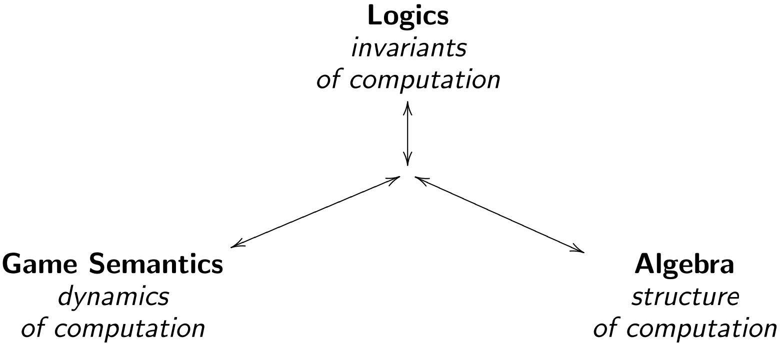
- More specifically, I will be interested here in the **causality** induced by fist-order connectives in logic.
- I will characterize definable strategies by giving a **presentation** of the monoidal category of games for this logic.

UNIFYING POINTS OF VIEW



CAUSALITY
IN FIRST ORDER
LOGIC

FIRST-ORDER LOGIC

- Formulas:

$$A \quad ::= \quad \exists x.A \quad | \quad \forall x.A \quad | \quad A \wedge A \quad | \quad A \vee A \quad | \quad P$$

  where P are given propositions depending on variables, e.g. $P(x, y) = (x = y)$.

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- Rules:

$$\frac{\Gamma \vdash P, \Delta}{\Gamma \vdash \forall x.P, \Delta}(\forall) \qquad \frac{\Gamma \vdash P[t/x], \Delta}{\Gamma \vdash \exists x.P, \Delta}(\exists)$$

(with $x \notin \mathrm{FV}(\Gamma, \Delta)$)

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}(\wedge) \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}(\vee)$$

$$\vdots$$

$+$ consistent axioms depending on the propositions

CAUSALITY IN PROOFS

$$\cfrac{\cfrac{\cfrac{\pi}{\Gamma \vdash A, B, \Delta}}{\Gamma \vdash A, \forall y.B, \Delta}(\forall)}{\Gamma \vdash \forall x.A, \forall y.B, \Delta}(\forall)$$

CAUSALITY IN PROOFS

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CAUSALITY IN PROOFS

$$\dfrac{\dfrac{\dfrac{\pi}{\Gamma \vdash A[t/x], B[t'/y], \Delta}}{\Gamma \vdash A[t/x], \exists y.B, \Delta}(\exists)}{\Gamma \vdash \exists x.A, \exists y.B, \Delta}(\exists) \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\pi}{\Gamma \vdash A[t/x], B[t'/y], \Delta}}{\Gamma \vdash \exists x.A, B[t'/y], \Delta}(\exists)}{\Gamma \vdash \exists x.A, \exists y.B, \Delta}(\exists)$$

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If $x \notin \mathrm{FV}(t)$!

Remark:

$$\Gamma \vdash \forall x.A, \Delta$$

is the same as

$$\Gamma, \exists x.A^* \vdash \Delta$$

CAUSALITY IN PROOFS

For instance, we can permute

$$\dfrac{\dfrac{\dfrac{}{\mathbb{N}(x) \vdash 0 = 0}}{\mathbb{N}(x) \vdash \exists y.(y = 0)}}{\exists x.\mathbb{N}(x) \vdash \exists y.(y = 0)} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{}{\mathbb{N}(x) \vdash 0 = 0}}{\exists x.\mathbb{N}(x) \vdash 0 = 0}}{\exists x.\mathbb{N}(x) \vdash \exists y.(y = 0)}$$

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but not

$$\dfrac{\dfrac{\overline{\mathbb{N}(x) \vdash (x - x) = 0}}{\mathbb{N}(x) \vdash \exists y.(y = 0)}}{\exists x.\mathbb{N}(x) \vdash \exists y.(y = 0)}$$

CAUSALITY IN PROOFS

Dependencies induced by proofs are of the form

$$\forall x \longrightarrow \exists y$$

where the witness t given for y has x as free variable.

