SYMMETRIES

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This is a small survey on the use of symmetry in:

- algebra: Galois theory
- geometry: deck transformations

CLASSICAL GALOIS THEORY

Starting point

We want to find the roots of the following polynomial in $\mathbb{Q}[X]$:

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Any equation involving those, with coefficients in $\mathbb{Q}$, is still valid if we permute a with b, or c with d:

$$a^2 + 1 = 0 \qquad a + b = 0 \qquad ac = bd \qquad \dots$$

Otherwise said, we have two automorphisms of $\mathbb{Q}(i, \sqrt{5})$ fixing $\mathbb{Q}$:

$$\begin{array}{rcl} i & \mapsto & -i \\ \sqrt{5} & \mapsto & \sqrt{5} \end{array} \qquad\qquad \begin{array}{rcl} i & \mapsto & i \\ \sqrt{5} & \mapsto & -\sqrt{5} \end{array}$$

Intermediate fields

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Given a field extension L/K, we write

$$L /\!\!/ K$$

for the poset of **intermediate extensions** $K \subseteq M \subseteq L$, the order being $\subseteq$.

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We also have a poset

$$\mathrm{Aut}(L /\!\!/ K)$$

of subgroups of $\mathrm{Aut}(L/K)$ ordered by $\subseteq$.

The Galois correspondence

Theorem

Given a field extension L/K, there is an adjunction

$$\begin{tikzcd}
\mathrm{Aut}(L/\!\!/K) \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & (L/\!\!/K)^{\mathrm{op}} \arrow[l, bend left, "G"]
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- G takes $K \subseteq M \subseteq L$ to the subgroup of $\mathrm{Aut}(L/K)$:

$$GM \quad = \quad \{f : L \to L \mid f \text{ fixes } M\}$$

- F takes a subgroup $A \subseteq \mathrm{Aut}(L/K)$ to

$$FA \quad = \quad \{x \in K \mid A \text{ fixes } x\}$$

We can check that they are functors such that

$$\frac{FA \supseteq B}{A \subseteq GB}$$

Galois correspondences

An adjunction between two posets

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In this case $T = G \circ F$ is a **closure operator**:

- *extensive*: $x \leq T(x)$
- *increasing*: $x \leq y$ implies $T(x) \leq T(y)$
- *idempotent*: $T(T(x)) = T(x)$

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We write $P^* \subseteq P$ for the set of its **fixpoints**:

$$P^* \quad = \quad \{x \in P \mid G \circ F(x) = x\}$$

and similarly for $Q^* \subseteq Q$.

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Proposition

We have an induced bijection

$$\begin{tikzcd} P^* \arrow[rr, bend left, "F"] & & Q^* \arrow[ll, bend left, "G"] \end{tikzcd}$$

Generality

Note that up to now, the fact that we consider fields was used nowhere: it would equally work with groups, rings, or whatever you like...

The fundamental theorem

In the case of

$$\begin{tikzcd}
\mathrm{Aut}(L /\!\!/ K) \arrow[rr, bend left, "F"] \arrow[rr, phantom, "\perp"] & & (L /\!\!/ K)^{\mathrm{op}} \arrow[ll, bend left, "G"]
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the correspondence is a bijection if (and only if) L/K is an extension which is *finite*, *separable* and *normal*.

In the general case, F is injective and G is surjective.

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Example

$$\begin{aligned}[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] &= [\mathbb{Q}(\sqrt{2})(\sqrt{3}) : \mathbb{Q}(\sqrt{2})] \times [\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] \\ &= 2 \times 2 = 4\end{aligned}$$

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- **Galois** when algebraic, normal and separable.

Why we need normal

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An element $f \in \mathbf{Aut}(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q})$ has to satisfy

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We have

$$G \circ F(\mathbb{Q}(\sqrt[3]{2})) \quad = \quad G \circ F(\mathbb{Q})$$

Let’s go through this slowly...

Generated extensions

Given an extension L/K and $A \subseteq L$ a subset, we write $K(A)$ for the **extension generated** by A, which can be described as

- the intersection of all extensions of K containing A,
- the subfield of L whose elements are of the form

  $$(P/Q)(a_1, \ldots, a_n)$$

  where P/Q is a rational fraction in $K(X_1, \ldots, X_n)$.

