KOSZULITY OF PROPERADS BY REWRITING

SAMUEL MIMRAM

CATHRE meeting

MAY 22, 2014

Préface

A work in progress

joint with Bruno Vallette and Sinan Yalin

but all errors are *mine*!

(thanks also to Éric Hoffbeck)

A CONVERGENT PRESENTATION OF THE FROBENIUS PRO

Rewriting systems

A **rewriting system** consists of rules which are pairs of “terms” (elements of some free stuff):

$$f \overset{r}{\Longrightarrow} g$$

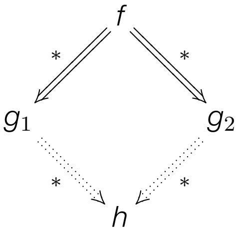
Given a context C, we say that $C[f]$ *rewrites to* $C[g]$, and write

$$C[f] \overset{C[r]}{\Longrightarrow} C[g]$$

Convergent rewriting systems

A rewriting system is

- **confluent** when



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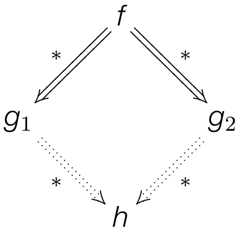

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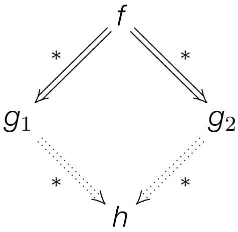

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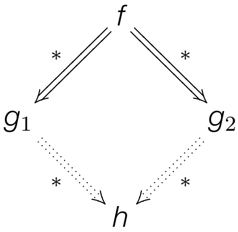

- **terminating** when there is no infinite rewriting sequence
- **convergent** when both terminating and confluent:
  in this case normal forms provide canonical representatives
  of terms modulo the congruence generated by the rules

Definition

A **PRO** is a monoidal category whose objects are integers and tensor product is given by addition.

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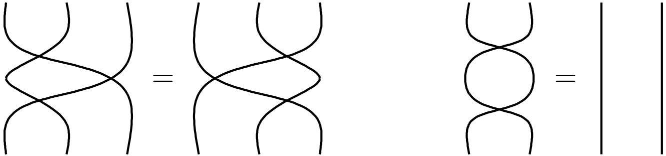
Example

The PRO **Bij** is presented by  $: 2 \to 2$ with relations



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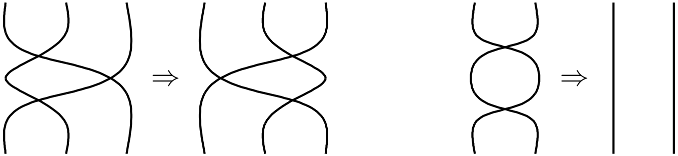
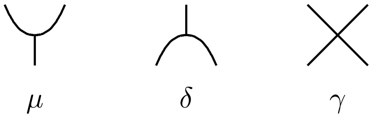


Theorem (Lafont)

This is a convergent presentation.

The PRO **FRO**

FRO is the PRO generated by



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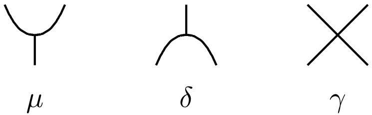
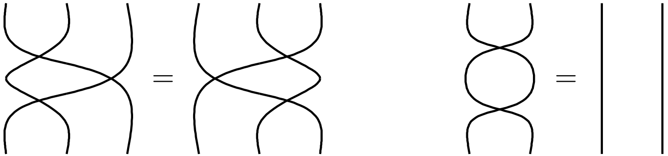


such that

- symmetry satisfies Yang-Baxter and is involutive



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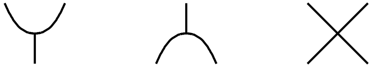
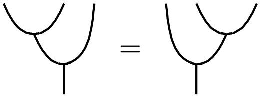
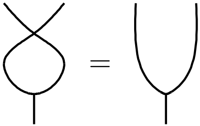


μ δ γ

such that

- symmetry satisfies Yang-Baxter and is involutive
- multiplication is associative and commutative





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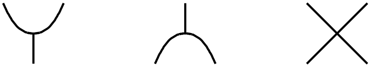

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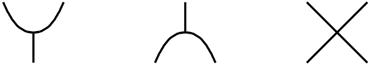

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- multiplication is associative and commutative
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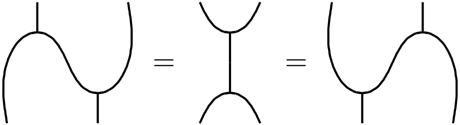
- Frobenius relations are satisfied



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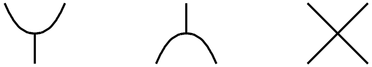

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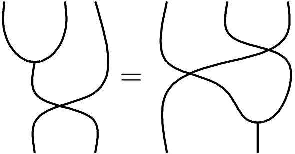
such that

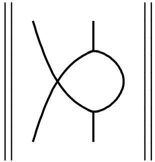
- symmetry satisfies Yang-Baxter and is involutive
- multiplication is associative and commutative
- comultiplication is associative and commutative
- Frobenius relations are satisfied
- symmetry is “natural” wrt multiplication and comultiplication:



etc.

Weighting diagrams

The **weight** of a diagram is the number of multiplications or comultiplications (but we do not count crossings):



$= \quad 2$

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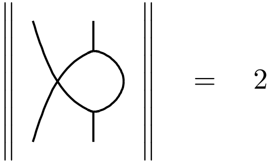

The relations are *homogeneous* and therefore the weight is also defined on the quotient.

