Koszul Duality for Algebras

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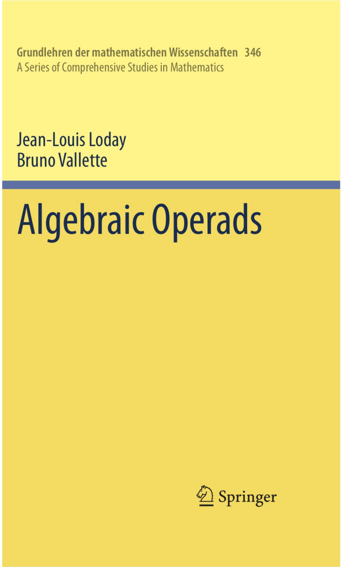
October 1, 2013

SO, I TRIED TO READ…



SOME DEFINITIONS

ALGEBRAS

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An **augmented** algebra is equipped with a morphism of algebras $\varepsilon : A \to \mathbb{K}$. In this case $A \cong \mathbb{K}1 \oplus \operatorname{Ker} A$.

$$\text{non-unital algebras} \quad \cong \quad \text{augmented unital algebras}$$

DERIVATIONS

A **derivation** $d : A \to M$ of a bimodule M over A satisfies

$$d(ab) \quad = \quad d(a)\ b + a\ d(b)$$

We write $\mathrm{Der}(A, M)$ for the space of derivations.

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Proposition

When $A = TV$, we have

$$\mathrm{Hom}(V, M) \quad \cong \quad \mathrm{Der}(TV, M)$$

COALGEBRAS

Dually, a **coalgebra** is a comonoid in **Vect**.

The **tensor coalgebra** over V is

$$T^c V \quad = \quad \mathbb{K}1 \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \dots$$

equipped with the deconcatenation tensor product

$$\Delta(v_1 \dots v_n) \quad = \quad \sum_{i=0}^{n} v_1 \dots v_i \otimes v_{i+1} \dots v_n$$

It is the *cofree conilpotent coalgebra*.
(conilpotent: $\forall v, \exists n \in \mathbb{N}, \overline{\Delta}^n(v) = 0$)

CODERIVATIONS

A **coderivation** $d : C \to C$ of a coalgebra (C, Δ) should satisfy

$$\Delta \circ d \quad = \quad (d \otimes \mathrm{id}) \circ \Delta + (\mathrm{id} \otimes d) \circ \Delta$$

Proposition

When $C = T^c V$, any coderivation is uniquely determined by its weight 1 component

$$T^c C \xrightarrow{d} T^c V \xrightarrow{\mathrm{proj}_V} V$$

GRADED VECTOR SPACES

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Its elements $v = v_1 \ldots v_n$ admit a *degree* $|v| = |v_1| + \ldots + |v_n|$ and a *weight* $\mathrm{weight}(v) = n$.

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The **suspension** is

$$sV = \mathbb{K}s \otimes V$$

with $|s| = 1$ (it increments the weight).

KOSZUL SIGN CONVENTION

The **symmetry** $\tau : V \otimes W \to W \otimes V$ is

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Similarly,

$$(f \otimes g)(v \otimes w) \quad = \quad (-1)^{|g||v|} f(v) \otimes g(w)$$

and

$$(f \otimes g) \circ (f' \otimes g') \quad = \quad (-1)^{|g||f'|} (f \circ f') \otimes (g \circ g')$$

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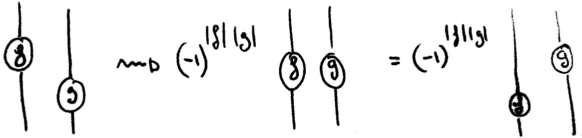

CODERIVATIONS

For instance, if (A, μ) is a graded algebra, we can extend multiplication as a coderivation on $T^c(A)$:

$$d^{(n)} = \sum_{i+j+2=n} \mathrm{id}_i \otimes \mu \otimes \mathrm{id}_j$$

which means

$$d(x_1 \otimes \ldots \otimes x_n) = \sum_{i=1}^{n-1} (-1)^{i-1} x_1 \otimes \ldots \otimes x_{i-1} \otimes \mu(x_i \otimes x_{i+1}) \otimes x_{i+2} \otimes \ldots \otimes x_n$$

when $|\mu| = -1$ and $|x_i| = 1$.

DGA ALGEBRAS

A **differential graded vector space** is (V, d) with $d : V \to V$ such that $|d| = -1$ and $d^2 = 0$.

The **tensor product** $V \otimes W$ has differential

$$d_{V \otimes W} \quad = \quad d_V \otimes \mathrm{id}_W + \mathrm{id}_V \otimes d_W$$

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A **differential graded associative algebra** (dga alg.) (A, d) is a monoid in dgvs: differential is a derivation for the product

$$d \circ \mu \quad = \quad \mu \circ (d \otimes \mathrm{id} + \mathrm{id} \otimes d)$$

It is *connected* when $A_0 = \mathbb{K}1$.