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Proposition

A simple extension $K(a)$ is either of the form

- $K(X)$ if a is transcendental over K,
- $K[X]/I$ if a is algebraic over K where I is the ideal

$$I \quad = \quad \{P \in K[X] \mid P(a) = 0\}$$

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Remark

I is the kernel of the evaluation at a (therefore an ideal) whose target is the field K, therefore it is a maximal ideal and $K[X]/I$ is a field.

Simple extensions

In fact, extensions are usually simple:

Theorem (Primitive element theorem)

If L/K is a separable extension of finite degree then it is simple.

Simple extensions

The ring $K[X]$ is always a PID, therefore

$$I \quad = \quad \{P \in K[X] \mid P(a) = 0\} \quad = \quad (\underline{P})$$

i.e. if a is a root of a polynomial over K then there is a **minimal polynomial** $\underline{P}$ with this property.

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Remark

$\underline{P}$ divides any polynomial Q such that $Q(a) = 0$.

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Remark

$\underline{P}$ is the only irreducible polynomial with a as root
(up to a non-zero multiplicative constant).

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Remark

This means that if we add a root of P to K,
we necessarily add all of them.

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Every element has a representative as $P(a)$
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Proposition

Given $K(a)/K$ a simple extension one has

- $[K(a) : K] = \infty$ if a is transcendental,
- $[K(a) : K] = \deg(P)$ where P is the minimal polynomial of a otherwise.

In fact, a basis of $K(a)/K$ consists of the a^i with $0 \leq i < \deg(P)$.

Finite extensions

Lemma

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Remark

Every finite extension is algebraic, but an algebraic extension can be infinite. For instance, the set $\mathbb{A}$ of **algebraic numbers**, i.e. those which are algebraic over $\mathbb{Q}$:

$$\mathbb{A} \quad = \quad \{x \in \mathbb{C} \mid P(x) = 0 \text{ for some } P \in \mathbb{Q}[X]\}$$

Ruler and compass

Consider the points you can construct in $\mathbb{R}^2$, starting from two points, by

1. drawing a straight line through two points,
2. drawing circles centered at a point and going through another point,

and taking points at intersections of those.

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Proposition

The elements x_{i+1} and y_{i+1} are zeros of polynomials of degree one or two in K_i. Therefore, $[K_i : K]$ is a power of two.

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- Contradiction.

Squaring the circle

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- We begin with $p_0 = (0, 0)$ and $p_1 = (1, 0)$.
- We should be able to construct $(\sqrt{\pi}, 0)$ and therefore $(\pi, 0)$.
- This would imply that $[\mathbb{Q}(\pi) : \mathbb{Q}]$ is a power of two, but we know that π is not algebraic over $\mathbb{Q}$.
- Contradiction.

For the rest, we should rely on subtler invariants than size...

We should take *symmetries* in account!

General idea

Suppose that we want to make a simple algebraic extension $K(a)/K$, where P is the minimal polynomial of a.

We may not be able to distinguish between a and another root of P: more precisely, there will be an action of the Galois group.

Splitting fields

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Lemma

For any $P \in K[X]$, the splitting field exists and is unique up to isomorphism.

Proof.

Iteratively formally add roots of irreducible factors Q of P to K, i.e. take $K[x]/(Q)$, until P splits. □

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In characteristic 0, every irreducible polynomial is separable.

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When K is separable, a finite extension L/K is normal if and only if L is the splitting field for some polynomial over K.

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Proposition (???)

A finite extension L/K is separated iff the action of $\mathrm{Aut}(L/K)$ on the conjuguates of an element of K is faithful.

The fundamental theorem

Theorem (Galois)

When L/K is a finite, separable and normal extension, we have an isomorphism

$$\begin{tikzcd}
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Given $f \in \mathrm{Aut}(L/K)$, and a such that $P(a) = 0$, we have

$$f(P(a)) = P(f(a)) = 0$$

so that f permutes the zeros of P: i.e. *exchanges conjugate elements*. Conversely, since L is generated by roots of P such a permutation determines an element of $\mathrm{Aut}(L/K)$.

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so that f permutes the zeros of P: i.e. *exchanges conjugate elements*. Conversely, since L is generated by roots of P such a permutation determines an element of $\mathrm{Aut}(L/K)$.