This models the **causality** in information.

DIALOGUE GAMES

$\forall x$ $\exists y$ $P(x, y)$

$\forall$belard and $\exists$loise

GAMES

Formulas

$$A \quad = \quad \exists x.A \quad | \quad \forall x.A \quad | \quad A \wedge A \quad | \quad A \vee A \quad | \quad \ldots$$

will be interpreted as games $(M, \lambda, \leq)$:

- a set M of *moves*,
- a partial order $\leq$ on M called *causality*,
- a function $\lambda : M \to \{\forall, \exists\}$ indicating *polarity* ($\forall$: Opponent, $\exists$: Player)

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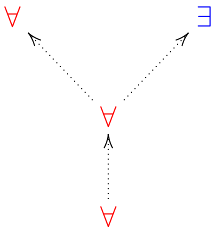
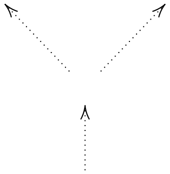


STRATEGIES

strategy = dependency relation on the moves of the game

$$\vdash \forall x.\forall y.(\forall z.P \vee \exists z'.Q)$$

$\rightsquigarrow$



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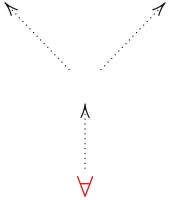

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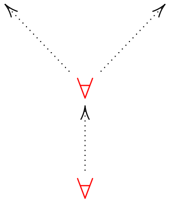

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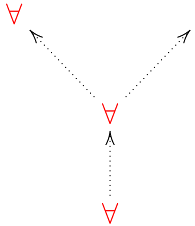

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$$
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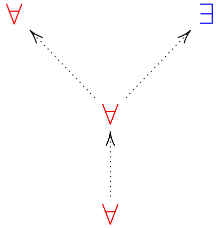

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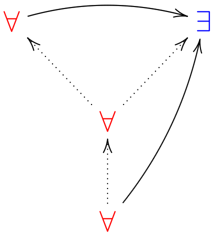

Free variables of t: $\{x, z\}$

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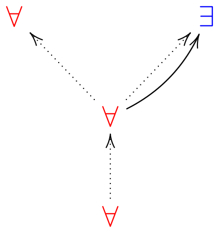

Free variables of t: $\{y\}$

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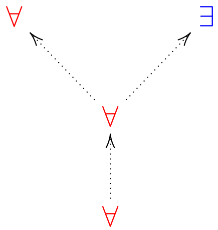

Free variables of t: $\emptyset$

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A strategy $\sigma : A$ should moreover satisfy the following properties

1. Polarity: if $m \; \sigma \; n$ then m opponent and n player move
2. Acyclicity: the relation $\leq_A \; \cup \; \sigma$ is **acyclic**

STRATEGIES

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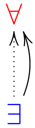
A strategy $\sigma : A$ should moreover satisfy the following properties

1. Polarity: if $m \ \sigma \ n$ then m opponent and n player move

Forbids:



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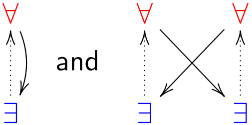

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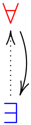
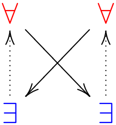


and



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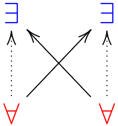


(similar to the correctness criterion of LL)

STRATEGIES

In this context, a strategy σ is a partial order relation on the moves of the game. However, we could easily reformulate σ as the set of plays ($=$ sequences of moves) which respect the partial order σ.