Our goal

We want to orient those rules and complete them in order to get a convergent presentation of the PRO.

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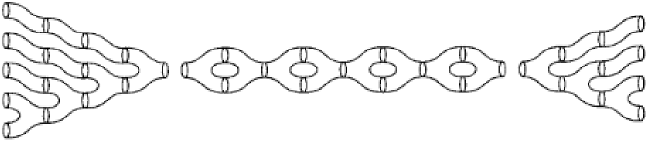
We add another requirement: rules should be homogeneous and in weight ≤ 2 (the usual requirement for showing koszulity).

Take 1

The “usual” normal form is



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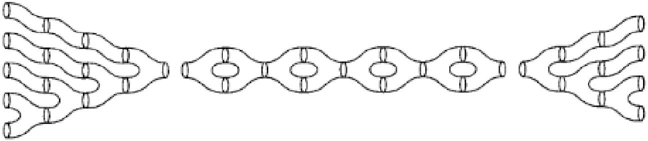

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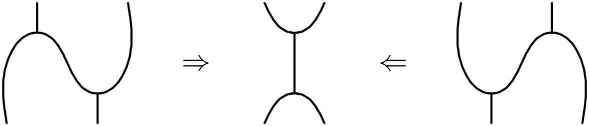

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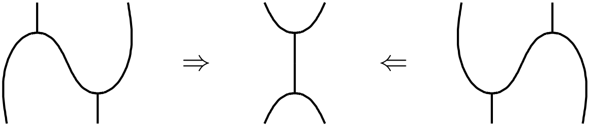
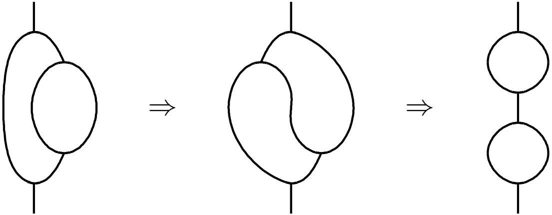


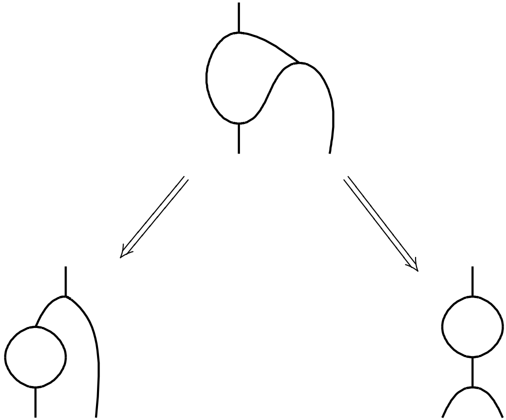
We should orient associativity and coassociativity as

so that



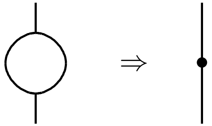
Take 1

During the completion a non-quadratic rule has to be added:



Take 2

We could add a new generator



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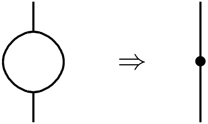

(I think that) this gives rise to a convergent quadratic + linear presentation, but this generator would have to be in weight 2 in order not to change the presented PRO.

Take 3

The idea of the preceding normal form is to have many wires in on top, many wire out at the bottom, and genus in the middle. Maybe can we find some other ways to do this.

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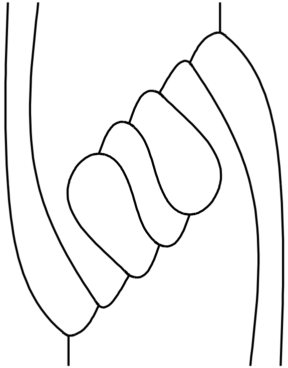
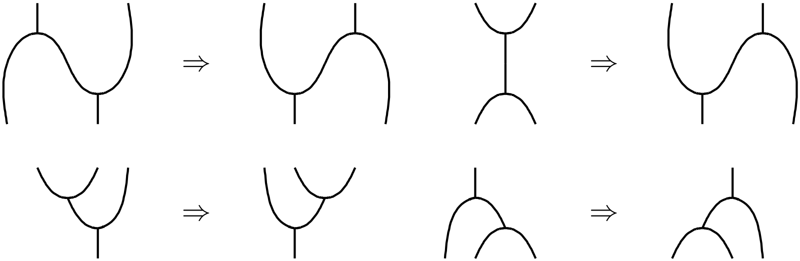


Take 3

This gives rise to the rewriting system



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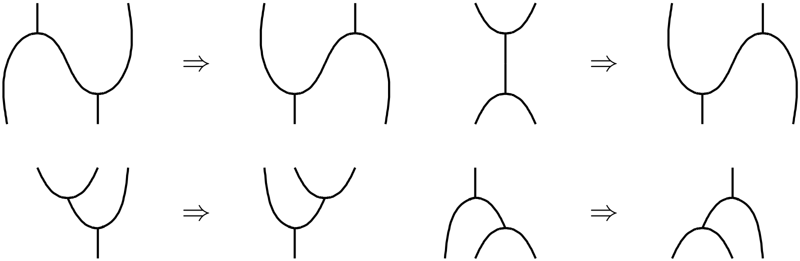