The group $\mathrm{Aut}(L/K)$ can thus be seen as a group of permutation of roots of P (this was Galois' original definition).

An example

Consider $P = X^4 - 2$ over $\mathbb{Q}$, and consider its splitting field $K/\mathbb{Q}$.

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Therefore

$$K \quad = \quad \mathbb{Q}(\sqrt[4]{2}, i)$$

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Its degree is

$$[K : \mathbb{Q}] = [\mathbb{Q}(\sqrt[4]{2}, i) : \mathbb{Q}(\sqrt[4]{2})][\mathbb{Q}(\sqrt[4]{2}) : \mathbb{Q}] = 2 \times 4 = 8$$

(minimal polynomials are respectively, $X^2 - 1$ and $X^4 - 2$).

An example

The elements $f \in \mathsf{Aut}(K/\mathbb{Q})$ have to satisfy

$$f(i) \in \{i, -i\} \qquad\qquad f(\sqrt[4]{2}) \in \left\{\sqrt[4]{2}, i\sqrt[4]{2}, -\sqrt[4]{2}, -i\sqrt[4]{2}\right\}$$

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and all possible combinations are suitable.

We can even work out a presentation with generators r, s

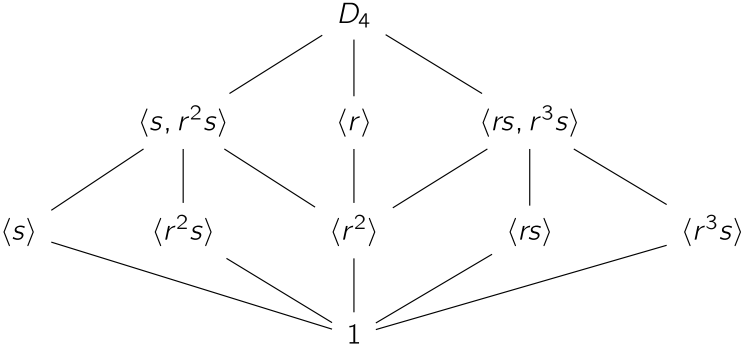
$$r(i) = i \qquad r(\sqrt[4]{2}) = i\sqrt[4]{2} \qquad\qquad s(i) = -i \qquad s(\sqrt[4]{2}) = \sqrt[4]{2}$$

namely

$$\mathrm{Aut}(K/\mathbb{Q}) \quad = \quad \langle r, s \mid r^4 = 1, s^2 = 1, srsr = 1 \rangle \quad = \quad D_4$$

An example

The subgroups of $\mathrm{Aut}(K/Q) = D_4$ are



An example

The intermediate extensions of K/Q are

$$\begin{tikzcd}
 & & \mathbb{Q}(i, \sqrt[4]{2}) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & & \\
 & \mathbb{Q}(\sqrt{2}) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \mathbb{Q}(i) \arrow[d, dash] & \mathbb{Q}(i\sqrt{2}) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\mathbb{Q}(\sqrt[4]{2}) \arrow[drr, dash] & \mathbb{Q}(i\sqrt[4]{2}) \arrow[dr, dash] & \mathbb{Q}(i, \sqrt{2}) \arrow[d, dash] & \mathbb{Q}(...) \arrow[dl, dash] & \mathbb{Q}(...) \arrow[dll, dash] \\
 & & \mathbb{Q} & &
\end{tikzcd}$$

Radical extensions

An extension L/K is **radical** when $L = K(a_1, \ldots, a_k)$ with

$$a_i^{n_i} \quad \in \quad K(a_1, \ldots, a_{i-1})$$

for some $n_i \in \mathbb{N}$, i.e. L can be obtained from K by adjoining a sequence of n_i-th roots.

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A polynomial $P \in K[X]$ is **solvable by radicals** when its splitting field is an intermediate field of a radical extension L/K.

Theorem

An separable extension L/K is radical if and only if $\mathrm{Aut}(L/K)$ *is solvable.*

Solvable groups

A group G is **solvable** if there exists subgroups

$$1 = G_0 \subseteq G_1 \subseteq \dots \subseteq G_n = G$$

such that

- G_i is a normal subgroup of G_{i+1},
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The relation "being a normal subgroup" is not transitive.