A FIRST STEP

We handle the case where connectives in formulas occur in leaves:

$$\forall x_1 . \forall x_2 . \exists x_3 . \forall x_4 . \forall x_5 . \ \dots \ P(x_{i_1}, \dots, x_{i_k})$$

so games will be *filiform* (= total orders)

INTERPRETING PROOFS

A formula

$$A$$

is interpreted by a game

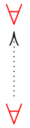
$$[\![A]\!]$$

Example

The formula

$$\forall x.\forall y.P$$

is interpreted by the game



INTERPRETING PROOFS

A sequent

$$A \vdash B$$

is interpreted by a game

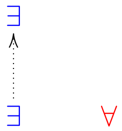
$$[\![A]\!]^* \parallel [\![B]\!]$$

Example

The sequent

$$\forall x.\forall y.P \vdash \forall z.P$$

is interpreted by the game



INTERPRETING PROOFS

A proof

$$\frac{\vdots}{A \vdash B}$$

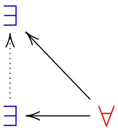
is interpreted by a strategy σ on the game

$$[\![A]\!]^* \parallel [\![B]\!]$$

Example

The proof *is interpreted by the strategy*

$$\dfrac{\dfrac{\dfrac{\dfrac{}{z = z \vdash z = z}}{\forall y.z = y \vdash z = z}}{\forall x.\forall y.x = y \vdash z = z}}{\forall x.\forall y.x = y \vdash \forall z.z = z}$$



A MONOIDAL CATEGORY OF GAMES

We thus build a category **Games**:

- Objects A are filiform games
- Morphisms $\sigma : A \to B$ are strategies on $A^* \| B$

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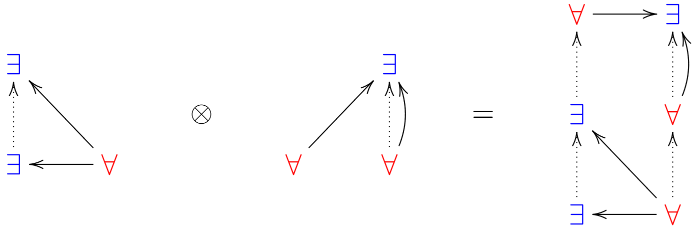

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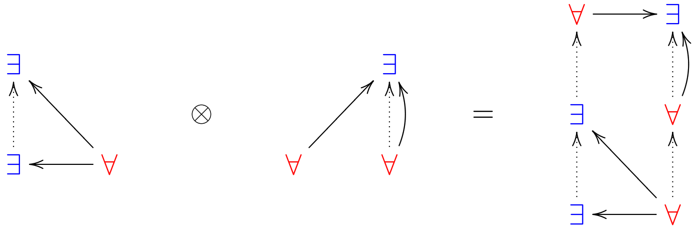

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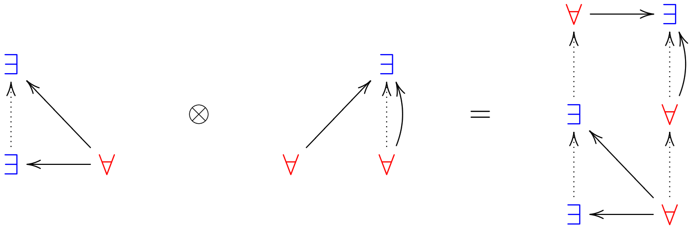

- Strategies $\sigma : A \to B$ are particular relations on $A \times B$, composition is defined as in **Rel**.
- *It is not obvious that the acyclicity condition of strategies is preserved by composition!*

SO WHAT?

This semantics is nice but

- Why do strategies compose?
- We claim that all the strategies are definable
  (i.e. are interpretation of proofs).
  How do we show this?
- What does it tell us about the structure of dependencies?

SO WHAT?

This semantics is nice but

- Why do strategies compose?
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In order to answer these questions, we build a **presentation** the monoidal category **Games**.

PRESENTING THE CATEGORY OF GAMES

ROADMAP

We will progressively introduce presentations of the following monoidal categories.

- **Mat**$_{\mathbb{N}}$ whose objects are integers and morphisms $M : m \to n$ are $m \times n$ matrices with coefficients in $\mathbb{N}$, with the direct sum as tensor product.
- **Rel** whose objects are integers and morphisms $R : m \to n$ are relations on $[m] \times [n]$.
- **Games**.

PRESENTING CATEGORIES

The usual notion of presentation of a monoid has been generalized by Burroni into the notion of n-polygraph which allows to present $(n-1)$-categories.

In particular, a (strict) monoidal category is the same as a 2-category with one 0-cell, so it can be presented by a 3-polygraph.