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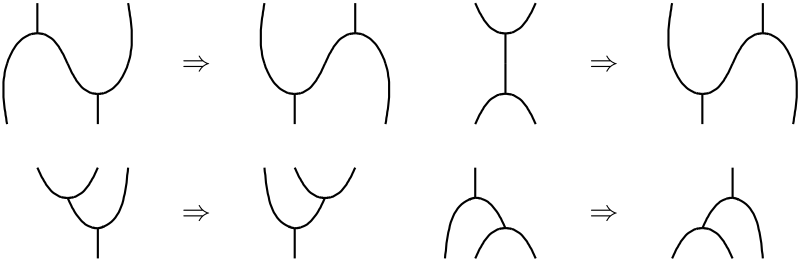
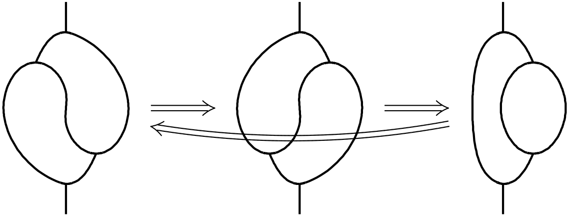
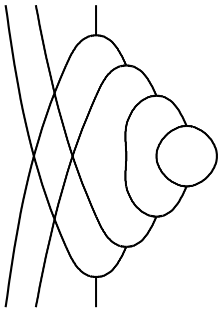


which can be completed into a quadratic one (?) but



A convergent presentation

Let’s try another idea for normal forms.



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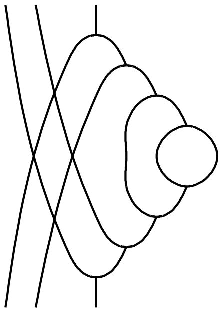
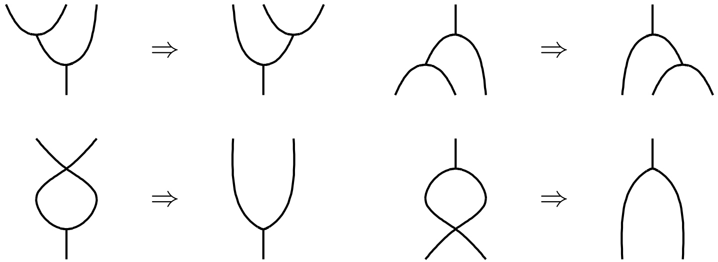


Now, we can orient relations and complete the rewriting system…

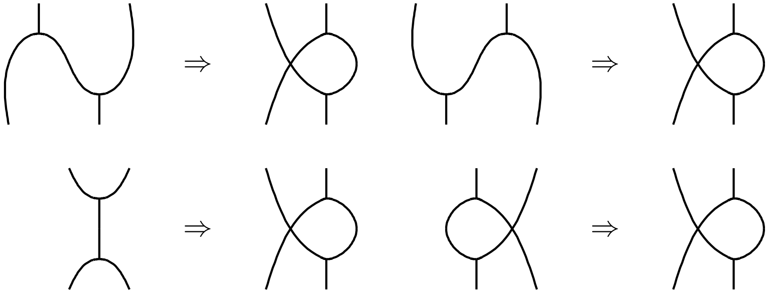
A convergent presentation

Associativity and commutativity:



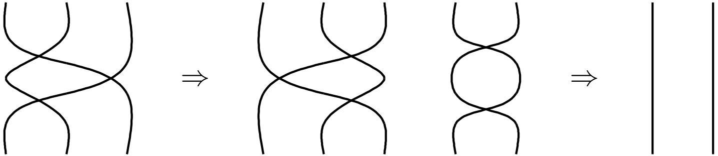
A convergent presentation

Frobenius laws:



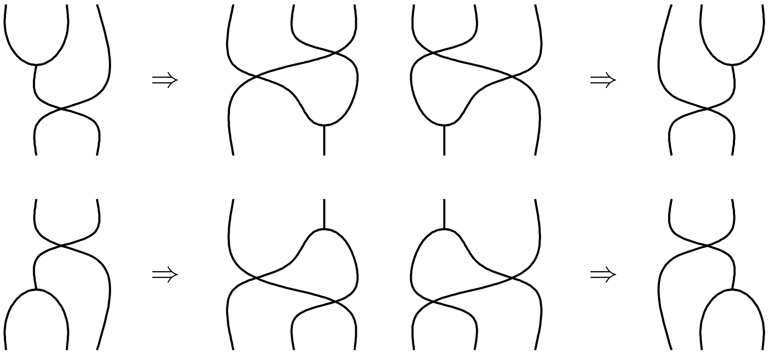
A convergent presentation

Symmetry:



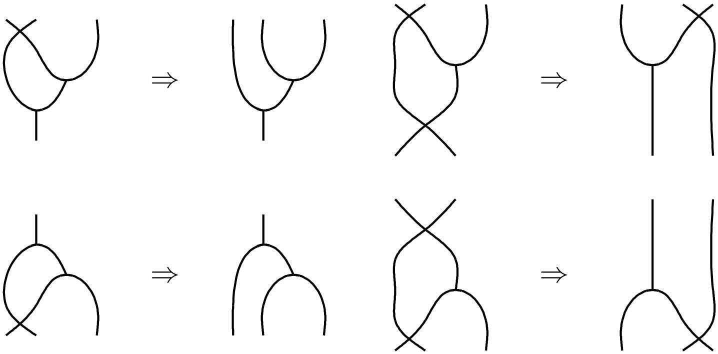
A convergent presentation

“Naturality” of symmetry:



A convergent presentation

Rules obtained by completion:



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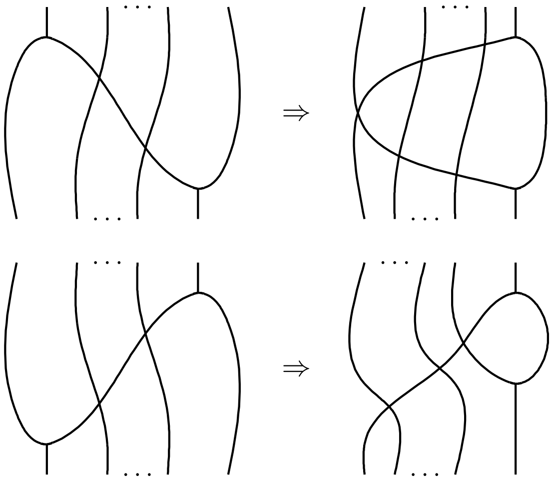

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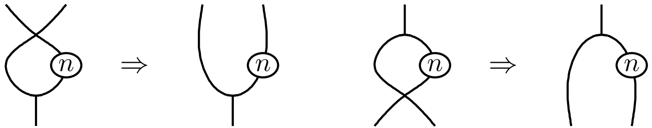
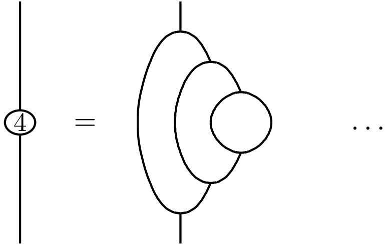


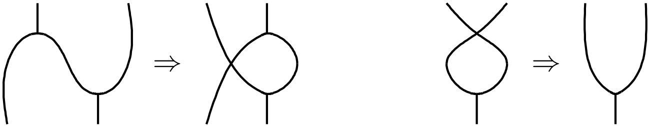
with



(admittedly this one is not subquadratic, but not too bad…)

Termination

Termination is not easy to show because of the interaction between Frobenius and commutativity:



- the usual argument commutativity decreases transpositions does not work because Frobenius increases it
- the usual interpretations as functions do not work
- etc.

Termination

We show termination as follows:

1. first we eliminate commutativity by interpreting a diagram as a relation
2. then we eliminate Frobenius rules by counting, for each (co)multiplication, the number of inputs or outputs of the global diagram its left branch is liked to
3. rules left can be shown terminating using standard techniques

The category **Rel**

We consider the category **Rel** whose objects are sets and morphisms $R : A \to B$ are relations $R \subseteq A \times B$.

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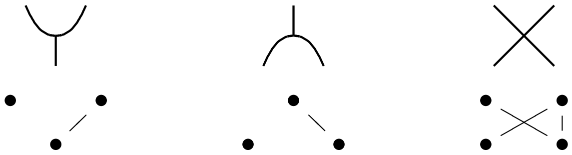
It is enriched in posets via inclusion of relations:

$$R \subseteq R' \quad \text{and} \quad S \subseteq S' \qquad \text{implies} \qquad S \circ R \subseteq S' \circ R'$$

and the order on hom-sets is well-founded.

Diagram as relations

We interpret the generators as the following relations:



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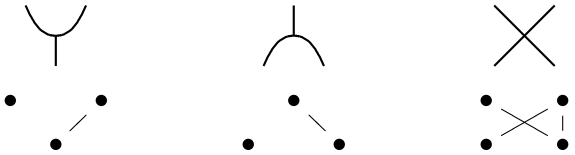

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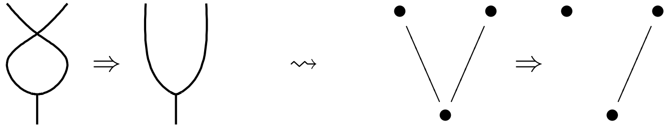

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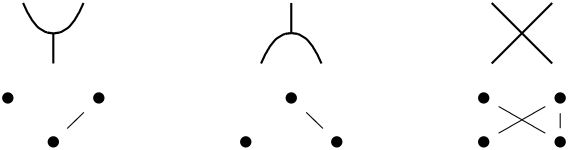

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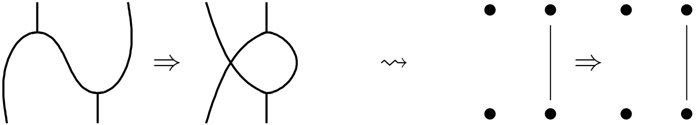

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In **Rel**, the relations link from inputs to outputs, we define a variant where there can also be links between inputs and inputs (and same for outputs).

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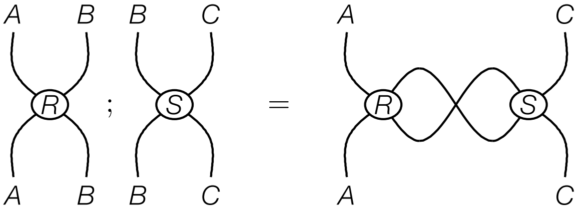
Notice that the category **Rel** is traced.

The category **IRel**

The category **IRel** has finite sets as objects. Morphisms $R : A \to B$ are relations $R : A \uplus B \to A \uplus B$ in **Rel**.