SPECTRAL SEQUENCES

Suppose that we have a chain complex

$$\ldots \to C_n \xrightarrow{d_n} C_{n-1} \to \ldots$$

together with a filtration

$$\ldots \quad \subseteq \quad F_p C_n \quad \subseteq \quad F_{p+1} C_n \quad \subseteq \quad \ldots$$

compatible with differential so that we have a chain complex $F_p C$.

$$\begin{array}{ccccccccc}
\ldots \xrightarrow{d_3} & F_2 C_2 & \xrightarrow{d_2} & F_2 C_1 & \xrightarrow{d_1} & F_2 C_0 & \xrightarrow{d_0} \ldots \\
& \bigcup & & \bigcup & & \bigcup & \\
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\end{array}$$

SPECTRAL SEQUENCES

Starting from $F_\bullet C_\bullet$,

- we define $E^0_{p,q} = F_p C_{p+q} / F_{p-1} C_{p+q}$ with $d^0 : E^0_{\bullet,\bullet} \to E^0_{\bullet,\bullet-1}$ induced by d
- $E^1 = H(E^0, d^0)$, there exists a boundary $d^1 : E^1_{\bullet,\bullet} \to E^1_{\bullet-1,\bullet}$
- $E^2 = H(E^1, d^1)$, there exists a boundary $d^1 : E^1_{\bullet,\bullet} \to E^1_{\bullet-2,\bullet+1}$
- etc.

we get **pages**

$$(E^r_{p,q}, d^r) \qquad \text{with} \qquad d^r : E^r_{\bullet,\bullet} \to E^r_{\bullet-r,\bullet+r-1}$$

Theorem

When F is bounded below ($\forall n, \exists k, \forall p \leq k, F_p C_n = 0$) and exhaustive ($C_n = \bigcup_p F_p C_n$) the sequence converges:

$$\operatorname{gr} H(C_\bullet, d) \quad = \quad E^\infty$$

i.e.

$$F_p H_{p+q}(C) / F_{p-1} H_{p+q}(C) \quad \cong \quad E^\infty_{p,q}$$

MINIMAL MODELS

A **model** $p : M \twoheadrightarrow A$ is a surjective morphism of dga alg which is a quasi-iso.

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A model is **minimal** when

1. M is quasi-free: for some V, $M \cong TV$ as a graded algebra
2. its differential is *decomposable*: $d : V \to TV^{(\geq 2)}$
3. V admits a decomposition into

$$V = \bigoplus_{k \geq 1} V^{(k)}$$

   such that

$$d\left(V^{(k+1)}\right) \subseteq T\left(\bigoplus_{i=1}^{k} V^{(i)}\right)$$

Such a model is unique up to (non-canonical) isomorphism.

MINIMAL MODELS

A projective (=free) resolution of M by A-modules is **minimal** when matrices corresponding to

$$\ldots \to A^{b_i} \to A^{b_{i-1}} \to \ldots$$

contain only *positive* entries in A.

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This means that all the boundary maps of

$$\ldots \to \mathbb{K} \otimes_A A^{b_1} \to \mathbb{K} \otimes_A A^{b_0} \to 0$$

are zero and thus

$$\mathrm{Tor}_i^A(\mathbb{K}, M) \quad = \quad \mathbb{K}^{b_i}$$

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An algebra is **Koszul** when the matrices of the boundary maps belong to A_1.

BAR AND COBAR CONSTRUCTIONS

BAR AND COBAR FUNCTORS

We are going to construct two functors

1. $B : \mathbf{DGAAlg} \to \mathbf{DGACoalg}$
2. $\Omega : \mathbf{DGACoalg} \to \mathbf{DGAAlg}$

such that

$$\Omega \quad \dashv \quad B$$

Moreover, the unit and counit

$$C \to B\Omega C \qquad\qquad \Omega BA \to A$$

will be quasi-iso, the counit thus providing a free resolution of A.

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In fact, we have more:

$$\mathbf{DGAAlg}(\Omega C, A) \quad \cong \quad \mathbf{Twisting}(C, A) \quad \cong \quad \mathbf{DGACoalg}(C, BA)$$

THE BAR RESOLUTION

Let's recap the case we all know.
We start from a monoid M and construct a free resolution

$$\dots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} \dots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

of $\mathbb{Z}$ by left $\mathbb{Z}M$-modules with

- $C_n = \mathbb{Z}M[M^n]$
- $\varepsilon : \mathbb{Z}M[M^0] \to \mathbb{Z}$ is the map such that $\varepsilon([]) = 1$,
  i.e. $\varepsilon(\sum_{u \in M} n_u u) = \sum_{u \in M} n_u$
- $\partial_n[a_1|\dots|a_n]$ is

$$a_1[a_2|\dots|a_n] + \sum_{i=1}^{n-1}(-1)^i[a_1|\dots|a_i a_{i+1}|\dots|a_n] + (-1)^n[a_1|\dots|a_{n-1}]$$

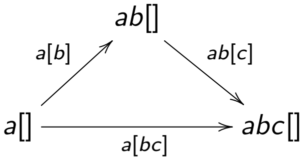
THE BAR RESOLUTION

Geometrically, we have

- vertices $a[]$
- edges $a[b]$ with $\partial_1(a[b]) = ab[] - a[]$

$$a[] \xrightarrow{a[b]} ab[]$$

- triangles $a[b|c]$ with $\partial_2(a[b|c]) = ab[c] - a[bc] + a[b]$



- tetrahedron $a[b|c|d]$
- etc.

THE BAR RESOLUTION

We can construct a contracting homotopy

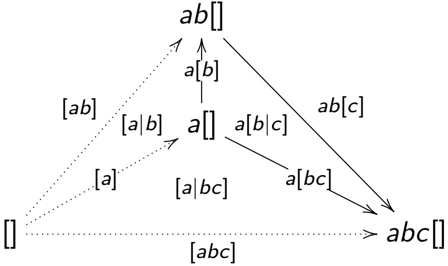
$$\cdots \underset{s_n}{\overset{\partial_{n+1}}{\rightleftarrows}} C_n \underset{s_{n-1}}{\overset{\partial_n}{\rightleftarrows}} \cdots \underset{s_1}{\overset{\partial_2}{\rightleftarrows}} C_1 \underset{s_0}{\overset{\partial_1}{\rightleftarrows}} C_0 \underset{\eta}{\overset{\varepsilon}{\rightleftarrows}} \mathbb{Z} \longrightarrow 0$$

where η and the s_i are $\mathbb{Z}$-linear (not $\mathbb{Z}M$!) and such that

$$\varepsilon\eta = \mathrm{id}_{\mathbb{Z}} \qquad \partial_1 s_0 + \eta\varepsilon = \mathrm{id}_{C_0} \qquad \partial_{n+2}s_{n+1} + s_n\partial_{n+1} = \mathrm{id}_{C_{n+1}}$$

by

$$\eta(1) = [] \qquad s_n(a_0[a_1|\ldots|a_n]) = [a_0|a_1|\ldots|a_n]$$



THE BAR RESOLUTION

- Because of the contracting homotopy, the sequence is exact (it's a free resolution of $\mathbb{Z}$ by $\mathbb{Z}M$-modules).
- Between two free resolutions there is a morphism which is unique up to homotopy.
- Thus, the homology of the complex obtained by $- \otimes_{\mathbb{Z}M} \mathbb{Z}$ does not depend on the choice of the free resolution (only on M).