Polygraphs are
a higher-dimensional generalization
of rewriting systems

POLYGRAPHS

A 0-signature

$$\begin{array}{c}\Sigma_0\\ \downarrow\\ \Sigma_0\end{array}$$

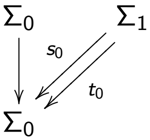
Example

signature

$x \qquad y$

POLYGRAPHS

A 1-polygraph

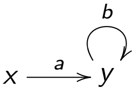


Example

signature

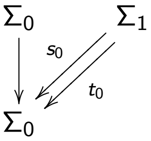
x y

rules



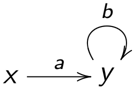
POLYGRAPHS

A 1-signature = a 1-polygraph



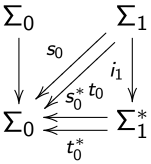
Example

signature



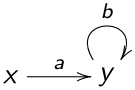
POLYGRAPHS

A 1-signature generates a *category*



Example

signature

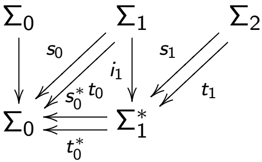


terms

$$x \xrightarrow{a} y \xrightarrow{b} y \xrightarrow{b} y$$

POLYGRAPHS

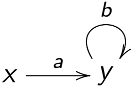
A 2-polygraph



such that $s_0^* \circ s_1 = s_0^* \circ t_1$ and $t_0^* \circ s_1 = t_0^* \circ t_1$

Example

signature



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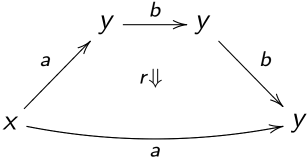
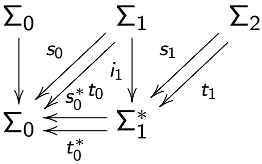


POLYGRAPHS

A 2-signature = a 2-polygraph



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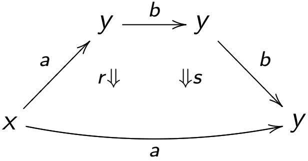
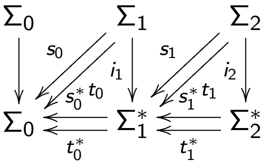


POLYGRAPHS

A 2-signature generates a 2-category



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Example

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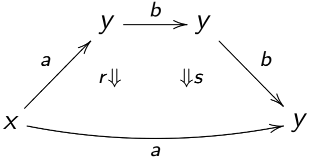
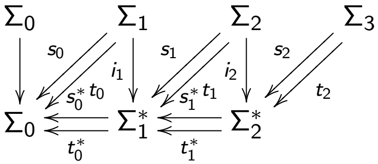


A 3-polygraph



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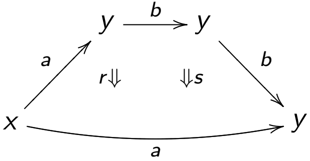
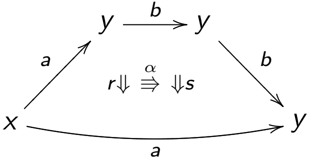
Example

signature



rules



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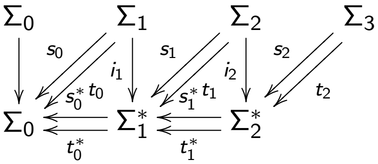

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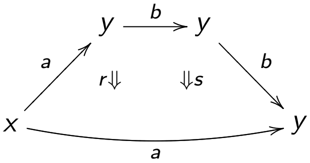

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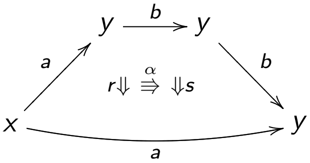
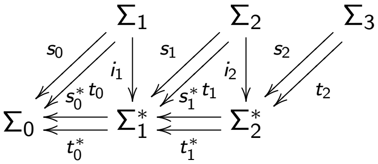


The presented 2-category: Σ^*/Σ_3.