Composition of $R : A \to B$ and $S : B \to C$ is given by



(this is essentially the Int construction / the composition of GoI)

Diagrams as relations

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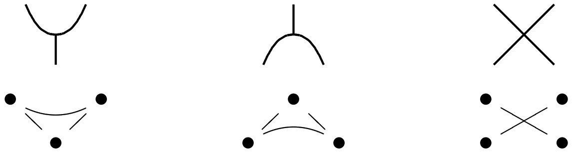

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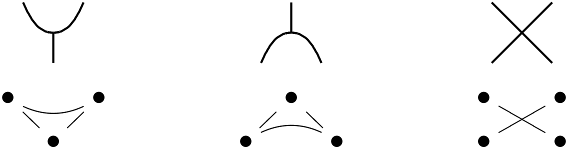
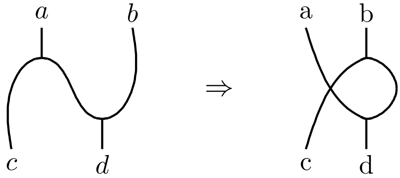


We interpret (co)multiplications as the multiset of inputs/outputs they are linked to:



$a + 2c$ $\quad\quad$ $a + c$

KOSZULITY
OF
PROPERADS

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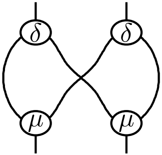
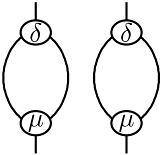
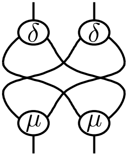
A **PROPerad** is (roughly) a PROP in which morphisms cannot be split as non-trivial connected components and only the compositions preserving this are allowed.



OK



NOT OK



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An **operad** is a PROPerad with operations of coarity 1.

PROPerads

A **PRO** is a monoidal category whose objects are integers and tensor product is given by addition.

A **PROP** is a PRO equipped with a symmetry.

A **PROPerad** is (roughly) a PROP in which morphisms cannot be split as non-trivial connected components and only the compositions preserving this are allowed.

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We generally consider the case where we are enriched over **Vect**.

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We have forgetful functors

Operads $\to$ **PROPerads** $\to$ **PROP** $\to$ **PRO**

A Σ**-bimodule** is a family $(M(m,n))$ of $\Bbbk$-modules $M(m,n)$ with a left action of Σ_n and a right action of Σ_m.

Symmetric bimodules

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Notice that in a PROP(erad) every Hom-set is a Σ-bimodule, and we have an adjunction

$$\mathbf{PROPerad} \underset{U}{\overset{T}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \Sigma\textbf{-Bimod}$$

i.e. we have a notion of free PROPerad on a Σ-bimodule.

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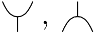
The formal definition of a PROPerad can be given as a monoid for a suitable tensor product in the category Σ**-Bimod**.

The PROPerad **Frob**

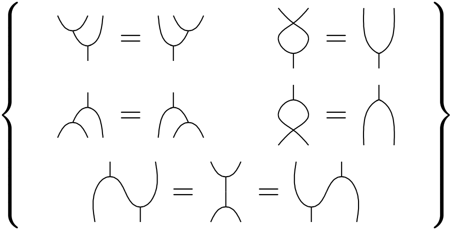
We define the following PROPerad:

$$\mathbf{Frob} \quad = \quad T\Big(\quad , \quad \Big)/(R)$$



with

$R =$



Weight

Given a PROPerad of the form

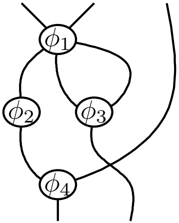
$$\mathcal{P} \quad = \quad TM/(R)$$

where R is homogeneous, the **weight** $|\phi|$ of an operation ϕ is the number of operations in M necessary to write it.

We will be in this case in the following.

The **bar construction** $B\mathcal{P}$ of a PROPerad $\mathcal{P}$ is the free (co)PROPerad on the underlying Σ-module of $\mathcal{P}$.

$\phi =$



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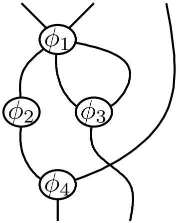

An operation has

- a *weight*: $|\phi| = \sum_i |\phi_1| + |\phi_2| + |\phi_3| + |\phi_4|$
- an *homological degree*: deg$(\phi) = 4$

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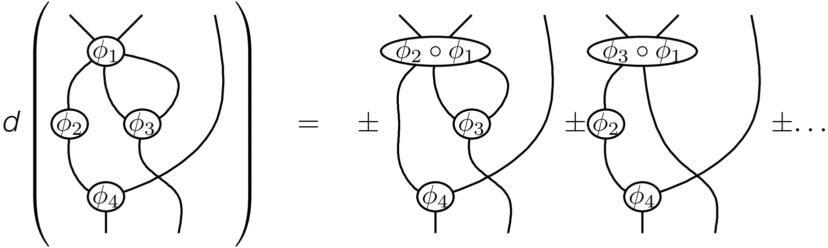

It is equipped with a differential $d : B\mathcal{P} \to B\mathcal{P}$ of degree -1 obtained as the sum of $\pm$ the composition of two adjacent operations (extends composition in degree 2 as a coderivaion)

Koszulity of PROPerads

Definition

A PROPerad is **koszul** when $H_n(B\mathcal{P})$ is concentrated in weight n, for every n.

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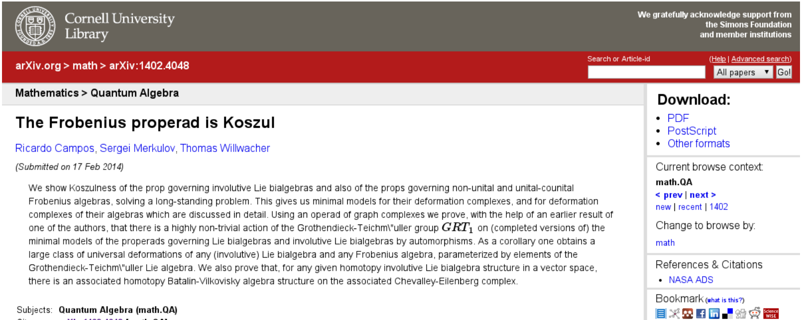

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Mathematics > Quantum Algebra