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Remark

We can get a slightly smaller resolution by setting

$$[a_1|\dots|1|\dots|a_n] = 0$$

This is the *normalized bar resolution*.

FREE RESOLUTIONS

A **resolution**:

$$\dots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} \dots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

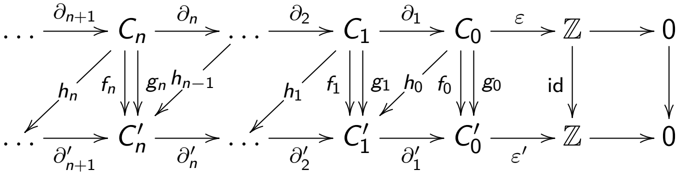
FREE RESOLUTIONS

A **morphism of resolutions**:

$$\begin{array}{ccccccccccccc}
\cdots & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & \cdots & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & C_0 & \xrightarrow{\varepsilon} & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_n} & & & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow \\
\cdots & \xrightarrow[\partial'_{n+1}]{} & C'_n & \xrightarrow[\partial'_n]{} & \cdots & \xrightarrow[\partial'_2]{} & C'_1 & \xrightarrow[\partial'_1]{} & C'_0 & \xrightarrow[\varepsilon']{} & \mathbb{Z} & \longrightarrow & 0
\end{array}$$

FREE RESOLUTIONS

An **homotopy** between morphism of resolutions:



such that

$$f_0 - g_0 = \partial'_1 h_0 \qquad\qquad f_n - g_n = \partial'_{n+1} h_n + h_{n+1}\partial'_n$$

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\end{array}
$$

Proposition

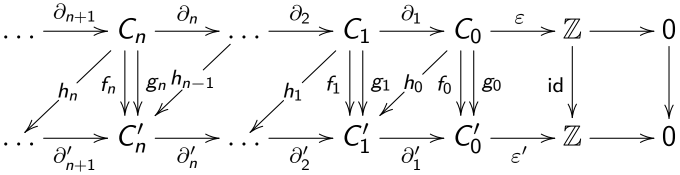
Between two free resolutions there is a morphism.

Proof.

f_0: ε' is surjective and C_0 free (projective).
f_{n+1}: for every x in a basis of C_{n+1}, $f_n \partial_{n+1}(x) \in \operatorname{Im} \partial'_{n+1} = \operatorname{Ker} \partial'_n$ because $\partial_n \partial_{n+1} = 0$ and $\partial'_n f_n = f_{n-1} \partial_n$. □

FREE RESOLUTIONS

An **homotopy** between morphism of resolutions:



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Proposition

Between two morphisms of free resolutions there is an homotopy.

Proof.

Similar. □

THE BAR CONSTRUCTION

We start from an augmented algebra $A = \mathbb{K}1 \oplus \overline{A}$ concentrated in degree 0. The differential graded coalgebra BA is $T^c(s\overline{A})$ with differential the coderivation $d_2 : BA \to BA$ extending

$$T^c(s\overline{A}) \twoheadrightarrow \mathbb{K}s \otimes \overline{A} \otimes \mathbb{K}s \otimes \overline{A} \xrightarrow{\mathrm{id} \otimes \tau \otimes \mathrm{id}} \mathbb{K}s \otimes \mathbb{K}s \otimes \overline{A} \otimes \overline{A} \xrightarrow{\Pi_s \otimes \mu_{\overline{A}}} \mathbb{K}s \otimes \overline{A}$$

with

- $\Pi_s(s \otimes s) = s$
- $\mu_{\overline{A}}$ is the restriction of μ to $\overline{A}$

THE BAR CONSTRUCTION

Lemma

Because μ is associative, we have $(d_2)^2 = 0$ (i.e. $(d_2)^2$ measures associativity).

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Lemma

The bar complex of A can be identified with the nonunital Hochschild complex of $\overline{A}$:

$$\dots \longrightarrow \overline{A}^{\otimes n} \xrightarrow{\partial_n} \overline{A}^{\otimes n-1} \longrightarrow \dots \longrightarrow \overline{A} \longrightarrow \mathbb{K} \longrightarrow 0$$

with

$$\partial_n[a_1|\dots|a_n] = \sum_{i=1}^{n-1}(-1)^{i-1}[a_1|\dots|\mu(a_i, a_{i+1})|\dots|a_n]$$

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Proposition

For $f : A \to A'$ morphism of aug dga alg, $Bf : BA \to BA'$ is a quasi-iso.

THE BAR CONSTRUCTION

If we start from a dga alg (A, d_A) there is an induced differential on $A^{\otimes n}$

$$d_1 \quad = \quad \sum_{i=1}^{n} (\mathrm{id}, \ldots, \mathrm{id}, d_A, \mathrm{id}, \ldots, \mathrm{id})$$

which induces a differential on $T^c A$. On can check

$$d_1 \circ d_2 + d_2 \circ d_1 = 0$$

(because μ_A is a morphism of dg vector spaces) and we define

$$BA \quad = \quad (T^c(s\overline{A}), d_1 + d_2)$$

THE COBAR CONSTRUCTION

We start from a coaug graded coalg $C = \overline{C} \oplus \mathbb{K}1$. The reduced coproduct $\overline{\Delta} : \overline{C} \to \overline{C} \otimes \overline{C}$ is such that $\Delta(x) = 1 \otimes x + \overline{\Delta}(x) + x \otimes 1$.