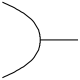
A FEW PARTICULAR CASES



- When $\Sigma_0 = \{*\}$, the presented 2-category has one 0-cell: it is a *monoidal category*.
- When moreover $\Sigma_1 = \{1\}$, the presented monoidal category has integers as objects with tensor given by addition: it is a *PRO*.

PRESENTING $\mathbf{Mat}_{\mathbb{N}}$

The category $\mathbf{Mat}_{\mathbb{N}}$ of matrices with coefficients in $\mathbb{N}$ contains the following morphisms:

- $\mu = \begin{pmatrix} 1 \\ 1 \end{pmatrix} : 2 \to 1$ pictured as 

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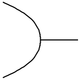
- $\mu = \begin{pmatrix} 1 \\ 1 \end{pmatrix} : 2 \to 1$ pictured as 
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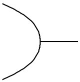
- $\mu = \begin{pmatrix} 1 \\ 1 \end{pmatrix} : 2 \to 1$ pictured as 
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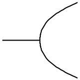
- $\delta = \begin{pmatrix} 1 & 1 \end{pmatrix} : 1 \to 2$ pictured as 

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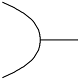
- $\mu = \begin{pmatrix} 1 \\ 1 \end{pmatrix} : 2 \to 1$ pictured as 
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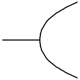
- $\delta = \begin{pmatrix} 1 & 1 \end{pmatrix} : 1 \to 2$ pictured as 
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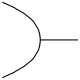
- $\mu = \begin{pmatrix} 1 \\ 1 \end{pmatrix} : 2 \to 1$ pictured as 
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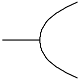
- $\delta = \begin{pmatrix} 1 & 1 \end{pmatrix} : 1 \to 2$ pictured as 
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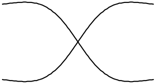
- $\gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ pictured as 

GENERATORS FOR $\mathbf{Mat}_{\mathbb{N}}$

Moreover, it seems that any matrix can be expressed as tensor and product of those generators:

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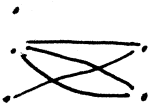

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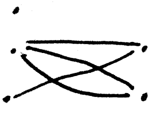
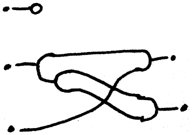


$$\rightsquigarrow$$



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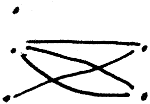

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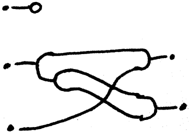


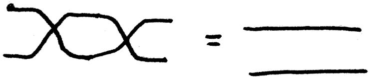
$$= (\mu \otimes \mu) \circ (1 \otimes \gamma \otimes 1) \circ (\gamma \otimes 1 \otimes 1) \circ (1 \otimes \delta \otimes 1) \circ (1 \otimes \delta \otimes \varepsilon)$$

So, we have hope to have a presentation with those generators.

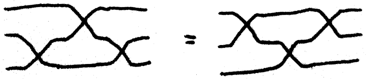
RELATIONS

It is easy to show that in $\mathbf{Mat}_{\mathbb{N}}$, the interpretations of the generators satisfy the following relations.

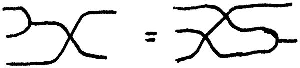
- γ induces a symmetry:
  - It is involutive:

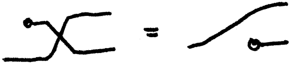


  - It satisfies the Yang-Baxter:



  - It is also "natural" wrt other generators:



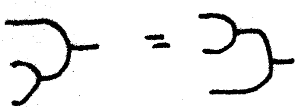


etc.

RELATIONS

It is easy to show that in $\mathbf{Mat}_{\mathbb{N}}$, the interpretations of the generators satisfy the following relations.

- γ induces a symmetry:
- (μ, η, γ) is a commutative monoid:
  - It is associative:



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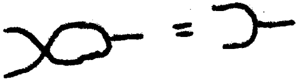

RELATIONS

It is easy to show that in $\mathbf{Mat}_{\mathbb{N}}$, the interpretations of the generators satisfy the following relations.