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Remark

The relation "being a normal subgroup" is not transitive.

Proposition

The symmetric groups S_n are solvable only for $n < 5$.

Idea of the proof

Lemma

Write L for the splitting field of $P = X^p - 1$ with p prime.
The Galois group $\mathrm{Aut}(L/K)$ is abelian.

Proof.

- Since $P' = pX^{p-1}$, P and P' have not zeros in common, therefore P has only simple zeros.
- Thus the group of roots of P is cyclic of order p.
- Writing a for a root, $L = K(a)$ and $f \in \mathrm{Aut}(L/K)$ is determined by $f(a)$ which should be in $\{a^i\}$.
- Writing f_i for the automorphism such that $f_i(a) = a^i$, we have $f_j \circ f_i(a) = f_i \circ f_j(a) = a^{ij}$ and therefore any two elements of $\mathrm{Aut}(L/K)$ commute. □

Idea of the proof

Lemma

Given a field K in which $X^n - 1$ splits, $a \in K$, and L the splitting field for $X^n - a$ over K, then $\mathrm{Aut}(L/K)$ is abelian.

Proof.

- Given b such that $b^n = a$, the roots of $X^n - a$ are of the form $u^i b$ with u a root of $X^n - 1$.
- Automorphisms are of the form f_i with $f_i(b) = u^i b$ and

  $$f_j \circ f_i(b) = u^{i+j} b$$

  they thus commute. ☐

Idea of the proof

Theorem

If a separable extension L/K is radical then $\mathrm{Aut}(L/K)$ is solvable.

Proof.

Given a radical extension L/K:

- we can suppose that we only take roots of prime powers,
- we take the normal closure of L,
- the splitting field of $X^n - a$ splits $X^n - 1$,
- we apply previous lemmas. ☐

An insoluble polynomial

Consider $P = X^5 - 6X + 3$ over $\mathbb{Q}$.

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Its Galois group is S_5, which is not solvable.

Therefore, P is not solvable by radicals.

Primitives

Note that this result is really due to the fact that our primitive for computing are **radicals**, i.e. roots of

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For instance, an **ultraradical** is the real solution of

$$X^5 + X - a$$

Every quintic is solvable with radicals and ultraradicals.

Can we use symmetry to show that
some tasks cannot be implemented?

The Galois task

A **task** is a polynomial P for which we want to find a root.

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A **process** is a program consisting of a loop which iteratively

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Can every task be solved by a process?

Where can we find symmetries?

Sources of symmetry:

- high-level programming languages can manipulate memory locations, but the implementation guarantees that the behavior will not depend on the chosen locations:

  invariance under action of the symmetric group on memory!

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- there can be a symmetry between the various inputs of a programs.

COVERING SPACES

Covering maps

A continuous map $p : E \to B$ between topological spaces is **covering** when every point $x \in B$ has an neighborhood U such that

$$p^{-1}(U) \quad \cong \quad \coprod_{i \in I} U$$

for some set I.

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for some set I.

The set $p^{-1}(x)$ is called the **fiber** over x.

Pointed covers

A *pointed space* (X, x) is a space X together with $x \in X$.

A *pointed morphism* $f : (X, x) \to (Y, y)$ is a morphism $f : X \to Y$ such that $f(x) = y$.

We write $\mathbf{Top}_\bullet$ for the resulting category.

A *pointed covering* is a pointed morphism which is also covering.

The universal cover

Given a pointed space (B, b), consider the full subcategory of

$$\mathbf{Top}_\bullet/(B, b)$$

whose objects are pointed covering $p : (E, e) \to (B, b)$.

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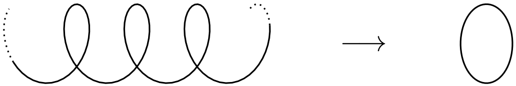
A **universal cover** is an initial object in this category.

Remark

It is not hard to show that morphisms of the above category are covering.

The universal cover

When B is "reasonable" (connected, locally path-connected and semilocally simply connected) the universal cover exists and can be described as the space whose points are homotopy classes of paths in B originating in b.