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The cobar construction is the aug alg $\Omega C = T(s^{-1}\overline{C})$ with differential the derivation $d_2 : \Omega C \to \Omega C$ extending

$$\mathbb{K}s^{-1}\otimes\overline{C} \xrightarrow{\Delta_s\otimes\overline{\Delta}} \mathbb{K}s^{-1}\otimes\mathbb{K}s^{-1}\otimes\overline{C}\otimes\overline{C} \xrightarrow{\mathrm{id}\otimes\tau\otimes\mathrm{id}} \mathbb{K}s^{-1}\otimes\overline{C}\otimes\mathbb{K}s^{-1}\otimes\overline{C}$$

with $\Delta_s(s^{-1}) = -s^{-1} \otimes s^{-1}$.

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with $\Delta_s(s^{-1}) = -s^{-1} \otimes s^{-1}$.

This extends as before to graded coalg by taking $d_1 + d_2$ as differential with, on $C^{\otimes n}$,

$$d_1 = \sum_{i=1}^{n} (\mathrm{id}, \ldots, \mathrm{id}, d_C, \mathrm{id}, \ldots, \mathrm{id})$$

THE BAR RESOLUTION

Proposition

We have an adjunction $\Omega \dashv B$.

Proposition

The counit $\varepsilon_A : \Omega BA \to A$ *is the bar-cobar resolution.*

Elements of (ΩBA) of weight n and degree p are

$$([a_1|\ldots|a_{k_1}] \quad \| \quad \ldots \quad \| \quad [a_1|\ldots|a_{k_n}])$$

with $(k_1 - 1) + \ldots + (k_n - 1) = p$.

- The differential does:
  - split a $[\ldots]$
  - multiply inside a $[\ldots]$
- $\varepsilon_A([a_1]\|\ldots\|[a_n]) = a_1 \ldots a_n$ (and 0 if not all unary)

TWISTING MORPHISMS

We have the following equivalences of categories:

$$\mathbf{DGAAlg}(\Omega C, A) \quad \cong \quad \mathbf{Twisting}(C, A) \quad \cong \quad \mathbf{DGACoalg}(C, BA)$$

Let's define twisting morphisms.

TWISTING MORPHISMS

Given a (dga) coalgebra (C, Δ) and a (dga) algebra (A, μ), the **convolution algebra** is $(\mathrm{Hom}(C, A), \star)$ with

$$(f \star g) \quad = \quad \mu \circ (f \otimes g) \circ \Delta$$

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The map

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{\mathrm{id} \otimes \alpha} C \otimes A$$

induces a unique derivation on $C \otimes A$. It's equal to the coderivation induced by

$$C \otimes A \xrightarrow{\alpha \otimes \mathrm{id}} A \otimes A \xrightarrow{\mu} A$$

and both are equal to

$$d_\alpha \quad = \quad (\mathrm{id}_C \otimes \mu) \circ (\mathrm{id}_C \otimes \alpha \otimes \mathrm{id}_A) \circ (\Delta \otimes \mathrm{id}_A)$$

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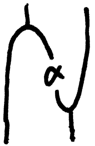
We have

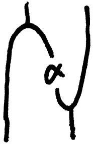
$$d_\alpha \circ d_\beta = d_{\alpha \star \beta}$$

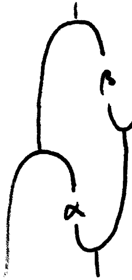
therefore $\alpha \star \alpha = 0$ implies $(d_\alpha)^2 = 0$.

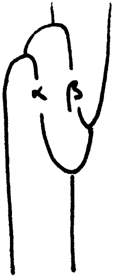
TWISTING MORPHISMS

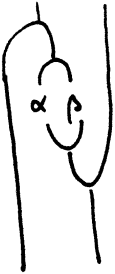
$d_\alpha \quad =$ 

TWISTING MORPHISMS

$d_\alpha \quad =$ 

$d_\alpha \circ d_\beta =$ 

$=$ 

$=$ 

$= d_{\alpha \star \beta}$

TWISTED DERIVATION

A **twisting morphism** is $\alpha : C \to A$ of degree -1 satisfying Maurer-Cartan:

$$\partial(\alpha) + \alpha \star \alpha \quad = \quad 0$$

with

$$\partial(\alpha) \quad = \quad d_A \circ \alpha + \alpha \circ d_C$$

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The **twisted derivation** is defined on $f : C \to A$ by

$$\partial_\alpha(f) \quad = \quad \partial(f) + [\alpha, f]$$

where

$$\partial(f) \quad = \quad d_C \circ f - (-1)^{|f|} f \circ d_A$$

and

$$[f, g] \quad = \quad f \star g - (-1)^{|f||g|} g \star f$$

This is a differential and a derivation wrt $\star$, $(\mathrm{Hom}(C, A), \star, \partial_\alpha)$ is thus a dga algebra.

TWISTED TENSOR PRODUCT

We define the **twisted tensor product** as

$$C \otimes_\alpha A \quad = \quad (C \otimes A, d_{C \otimes A} + d_\alpha)$$

with

$$d_{C \otimes A} \quad = \quad d_C \otimes \mathrm{id}_A + \mathrm{id}_C \otimes d_A$$

and d_α is the lifting of $\alpha : C \to A$ as a (co)derivation.

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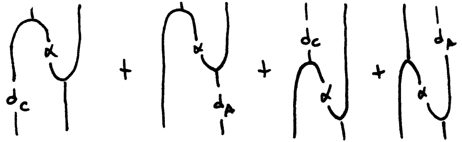
and d_α is the lifting of $\alpha : C \to A$ as a (co)derivation.