- γ induces a symmetry:
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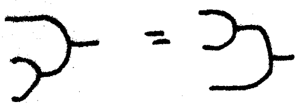

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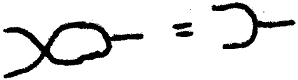

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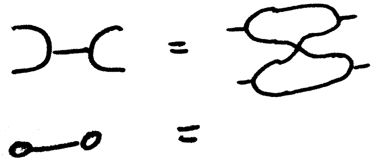
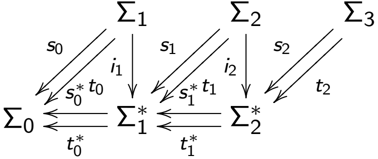
- $(\mu, \eta, \delta, \varepsilon, \gamma)$ is a bialgebra:



A CANDIDATE PRESENTATION

So, we have a candidate for presenting $\mathbf{Mat}_{\mathbb{N}}$:



with

$$\begin{array}{lcl} \Sigma_0 & = & \{*\} \\ \Sigma_1 & = & \{1 : * \to *\} \\ \Sigma_2 & = & \{\mu : 2 \to 1, \eta : 0 \to 1, \delta : 2 \otimes 1, \varepsilon : 1 \to 0, \gamma : 2 \to 2\} \\ \Sigma_3 & = & \{\dots\} \end{array}$$

where 2 is a notation for $1 \otimes 1$.

CANONICAL FORMS

- To show that the polygraph is a presentation of $\mathbf{Mat}_{\mathbb{N}}$, we could
  - orient the relations,
  - show that the rewriting system is terminating,
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  - the normal forms can be difficult to describe
- Here we adopt a semantic approach (adapted from Lafont) by
  - defining (manually) normal forms,
  - show that any morphism rewrites to a normal form,
  - show that the interpretation of normal forms are in bijection with matrices

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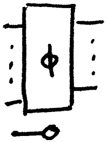
$E\phi \quad =$ 

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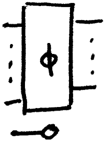
$E\phi \quad =$ 

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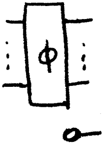
$H\phi \quad =$ 

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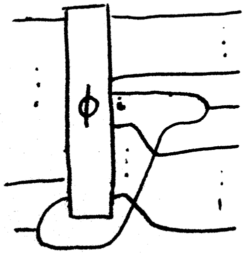

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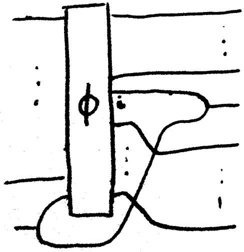
$W_i\phi \quad =$ 

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So, **pcf** are well-typed morphisms generated by the grammar

$$\phi \quad ::= \quad Z \;\mid\; E\phi \;\mid\; H\phi \;\mid\; W_i\phi$$

REDUCING TO PCF

Proposition

Every morphism ϕ is equivalent (by the rules of the presentation) to a pre-canonical form.

Proof.

We use Lafont's "tetris" method, by induction on the *size* of ϕ. Easy for the identity. Otherwise ϕ can be decomposed as $\xi \circ \psi$ where ξ contains exactly one generator and by IH we can suppose that ψ is a pcf. We proceed by case analysis:

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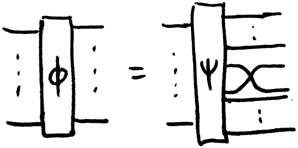

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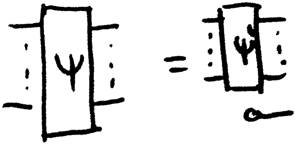

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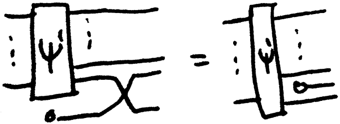

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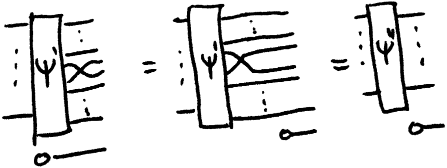

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□

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This means that every matrix M can be described by the following operations:

- $Z : 0 \to 0$: the empty matrix
- EM: add an empty row to the matrix
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However, matrices are not necessarily described in an unique way by this method.