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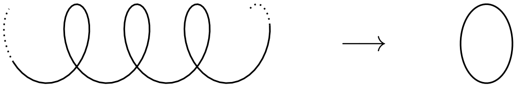

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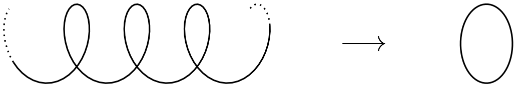

This construction does not depend on the choice of b in its connected component.

It can be characterized as the *simply connected* pointed cover of (B, b).

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We will be in this case in the following and forget about the base point (otherwise consider connected components).

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We will be in this case in the following and forget about the base point (otherwise consider connected components).

We will also suppose that covering spaces we consider are connected.

Deck transformations

A **deck transformation** of a covering $p : E \to B$ is a homeomorphism $f : E \to E$ such that

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This means that f permutes points within fibers.

We write $\mathrm{Aut}(p)$ for the **deck group**.

The fundamental theorem

Given an universal cover $p : (\tilde{X}, \tilde{x}) \to (X, x)$, there is an isomorphism

$$\pi_1(X, x) \quad \cong \quad \mathsf{Aut}(p)$$

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There is a bijective correspondence between

- subgroups of $\pi_1(X, x)$,
- coverings of (X, x).

$$\begin{tikzcd}
\mathrm{Aut}(p) \arrow[rr, bend left, "F"] & & (\mathbf{Cov}(X))^{\mathrm{op}} \arrow[ll, bend left, "G"]
\end{tikzcd}$$

Let’s detail this…

The fundamental groupoid

Given a space X, its **fundamental groupoid** $\Pi_1(X)$ is the category whose objects are points in X and morphisms

$$f \quad : \quad x \quad \twoheadrightarrow \quad y$$

are paths up to homotopy.

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X being connected, for every points x, y, there exists a morphism $f : x \twoheadrightarrow y$. It induces a group isomorphism

$$\begin{array}{rcl} \pi_1(X, x) & \to & \pi_1(X, y) \\ g & \mapsto & f \circ g \circ f^{-1} \end{array}$$

The fundamental groupoid

This construction is functorial

$$\mathbf{Top} \quad \to \quad \mathbf{Gpd}$$

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and thus a group morphism

$$f_* \quad : \quad \pi_1(X, x) \quad \to \quad \pi_1(Y, f(x))$$

The homotopy lifting property

Proposition

Given a covering $p : E \to B$, a homotopy $f : X \times I \to B$ and a lifting $\tilde{f}_0 : X \to E$, there exists a homotopy $\tilde{f} : X \times I \to E$ such that

$$\begin{tikzcd}
X \arrow[r, "\tilde{f}_0"] \arrow[d, "\mathrm{id} \times 0"'] & E \arrow[d, "p"] \\
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\end{tikzcd}$$

For instance,

- with $X = \{*\}$, we get the **path lifting property**.
- with $X = I$, we can lift homotopies between paths,
- etc.

Faithfulness

The induced functor is always faithful:

$$p_* \quad : \quad \Pi_1(E) \quad \to \quad \Pi_1(B)$$

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We can thus see $\pi_1(E, e)$ as a subgroup of $\pi_1(B, b)$.

The action of $\pi_1(X)$

By the path lifting property, every path

$$f \quad : \quad x \quad \twoheadrightarrow \quad y$$

induces a function

$$p^{-1}(x) \quad \to \quad p^{-1}(y)$$

sending $\tilde{x} \in p^{-1}(x)$ to the endpoint $\tilde{y}$ of the path

$$\tilde{f} \quad : \quad \tilde{x} \quad \twoheadrightarrow \quad \tilde{y}$$

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By the homotopy lifting property, two homotopic paths give rise to the same function.

The action of $\pi_1(X)$

We thus get a functor

$$p^* \quad : \quad \Pi_1(X) \quad \to \quad \mathbf{Set}$$

such that

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and for $f : x \twoheadrightarrow y$ the function

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Since X is connected, we obtain for instance that any two fibers are isomorphic: their cardinal is called the **degree** of the cover.

Lifting morphisms

An automorphism $g \in \mathsf{Aut}(p)$ gives rise to an isomorphism on the set $p^{-1}(x)$, i.e. we have a group morphism

$$\mathsf{Aut}(p) \quad \to \quad \mathsf{Iso}(p^{-1}(x))$$

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Which isomorphisms come from such an automorphism?