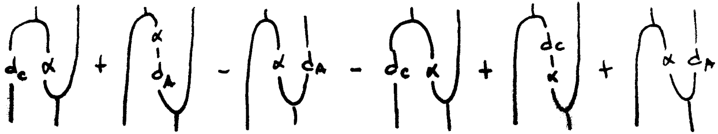
We get a differential when α is a twisting morphism:

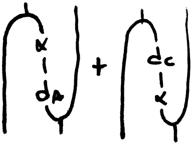
$$\begin{aligned} (d_{C\otimes A} + d_\alpha)^2 &= d^2_{C\otimes A} + d_{C\otimes A} \circ d_\alpha + d_\alpha \circ d_{C\otimes A} + d^2_\alpha \\ &= d_{d_A \circ \alpha + \alpha \circ d_C} + d_{\alpha \star \alpha} \\ &= d_{\partial(\alpha) + \alpha\star\alpha} = 0 \end{aligned}$$

TWISTED TENSOR PRODUCT

$d_{C\otimes A} \circ d_\alpha + d_\alpha \circ d_{C\otimes A}$

$=$ 

$=$ 

$=$ 

$= d_{d_A \circ \alpha + \alpha \circ d_C}$

THE ADJUNCTION

$$\mathbf{DGAAlg}(\Omega C, A) \quad \cong \quad \mathbf{Twisting}(C, A) \quad \cong \quad \mathbf{DGACoalg}(C, BA)$$

The first equivalence:

- A map

$$\phi \quad : \quad \Omega C = T(s^{-1}C) \to A$$

  is characterized by its restriction $\overline{\varphi} : \overline{C} \to A$ (T is left adjoint) and thus by

$$\varphi \quad = \quad c \mapsto \overline{\varphi}(s^{-1}c)$$

- ϕ commutes with differentials:

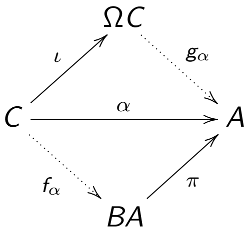
$$\begin{aligned} d_A \circ \phi &= \phi \circ (d_1 + d_2) \\ d_A \circ \varphi &= -\varphi \circ d_C - \varphi \star \varphi \\ 0 &= d_A \circ \varphi + \varphi \circ d_C + \varphi \star \varphi \\ 0 &= \partial(\varphi) + \varphi \star \varphi \end{aligned}$$

THE BAR RESOLUTION

By the equivalence of categories there is a (universal) twisting morphism

$$\pi : BA \to A$$

given by



The twisted tensor product $BA \otimes_\pi A$ is the nonunital Hochschild complex with coefficients in A and is acyclic!

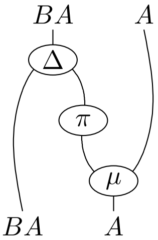
BUG: we've lost the first term!...

THE BAR RESOLUTION

We have that $BA \otimes_\pi A$ is $BA \otimes A$ as a graded space.

It differential (forgetting about suspensions):

- $\pi : BA \to A$ is the corestriction
- $d_\pi = (\mathrm{id}_{BA} \otimes \mu) \circ (\mathrm{id}_{BA} \otimes \pi \otimes \mathrm{id}_A) \circ (\Delta \otimes \mathrm{id}_A)$



- $0 = \partial(\pi) + \pi \star \pi = \pi \circ d_{BA} + \pi \star \pi$

COMPARISON THEOREM

Theorem

Let A (resp. C) be a connected wdga algebra (resp. coalgebra), for any twisting morphism the following are equivalent:

1. $C \otimes_\alpha A$ *is acyclic*
2. $A \otimes_\alpha C$ *is acyclic*
3. *the canonical dga coalgebra morphism* $C \to BA$ *is a quasi-iso*
4. *the canonical dga algebra morphism* $\Omega C \to A$ *is a quasi-iso*

Proof.

Using spectral sequences. . . □

KOSZUL DUALITY FOR QUADRATIC ALGEBRAS

COMPARISON THEOREM

In good cases, we can construct from an algebra A a dg coalgebra $A^{¡}$ and a twisting morphism $\kappa : A^{¡} \to A$ such that

Theorem

The following are equivalent:

1. $A^{¡} \otimes_{\kappa} A$ *is acyclic*
2. $A \otimes_{\kappa} A^{¡}$ *is acyclic*
3. $A^{¡} \rightarrowtail BA$ *is a quasi-iso*
4. $\Omega A^{¡} \twoheadrightarrow A$ *is a quasi-iso*

and when these hold, $A^{¡}$ *gives a minimal resolution of* A*.*

In other words, we are looking for a "small" coalgebra $A^{¡}$ playing the same role as BA.

QUADRATIC ALGEBRAS

A **quadratic algebra** A is

$$A \quad = \quad A(V,R) \quad = \quad TV/(R)$$

where (R) is the two-sided ideal generated by $R \subseteq V^{\otimes 2}$:

$$A \quad = \quad \mathbb{K}1 \oplus V \oplus \Big(V^{\otimes 2}/R\Big) \oplus \ldots \oplus \left(V^{\otimes n}/ \sum_{i+2+j=n} V^{\otimes i} \otimes R \otimes V^{\otimes j}\right) \oplus \ldots$$

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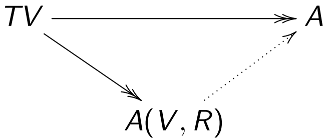
It is universal among subalgebras A of TV such that

$$R \rightarrowtail TV \twoheadrightarrow A \quad = \quad 0$$

i.e.