The Frobenius properad is Koszul

Ricardo Campos, Sergei Merkulov, Thomas Willwacher

(Submitted on 17 Feb 2014)

We show Koszulness of the prop governing involutive Lie bialgebras and also of the props governing non-unital and unital-counital Frobenius algebras, solving a long-standing problem. This gives us minimal models for their deformation complexes, and for deformation complexes of their algebras which are discussed in detail. Using an operad of graph complexes we prove, with the help of an earlier result of one of the authors, that there is a highly non-trivial action of the Grothendieck-Teichm\"uller group GRT_1 on (completed versions of) the minimal models of the properads governing Lie bialgebras and involutive Lie bialgebras by automorphisms. As a corollary one obtains a large class of universal deformations of any (involutive) Lie bialgebra and any Frobenius algebra, parameterized by elements of the Grothendieck-Teichm\"uller Lie algebra. We also prove that, for any given homotopy involutive Lie bialgebra structure in a vector space, there is an associated homotopy Batalin-Vilkovisky algebra structure on the associated Chevalley-Eilenberg complex.

Subjects: Quantum Algebra (math.QA)

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Koszulity of PROPerads

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Thus the question à 1000 Frs:

Is the Frobenius PROPerad koszul ?

Instead of lots of spectral sequences,
can we use the rewriting method?

The PROPerad **Frob** is difficult

There is a well-studied theory of koszulity for operads (Ginzburg-Kapranov, Getzler-Jones, …).
In particular, a convergent quadratic presentation of the (associated shuffle) operad implies koszulity (Hoffbeck, Dotsenko-Khoroshkin, …).

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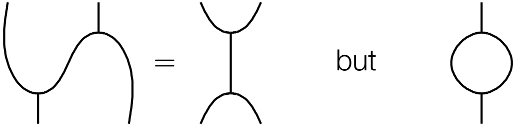

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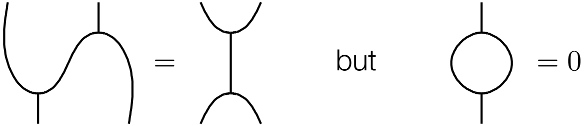

In fact, *involutive* Frobenius algebras are easily shown to be koszul by this method.

Monomial PROPerads

Given a PROPerad

$$\mathcal{P} \quad = \quad TM/(R)$$

presented by a convergent rewriting system, we write

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In the case of algebras or operads, a monomial quadratic algebra is always koszul, and this can be used to show that the existence of a quadratic convergent presentation implies koszulity.

The general plan

In order to show that **Frob** is koszul:

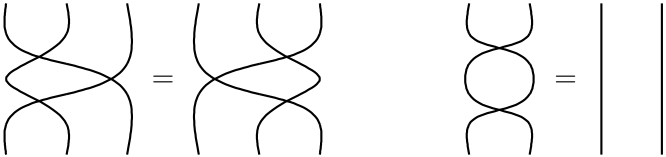
1. we “remark” that we have a convergent “quadratic” rewriting system for **FRO** (the underlying PRO)
2. we construct from it a filtration of the bar construction and show that it is enough to show that the associated monomial PROPerad is koszul
3. we show that this is the case

Free PRO(P)(erads)

A Σ-bimodule generates a PRO whose generators are

- $\phi : m \to n$ where ϕ is in a basis of $M(m, n)$
- $\gamma : 2 \to 2$ drawn as 

such that



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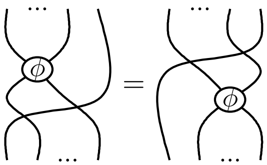

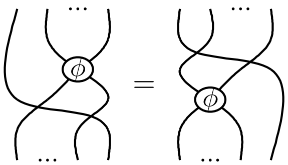

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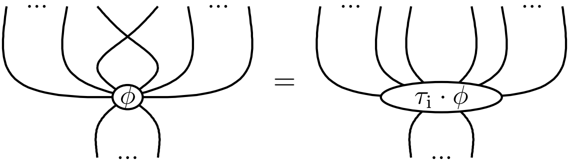

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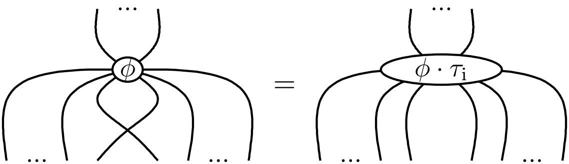

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Proposition

The category thus constructed is the free PROP on the Σ-module M and the free PROPerad generated by M embeds faithfully in it.

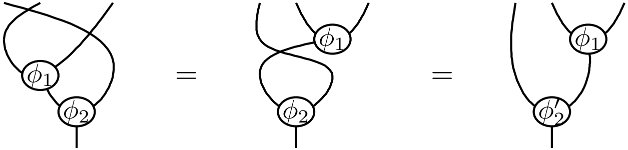
Shuffle operads

In the case of free operads on M with $M(0,1) = 0$, terms modulo equations are in bijection with planar trees with

- nodes labeled by *any* element ϕ of a fixed basis of M
- n leaves whose set of labels is $\{1, \ldots, n\}$ such that given a node $\phi(t_1, \ldots, t_n)$, the min of labels of t_i is less than the min of labels of t_j when $i < j$

Shuffle operads where defined by Dotsenko and Khoroshkin, starting from this observation from Hoffbeck.