In fact, we will see that such an automorphism is determined by the image of one element of $p^{-1}(x)$.

Lifting morphisms

Theorem

Given a covering $p : E \to B$, a continuous $f : X \to B$, $x \in X$ and $e \in p^{-1}(f(x))$,

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, dotted, "g"] \arrow[r, "f"'] & B \end{tikzcd}$$

there exists g such that $p \circ g = f$ and $g(x) = e$ if and only if

$$f_*(\pi_1(X, x)) \subseteq p_*(\pi_1(E, e))$$

and in this case g is unique.

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Proof.

Given a path $x \twoheadrightarrow y$ in X, its image by f has a lifting $\tilde{x} \twoheadrightarrow \tilde{y}$ under p and we set $g(x) = \tilde{y}$. The condition ensure that this does not depend on the path. □

Lifting morphisms

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and in this case g *is unique.*

Remark

When X is simply connected, the condition is always satisfied!

Lifting morphisms

We can apply the theorem to

$$\begin{tikzcd}
 & E \arrow[d, "p"] \\
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\end{tikzcd}$$

and deduce that, given $x \in E$, a p-automorphism g is uniquely determined by the image $g(x)$, and any $y \in E$ such that

$$p_*(\pi_1(E, x)) \quad = \quad p_*(\pi_1(E, y))$$

is possible as value for $g(x)$.

Automorphisms of the universal covering

In particular, if $p : E \to B$ is the universal covering,

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We thus have

$$\mathrm{Aut}(p) \quad \cong \quad \pi_1(X, x)$$

Galois theory

Theorem

There is a bijective correspondence between

- *subgroups of* $\pi_1(X, x)$,
- *coverings of* (X, x).

$$\mathrm{Aut}(p) \quad \cong \quad (\mathbf{Cov}(X))^{\mathrm{op}}$$

Proof.

To a subgroup of $G \subseteq \mathrm{Aut}(p)$, we associate the covering

$$p/G \quad : \quad \tilde{X}/G \to X$$

where $p : \tilde{X} \to X$ is the universal covering.

To a covering $q : Y \to X$, we associate $q_*(\pi_1(Y, y))$ for some $y \in p^{-1}(x)$. ☐

Degree and index

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The **index** $|G : H|$ of a subgroup $H \subseteq G$ is the number of cosets gH of H in G:

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Proposition

The degree of a covering is the index of the corresponding subgroup in $\pi_1(X, x)$.

Normal covering

A covering $p : (\tilde{X}, \tilde{x}) \to (X, x)$ is **normal** if its action on the fiber $p^{-1}(x)$ is **transitive**:

$$\forall y, z \in p^{-1}(x), \exists f \in \mathrm{Aut}(p), \quad y \cdot f = z$$

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The intuition of a normal covering: a given loop gets unfolded a given number of time, uniformly.

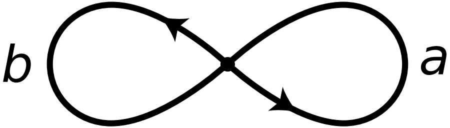
Separated covering

A covering is **separated** when the action is free...

...which is always the case (as we have seen).

An example

Consider the space $X = S_1 \vee S_1$:

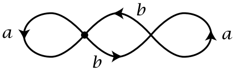


Its fundamental group is

$$\pi_1(X) \quad = \quad \langle a, b \mid \, \rangle$$

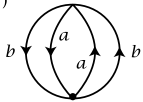
Some fundamental groups

(1)



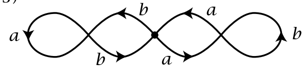
$\langle a,\ b^2, bab^{-1}\rangle$

(2)



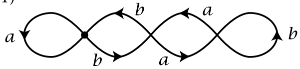
$\langle a^2, b^2, ab\rangle$

(3)



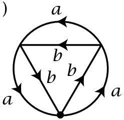
$\langle a^2, b^2, aba^{-1}, bab^{-1}\rangle$

(4)



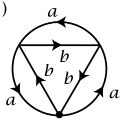
$\langle a, b^2, ba^2b^{-1}, baba^{-1}b^{-1}\rangle$

(5)



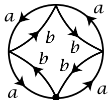
$\langle a^3, b^3, ab^{-1}, b^{-1}a\rangle$