TOWARDS A MINIMAL MODEL

Given $A(V, R)$ quadratic we want to construct a quasi-free resolution:

- it is of the form $(T(W), d)$
- $d : W \to \bigoplus_{i \geq 2} W^{\otimes i}$

Such that there is a quasi-iso $(T(W), d) \twoheadrightarrow A$.

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Notice that A is weighted (length of words) but seen as a dga concentrated in degree 0.

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So, let's try:

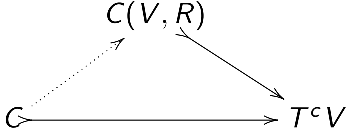
- we start from $W = V$: $TV \twoheadrightarrow A$
- we need to kill relations: $W = V \oplus R$
- we need to kill relations between relations: $W = V \oplus R \oplus (R \otimes V \cap V \otimes R)$
- etc.

QUADRATIC COALGEBRAS

The **quadratic coalgebra** is $C(V,R)$ with $R \subseteq V^{\otimes 2}$ is the universal subcoalgebra of $T^c V$ such that

$$C \rightarrowtail T^c V \twoheadrightarrow V^{\otimes 2}/R \quad = \quad 0$$

i.e.



Explicitly,

$$C \quad = \quad \mathbb{K}1 \oplus V \oplus R \oplus \ldots \oplus \left(\bigcap_{i+2+j=n} V^{\otimes i} \otimes R \otimes V^{\otimes j} \right) \oplus \ldots$$

KOSZUL DUAL COALGEBRA

The **Koszul dual** of $A(V, R)$ is the coalgebra

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In the case where R is generated by monomials, elements of degree n look like critical n-uples.

KOSZUL DUAL COALGEBRA

If we consider $\langle x, y \mid xx - yy \rangle$, elements of degree 3 are of the form

$$a(xx - yy)x + b(xx - yy)y \quad = \quad cx(xx - yy) + dy(xx - yy)$$

So

$$a = b = c = d = 0$$

KOSZUL DUAL ALGEBRA

The **Koszul dual algebra** $A^!$ is defined by

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$$(A^!)^{(n)} \quad = \quad s^n(A^{¡*})^{(n)}$$

If we dualize the exact sequence

$$0 \to R \rightarrowtail V^{\otimes 2} \twoheadrightarrow V^{\otimes 2}/R \to 0$$

we get

$$0 \leftarrow R^* \twoheadleftarrow (V^*)^{\otimes 2} \leftarrowtail R^{\perp} \leftarrow 0$$

$R^{\perp}$ is the image of $(V^{\otimes 2}/R)^*$ in $(V^{\otimes 2})^* \cong V^* \otimes V^*$ (finite dim.), i.e. functions which cancel on $R^{\perp}$. We have

$$A^{¡*} \quad = \quad A(s^{-1}V, s^{-2}R^{\perp}) \qquad\qquad A^! \quad = \quad A(V^*, R^{\perp})$$

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$$A^{¡*} \quad = \quad A(s^{-1}V, s^{-2}R^{\perp}) \qquad\qquad A^! \quad = \quad A(V^*, R^{\perp})$$

When A is f.d., $(A^!)^! = A$.

KOSZUL DUAL ALGEBRA

- $\langle x, y \mid \rangle$: an element $v \in R^{\perp}$ is $v \in V^* \otimes V^*$ satisfying

$$0 = v0$$

$A^! = \mathbb{K}1 \oplus \mathbb{K}\{x^*, y^*\}$ is the algebra of *dual numbers*.

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$A^! = \mathbb{K}1 \oplus \mathbb{K}\{x^*, y^*\}$ is the algebra of *dual numbers*.

- $\langle x, y \mid xy - yx \rangle$: an element $v \in R^\perp$ is of the form

$$v = a(x^*x^*) + b(x^*y^* - y^*x^*) + c(x^*y^* + y^*x^*) + d(y^*y^*)$$

with $b = 0$. Therefore

$$A^! = T(\mathbb{K}\{x^*, y^*\})/(x^*x^*, x^*y^* + y^*x^*, y^*y^*)$$

KOSZUL DUAL ALGEBRA

- $\langle x, y \mid xx - yy \rangle$:

$$A^! = T(\mathbb{K}\{x^*, y^*\})/(x^*x^* + y^*y^*, x^*y^*, y^*x^*)$$

- etc.

BACK TO BAR

We consider $BA = T^c(s\overline{A})$ over $A(V, R)$ quadratic. We have three gradings on BA:

- the *homological degree* of $[u_1|\dots|u_n]$ is n
- the *weight grading* of $[u_1|\dots|u_n]$ is the sum of lengths of u_i
- the *syzygy degree* of $[u_1|\dots|u_n]$ is the weight grading minus n

The differential on BA is d_2 which is of weight degree 0 and syzygy degree 1, so we have a cochain complex

$$
\begin{array}{ccccccccc}
0 & \leftarrow & V^3/(VR+RV) & \leftarrow & (V^2/R\otimes V)\oplus(V\otimes V^2/R) & \leftarrow & V\otimes V\otimes V & & (3)\\
 & & 0 & \leftarrow & V^2/R & \leftarrow & V\otimes V & & (2)\\
 & & & & 0 & \leftarrow & V & & (1)\\
 & & & & & & \mathbb{K} & & (0)\\
3 & & 2 & & 1 & & 0 & &
\end{array}
$$

Column: syzygy / row: weight

BACK TO BAR

$$
\begin{array}{ccccccccc}
0 & \longleftarrow & V^3/(VR+RV) & \longleftarrow & (V^2/R\otimes V)\oplus(V\otimes V^2/R) & \longleftarrow & V\otimes V\otimes V & & (3)\\
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3 & & 2 & & 1 & & 0 & &
\end{array}
$$

First column is $T^c(sV)$ of which A^{i} is a subspace.

BACK TO BAR

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 & & & & & & \mathbb{K} & & (0)\\
3 & & 2 & & 1 & & 0 & &
\end{array}$$

First column is $T^c(sV)$ of which $A^{¡}$ is a subspace.