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Canonical forms are pre-canonical forms

$$\phi \quad ::= \quad Z \;\mid\; E\phi \;\mid\; H\phi \;\mid\; W_i\phi$$

(i.e. words over $\{Z, E, H, W_i\}$) which are normal wrt the rewriting system

$$\begin{array}{rcll} HW_i & \Longrightarrow & W_{i+1}H & \\ HE & \Longrightarrow & EH & \\ W_iW_j & \Longrightarrow & W_jW_i & \text{when } i < j \end{array}$$

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Two pcf ϕ and ψ such that $\phi \Longrightarrow \psi$ are equivalent wrt the relations:

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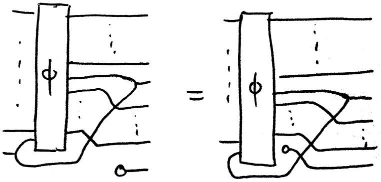

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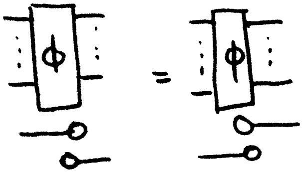

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The canonical forms are of the form

$$W_{n-1}^{k(m-1,n-1)} \dots W_0^{k(m-1,0} E \dots\dots\dots EW_{n-1}^{k(0,n-1)} \dots W_0^{k(0,0)} E \underbrace{H \dots H}_{n \text{ times}} Z$$

where

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These results had essentially been proved earlier by Lafont, however the "two-staged method" (pcf/cf)

- avoids having to introduce extra generators
- makes easy to see that nf correspond to matrices
- generalize easily to a presentation of **Games**
  (of course other methods could be used to show this result)

A PRESENTATION OF Rel

The category **Rel** can be presented by the same polygraph with the following extra relation added:

(1)

A bialgebra with (1) is called **qualitative**.

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The fact that is a presentation can be shown directly or by showing that

$$\mathbf{Rel} \quad \cong \quad \mathbf{Mat}_{\mathbb{N}} / \approx$$

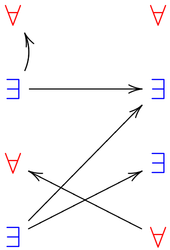
where $\approx$ identifies two matrices with the same non-null coefficients, i.e.

$$\mathbf{Rel} \quad \cong \quad \mathbf{Mat}_{\mathbb{Z}_2}$$

and showing that adding (1) precisely does this quotienting.

PRESENTING Games

Similarly, we can find a presentation for the category of strategies:



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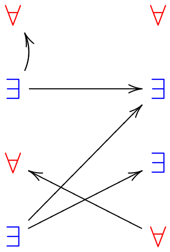
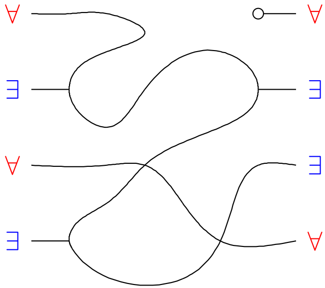


$\rightsquigarrow$



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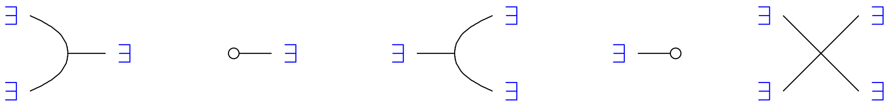
It can be shown that our initial category of games and strategies can be presented by a 3-polygraph such that

$$\Sigma_0 = \{*\} \qquad \Sigma_1 = \{\exists, \forall\}$$

where Σ_2 contains generators



with relations making it a qualitative bicommutative bialgebra (as in the previous presentation)

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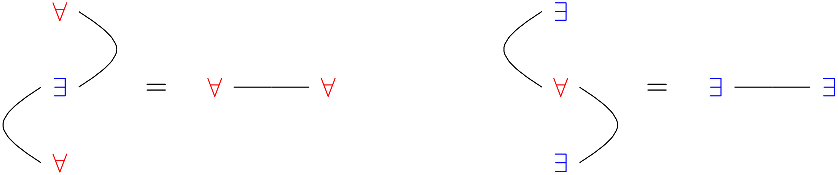