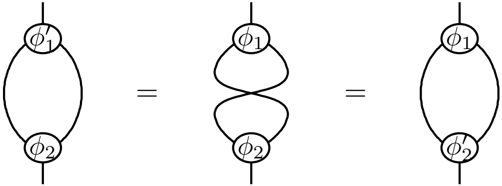
For instance, we have



$$\phi_2(\phi_1(2,3),1) \quad = \quad \phi_2(\phi_1(2,3),1) \quad = \quad \phi_2'(1,\phi_1(2,3))$$

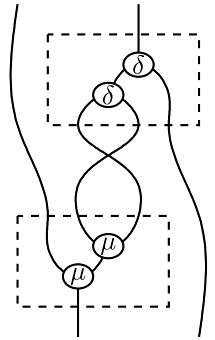
Shuffle PROPerads?

Unfortunately, there is no such notion as a *shuffle PROPerad*:



The bar construction

The bar construction on **FRO** has operations of the form



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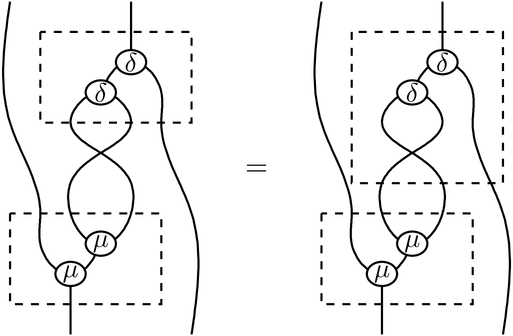

such that

- **FRO** axioms are satisfied inside the boxes
- there is an external symmetry
- external symmetry coincides with internal one

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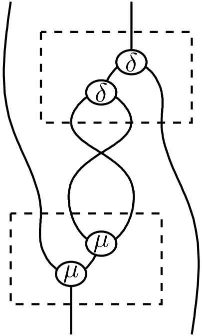

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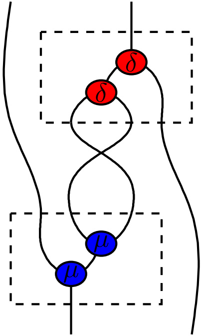

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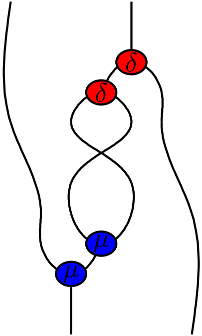


such that

- **FRO** axioms are satisfied for same colors

A convergent presentation of the bar construction

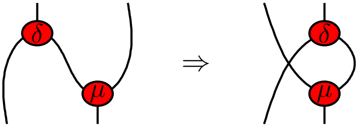
We thus have a convergent presentation of the PRO for the bar construction:

- it is generated my

$$\mu_c : 2 \to 1 \qquad \delta_c : 2 \to 1 \qquad \gamma : 2 \to 2$$

  with $c \in \mathbb{N}$ a color

- quotiented by relations of Frobenius with same colors for δ and μ



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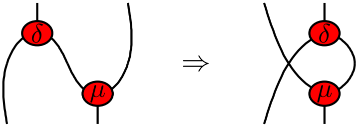
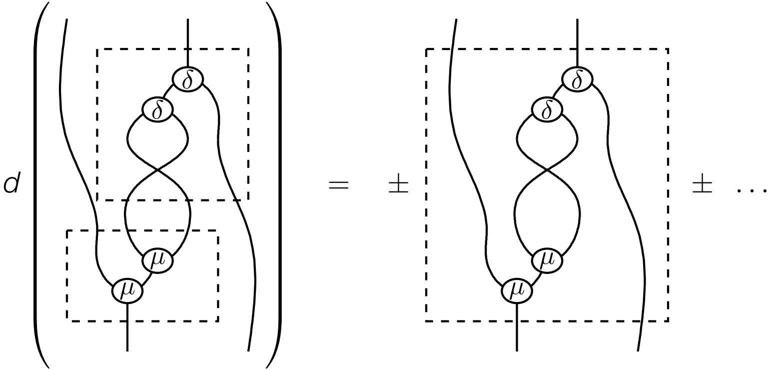


(the resulting PROP has to be taken up to renaming of colors and two disconnected components must have different colors)

Differential of the bar construction

The differential is given by



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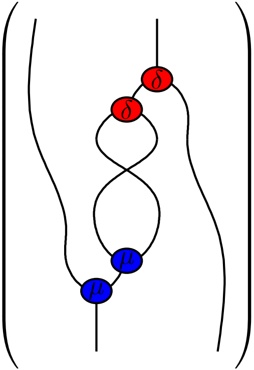
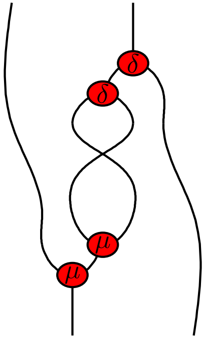
d



$= \pm$



$\pm \; \dots$

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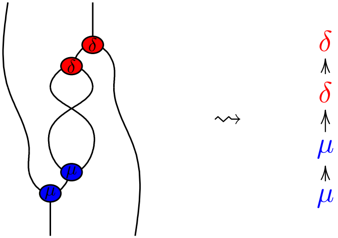
Differential of the bar construction

A morphism of the bar construction is represented by a diagram in $T(\delta_c, \mu_c, \gamma)$.

To such a diagram, we can associate a partial order whose elements are the instances of δ_c and μ_c (but not γ) and dependencies correspond to composition in the “expected way”:



Two operations which are immediate predecessor / successor are *adjacent* and their colors can be merged by the differential.