Proposition

The inclusion $A^{¡} \rightarrowtail BA$ induces an iso of graded coalg:

$$A^{¡} \xrightarrow{\sim} H^0(B^\bullet A) \qquad \textit{i.e.} \qquad A^{¡\,(n)} \cong H^0(B^\bullet A)^{(n)}$$

where $B^\bullet A$ is graded by syzygy degree and $(-)^{(n)}$ is the weight.

Proof.

The inclusion $A^{¡\,(n)} \rightarrowtail (sV)^{\otimes n}$ is the kernel of the differential. □

SIMILARLY FOR COBAR

$$
\begin{array}{ccccccccl}
0 & \longrightarrow & VR \cap RV & \longrightarrow & (V \otimes R) \oplus (R \otimes V) & \longrightarrow & V \otimes V \otimes V & & (3) \\
 & & 0 & \longrightarrow & R & \longrightarrow & V \otimes V & & (2) \\
 & & & & 0 & \longrightarrow & V & & (1) \\
 & & & & & & \mathbb{K} & & (0) \\
3 & & 2 & & 1 & & 0 & &
\end{array}
$$

$$H_0(\Omega_\bullet C) \quad \cong \quad C^{¡}$$

(for the "obvious" definition of $C^{¡}$)

THE KOSZUL COMPLEX

We define $\kappa : C(sV, s^2R) \to A(V, R)$ as

$$C(sV, s^2R) \twoheadrightarrow sV \xrightarrow{s^{-1}} V \rightarrowtail A(V, R)$$

It's twisting: $\kappa \star \kappa = 0$

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It's twisting: $\kappa \star \kappa = 0$

The **Koszul complex** $A^{¡} \otimes_\kappa A$ is weight graded. In weight n:

$$0 \to A^{¡(n)} \to A^{¡(n-1)} \otimes A^{(1)} \to \ldots \to A^{¡(1)} \otimes A^{(n-1)} \to A^{(n)} \to 0$$

COMPARISON THEOREM

Theorem

The following are equivalent:

1. $A^{¡} \otimes_\kappa A$ *is acyclic*
2. $A \otimes_\kappa A^{¡}$ *is acyclic*
3. $A^{¡} \rightarrowtail BA$ *is a quasi-iso*
4. $\Omega A^{¡} \twoheadrightarrow A$ *is a quasi-iso*

and when these hold, $A^{¡}$ *gives a minimal resolution of* A*.*

Proof.

We use previous comparison theorem with A and $C = A^{¡}$.
$\Omega A^{¡}$ is a free graded algebra, differential $d_{\Omega A^{¡}} = d_2$ satisfies $d(W) \subseteq W^{\geq 2}$ by construction, by "dual" of previous proposition we have $H_0(\Omega_\bullet A^{¡}) = A$ and the map $\Omega A \twoheadrightarrow A$ is a quasi-iso. □

KOSZUL ALGEBRAS

A quadratic algebra A is **Koszul** if

- its Koszul complex $A^{\text{i}} \otimes_\kappa A$ is acyclic
- $H^d(B^\bullet A) = 0$ when $d > 0$
- $H_d(\Omega_\bullet A^{\text{i}}) = 0$ when $d > 0$
- $H^\bullet(BA)$ is a subcoalgebra of $T^c(sV)$
- $A^!$ is Koszul
- ...

KOSZUL ALGEBRAS ARE QUADRATIC

Koszul: $H^d(B^\bullet A) = 0$ when $d > 0$

$H^1(B^\bullet A)^{(3)} = 0$ means that

$$V \otimes V \otimes V \quad \to \quad (V^2/R \otimes V) \oplus (V \otimes V^2/R) \quad \to \quad V^3/(RV + VR)$$

is exact.

KOSZUL ALGEBRAS ARE QUADRATIC

Koszul: $H^d(B^\bullet A) = 0$ when $d > 0$

$H^1(B^\bullet A)^{(3)} = 0$ means that

$$\begin{array}{ccccc} V \otimes V \otimes V & \to & (V^2/R \otimes V) \oplus (V \otimes V^2/R) & \to & V^3/(RV + VR) \\ [x|y|z] & \mapsto & [xy|z] - [x|yz] & \mapsto & 0 \end{array}$$

is exact.

KOSZUL ALGEBRAS ARE QUADRATIC

Koszul: $H^d(B^\bullet A) = 0$ when $d > 0$

Suppose R has elements of degree 2 and 3, $R_3 = \{xxx - yyy\}$.

$H^1(B^\bullet A)^{(3)} = 0$ means that

$$\begin{array}{ccccc} V \otimes V \otimes V & \to & (V^2/R_2 \otimes V) \oplus (V \otimes V^2/R_2) & \to & V^3/(R_2 V + R_3 + VR_2) \\ ? & \mapsto & ? & \mapsto & xxx - yyy \end{array}$$

is exact.

AN EXAMPLE

The dual of the **symmetric algebra** on X

$$S(\mathbb{K}X) \quad = \quad \mathbb{K}X/(x_i x_j - x_j x_i, i < j)$$

is the **exterior coalgebra**

$$\Lambda^c(s\mathbb{K}X) \quad = \quad \mathbb{K}X/(x_i x_j + x_j x_i, i \leq j)$$

and we have $\Lambda^c(s\mathbb{K}X) \otimes_\kappa S(\mathbb{K}X)$ acyclic.

Proof.
Define a contracting homotopy. □

Therefore $S(\mathbb{K}X)$ is Koszul (and $\Lambda^c(sX)$ too).

QUADRATIC-LINEAR ALGEBRAS

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Previous constructions can be generalized to the **quadratic-linear** case where $R \subseteq V \oplus V^{\otimes 2}$.