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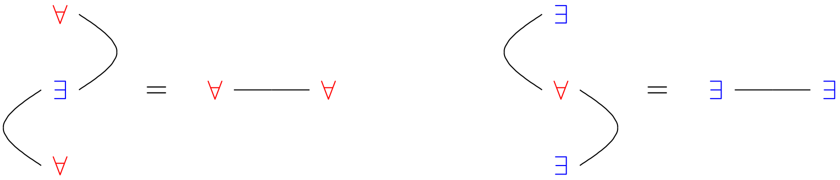
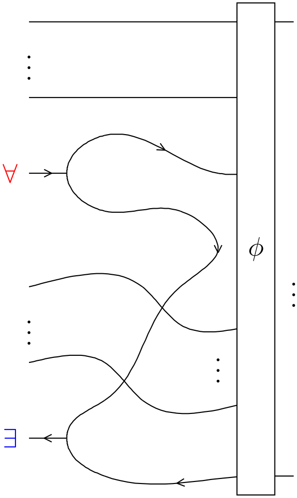


(by duality $\forall$ also has a structure of qualitative bicom. bialgebra)

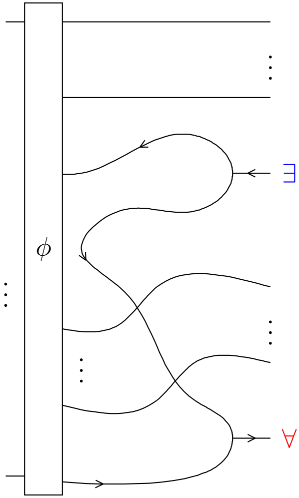
PRE-NORMAL FORMS

Similar to before (but typed) with the following productions added:

$A_i\phi =$

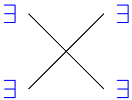


$B_i\phi =$



INTERPRETATION OF THE GENERATORS

For instance



gets interpreted as

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x = y \vdash y = x}}{x = y \vdash \exists v.y = v}}{x = y \vdash \exists u.\exists v.u = v}}{\exists y.x = y \vdash \exists u.\exists v.u = v}}{\exists x.\exists y.x = y \vdash \exists u.\exists v.u = v}$$

etc.

The category **Games**
is the
free monoidal category
containing a
qualitative bicommutative bialgebra
and its dual.

TECHNICAL BYPRODUCTS

From this presentation we deduce that

- strategies do **compose**
  (the acyclicity condition is preserved by composition)
- strategies are **definable**
  (i.e. are the interpretations of proofs)

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We have replaced an *external* definition of the category **Games**:

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by an *internal* definition:

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- **generating**
- local correctness

We ought to be able to do the same with the correctness criterion of linear logic.

ABOUT PROOFS

- The proofs are very boring, systematic and involves considering lots of cases. . .

ABOUT PROOFS

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- ...which is good news: we can hope *mechanization*

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- This methodology can be applied to give presentation of many categories where we know how to canonically “enumerate” the morphisms, without having to prove termination or local confluence.

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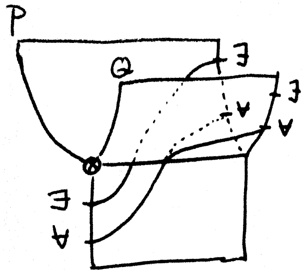
- However, we can in many cases get a convergent rewriting system.
- If we restrict the presentation of $\mathbf{Mat}_{\mathbb{N}}$ to (μ, η, γ), we get essentially a presentation of the terminal operad *Com*: having an explicit symmetry avoids having to consider shuffle operads.

FORMULAS WITH CONNECTIVES

We should be able to extend this to formulas with connectives (ongoing work), this requires going up one dimension:

$$\frac{\cdots}{\forall x.\exists y.(P \otimes Q) \vdash (\forall s.\exists t.P) \otimes (\forall u.\exists v.Q)}$$

gets interpreted by the following surface diagram



First order connectives somehow act as the tensorial negation in current Melliès work!...

Thanks!

Any question?