Filtering the bar construction

By “forgetting about colors”, we have a forgetful functor

$$U \quad : \quad T(\delta_c, \mu_c, \gamma)/(R') = B(\mathbf{FRO}) \quad \to \quad \mathbf{FRO} = T(\delta, \mu, \gamma)/(R)$$

(before and after quotienting by relations)

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We can thus define a filtration of $T(\delta_n, \mu_n, \gamma)$ indexed by $T(\delta, \mu, \gamma)$:

$$F_\phi(T(\delta_c, \mu_c, \gamma)) \quad = \quad \{\psi \in T(\delta_c, \mu_c, \gamma) \ | \ U(\psi) \preceq \phi\}$$

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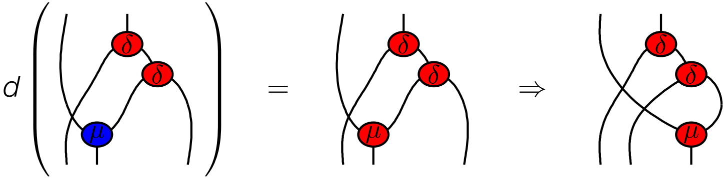
which induces a filtration on the quotient $B(\mathbf{FRO})$.

A good filtration

This filtration has nice properties:

- differential is stable



- it is exhaustive
- the partial order $\preceq$ can be extended as a total order isomorphic to $(\mathbb{N}, \leq)$

Reducing to the monomial case

We can therefore use a spectral sequence, whose first page is

$$E^0_\phi(B(\mathbf{FRO})) \quad = \quad F_\phi(B(\mathbf{FRO}))/F_{\phi-1}(B(\mathbf{FRO}))$$

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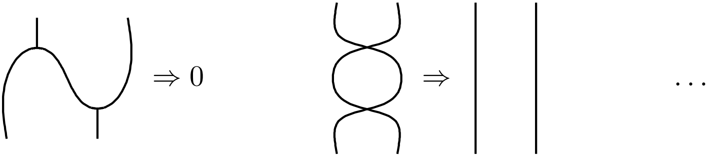
where $\overset{\circ}{\mathbf{FRO}}$ is the monomial version of **FRO**:



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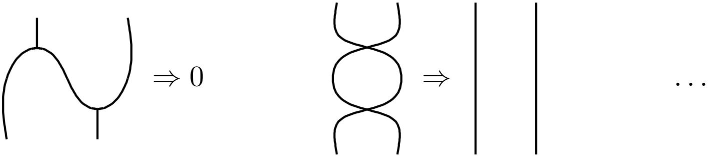
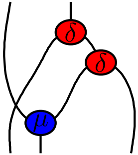


It is therefore enough to show that this one is koszul.

An operation in $B(\mathring{\textbf{FRO}})$ consists of a diagram in $T(\delta_n, \mu_n, \gamma)$ in normal form:



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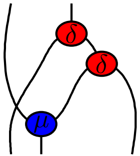
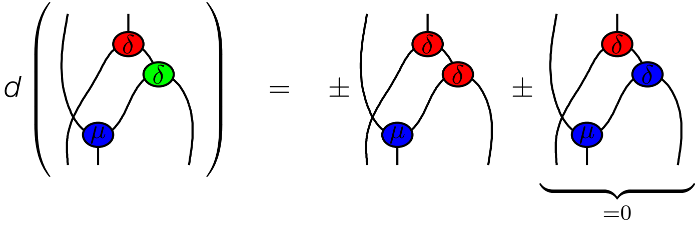
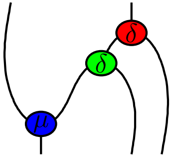


The differential takes two adjacent operations and merges their color: either the resulting diagram is normal, or the result is zero.



Koszulity of $\mathring{\mathbf{FRO}}$

Consider an element of $B(\mathring{\mathbf{FRO}})$, where all operations are colored differently:



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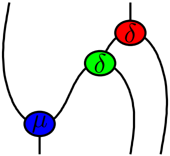

An element like this is in the diagonal: it has the same

- degree: number of used colors
- weight: number of operations

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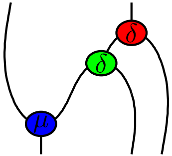

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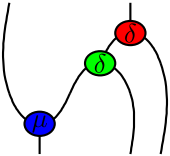
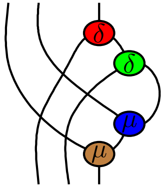
$B(\mathring{\textbf{FRO}})$ splits as a direct sum of complexes obtained as colorings of such elements.

Consider an element of $B(\mathring{\mathbf{FRO}})$:



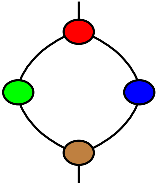
If the image of the differential is 0 then we are done, since it is on the diagonal.

Consider an element of $B(\mathring{\textbf{FRO}})$:



If the mergings of adjacent colors are independent then the complex is acyclic since it is isomorphic to the one of Δ^{k-1} ($k = 3$ in the above example).

Consider an element of $B(\mathring{\mathbf{FRO}})$:



If the mergings of adjacent colors are not independent, i.e. the morphism contains a “diamond” as above, then we cannot conclude…

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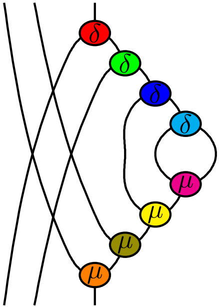

If the mergings of adjacent colors are not independent, i.e. the morphism contains a “diamond” as above, then we cannot conclude… but this does not happen because if a morphism contains a diamond, then its normal form too!

Conclusion

[Once every detail checked] we have shown that the PROPerad **Frob** is koszul.

Explicit handling of symmetries seems to be fruitful!

This is a motivating example for developing a theory of higher linear polygraphic rewriting system.