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Canonical example: universal enveloping alg of a Lie alg

$$U(\mathfrak{g}) \quad = \quad T(\mathfrak{g})/(x \otimes y - y \otimes x - [x, y])$$

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We write $q : TV \twoheadrightarrow V^{\otimes 2}$ so that $qA = A(V, qR)$ is quadratic.

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From now on, we suppose

- (ql_1): $R \cap V = 0$ (no superfluous generator)

In this case there exists a map $\varphi : qR \to V$ such that

$$R \quad = \quad \{X - \varphi(X) \mid X \in qR\}$$

A DIFFERENTIAL

The map $\tilde{\varphi}$

$$(qA)^{\mathsf{i}} = C(sV, s^2qR) \twoheadrightarrow s^2qR \xrightarrow{s^{-1}\varphi} sV$$

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If we suppose

- (qI_2): $(R \otimes V + V \otimes R) \cap V^{\otimes 2} \subseteq R \cap V^{\otimes 2}$

we have that

- the image of d_φ is in $(qA)^{\mathsf{i}} \subseteq T^c(sV)$
- $d_\varphi^2 = 0$

KOSZUL DUAL DGA COALGEBRA

Given $A = A(V, R)$ quadratic-linear satisfying (ql_1) and (ql_2), the **Koszul dual dga coalgebra** is

$$A^{\text{i}} \quad = \quad \left((qA)^{\text{i}}, d_\varphi\right) \quad = \quad \left(C\left(T^c(sV), s^2R\right), d_\varphi\right)$$

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The conditions (ql_1) and (ql_2) are equivalent to

$$(R) \cap V = \{0\} \qquad \text{and} \qquad R = (R) \cap \{V \oplus V^{\otimes 2}\}$$

A GRADED ALGEBRA

A quadratic-linear $A(V, R)$ is filtered by

$$F_n A \quad = \quad \mathrm{Im}(\oplus_{k \leq n} V^{\otimes k} \twoheadrightarrow A)$$

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Theorem (PBW)

When A q-l algebra is Koszul, the epimorphism

$$p \quad : \quad qA \twoheadrightarrow \operatorname{gr} A$$

is an isomorphism of graded algebras.

Proof.

Spectral sequences. . . □

THE REWRITING METHOD

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Given a quadratic algebra $A(V, R)$, we can

1. order a basis of V,
2. extend this order to monomials of TV (generally using deglex)
3. choose a basis of R and see it as rewriting rules $r_{lead} \to (r - r_{lead})$
4. check that critical pairs are confluent

In this case, the algebra is Koszul!

REDUCTION BY FILTRATION

When $A = A(V, R)$ admits a nice filtration, there exists a morphism

$$\mathring{A} = A(V, R_{lead}) \twoheadrightarrow \operatorname{gr} A$$

if $\mathring{A}$ is Koszul (often easier to show) and the map is an iso (in weight 3) then A is also Koszul.

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The deglex ordering induces a grading on TV which refines the weight grading. We consider the associated filtration

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(elements below the p-th element in the deglex ordering).
We can consider its image under $TV \twoheadrightarrow A$ and define

$$\operatorname{gr}_p A = F_p A / F_{p-1} A$$

REDUCTION BY FILTRATION

Proposition

If the algebra $\operatorname{gr} A$ *is Koszul then so is* A*.*

Proof.

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We have

$$\begin{array}{ccc} TV & & \\ \downarrow & \searrow & \\ \mathring{A} = TV/(R_{lead}) & \xrightarrow[\psi]{} & \operatorname{gr} A \end{array}$$

with ψ iso in weights 0, 1 and 2.

If ψ is iso and $\mathring{A}$ is Koszul, so is A.

THE DIAMOND LEMMA

Theorem

Suppose that $A = A(V, R)$ *quadratic such that* $\overset{\circ}{A}$ *is Koszul. If* $\overset{\circ}{A} \twoheadrightarrow \operatorname{gr} A$ *is injective in weight 3 then it is an isomorphism (in every weight). And A is Koszul.*

Proof.

Spectral sequences. . . □

MONOMIAL ARE KOSZUL

Theorem

Any quadratic monomial algebra $A = A(V, R)$ is Koszul.

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Proof.

We fix a basis Σ_V of V and Σ_R of R and define

- $L^{(n)}$: *normal forms* $a_1 \dots a_n$ such that $\forall i, a_i a_{i+1} \notin \Sigma_R$
- $\overline{L}^{(n)}$: *critical pairs* $a_1 \dots a_n$ such that $\forall i, a_i a_{i+1} \in \Sigma_R$

L is a basis of A and $\overline{L}$ is a basis of A^{i}. A basis of $A^{\text{i}} \otimes_\kappa A$ is $a_1 \dots a_m \otimes b_1 \dots b_n$ with $a_1 \dots a_m \in \overline{L}^{(m)}$ and $b_1 \dots b_n \in L^{(n)}$ and we have

$$\begin{aligned} d(a_1 \dots a_m \otimes b_1 \dots b_m) &= a_1 \dots a_{m-1} \otimes a_m b_1 \dots b_n \quad \text{if } a_m b_1 \notin \Sigma_R \\ &= 0 \quad \text{if } a_m b_1 \in \Sigma_R \end{aligned}$$

In the latter case, this is the boundary of $a_1 \dots a_m b_1 \otimes b_2 \dots b_n$ and the Koszul complex is acyclic. □

PBW BASIS

Now, consider $\overset{\circ}{A}$ associated to a quadratic algebra: it is monomial and thus Koszul.

The elements of L form a basis of $\overset{\circ}{A}$ and their image under $\psi : \overset{\circ}{A} \twoheadrightarrow \operatorname{gr} A \cong A$ span A. When they are independent they form a **PBW basis** (or **Gröbner basis**) of A and ψ is an iso.

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It's enough to check this in weight 3.