(6)



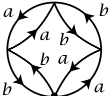
$\langle a^3, b^3, ab, ba\rangle$

(7)



$\langle a^4, b^4, ab, ba, a^2b^2\rangle$

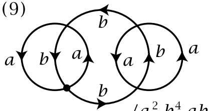
(8)



$\langle a^2, b^2, (ab)^2, (ba)^2, ab^2a\rangle$

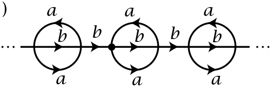
Some fundamental groups

(9)



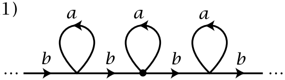
$\langle a^2, b^4, ab, ba^2b^{-1}, bab^{-2}\rangle$

(10)



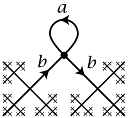
$\langle b^{2n}ab^{-2n-1}, b^{2n+1}ab^{-2n} \mid n \in \mathbb{Z}\rangle$

(11)



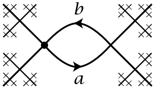
$\langle b^nab^{-n} \mid n \in \mathbb{Z}\rangle$

(12)



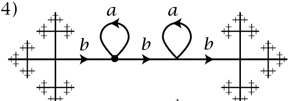
$\langle a\rangle$

(13)



$\langle ab\rangle$

(14)



$\langle a, bab^{-1}\rangle$

Let's try to drop
connectedness assumptions
(on E and on B)

Subgroups vs transitive actions

We have seen that

$$\begin{array}{c} \text{connected covering spaces} \\ \text{of } B \end{array} \quad \cong \quad \begin{array}{c} \text{subgroups} \\ \text{of } \pi_1(B) \end{array}$$

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This can be reformulated as

$$\begin{array}{c}\text{connected covering spaces}\\ \text{of } B \text{ with fiber } F\end{array} \quad \cong \quad \begin{array}{c}\text{transitive actions}\\ \text{of } \pi_1(B) \text{ on } F\end{array}$$

Subgroups vs transitive actions

The two points of view are the same on connected coverings:

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- given an action $\pi_1(B) \times F \to F$, we define

$$H \quad = \quad \mathsf{Stab}(x) \quad = \quad \{y \in \pi_1(B) \mid y \cdot x = x\}$$

  for some $x \in F$.

Non-connected covering spaces

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connected covering spaces of B with fiber F	$\cong$	transitive actions of $\pi_1(B)$ on F

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If we consider non-connected covers (but B still is), we get

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Non-connected covering spaces

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If we consider non-connected E and B, we get

$$\begin{array}{c}\text{covering spaces}\\ \text{of } B \text{ with fiber } F\end{array} \quad \cong \quad \begin{array}{c}\text{functors}\\ \Pi_1(B) \to \mathbf{Set}\end{array}$$

More categorically

In fact, this has mostly nothing to do with topology: everything can be done at the level of the fundamental groupoid $\pi_1(X)$.

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A functor $F : \mathcal{E} \to \mathcal{B}$ between groupoids is **covering**, or a **discrete opfibration**, when for every $f : x \to y$ in $\mathcal{B}$ and $\tilde{x} \in \mathcal{E}$ with $p(\tilde{x}) = x$, there exists a unique $\tilde{f} : \tilde{x} \to \tilde{y}$ such that $F(\tilde{f}) = f$.

$$\begin{tikzcd}
\tilde{x} \arrow[r, dotted, "\tilde{f}"] & \tilde{y} \\
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Typical example: when $p : E \to B$ is a covering map,

$$p_* : \pi_1(E) \to \pi_1(B)$$

is a covering functor.

Covering functors

All previous theorem can be shown in this setting.

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Theorem

There is an equivalence between the categories of

- *discrete opfibrations over* $\mathcal{B}$
- *covariant presheaves over* $\mathcal{B}$

This means that a covering functor

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Bonus: this works even when $\mathcal{B}$ is a category (not a groupoid)!

RELATING THOSE

Duality between geometry and algebra

Fix a field K.

To any space X, one can associate the commutative algebra

$$\mathcal{O}(X) \quad = \quad X \Rightarrow K$$

For instance:

$$(f+g)(x) \quad = \quad f(x)+g(x)$$

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