Coherent Tietze Transformations of 1-Polygraphs in Homotopy Type Theory

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Part I

Tietze transformations in group theory

Presentations of groups

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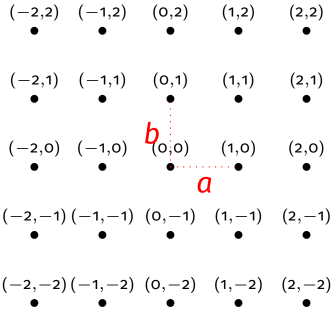

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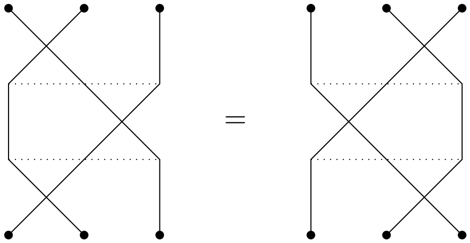

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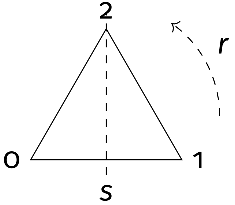

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- *semantic approach*:
  compute the presented group and construct an isomorphism
- *syntactic approach*:
  transform one presentation into the other in a way which preserves the presented group

Tietze transformations

The **Tietze transformations** are

(T0) add a definable generator:

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- *complete: two finite presentations of the same group are related by transformations.*

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Part II

Polygraphs in homotopy type theory

Higher inductive types

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data Nat : Type where
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data Circle : Type where
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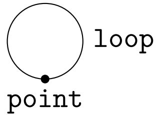

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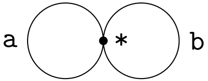

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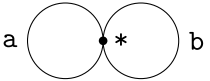

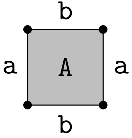

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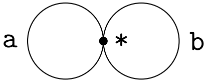

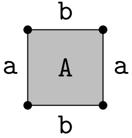

Proposition

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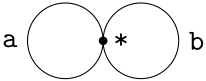

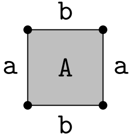

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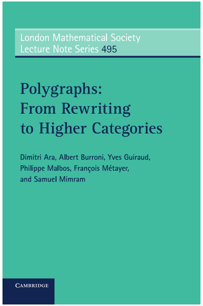
Those have been studied extensively for strict ω-categories, and used recently in HoTT by Kraus and von Raummer



Part III

1-polygraphs in homotopy type theory

Presenting sets

In order to simplify things, we consider here presentations of sets

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with $R \subseteq X \times X$. The presented set is

$$A = X/R$$

For instance,

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This is akin abstract rewriting systems vs string/term rewriting systems.

Claim: all the developments should generalize in higher dimensions.

1-polygraphs

A **1-polygraph** is a pair consisting of

- a type $P' : \mathcal{U}$ of **0-generators**,
- a family $P : \Sigma((x, y) : P' \times P') \to \mathcal{U}$ of **1-generators**.

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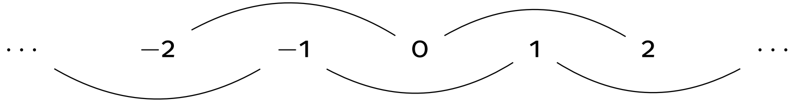

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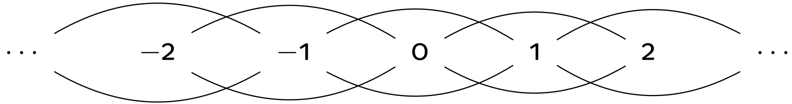

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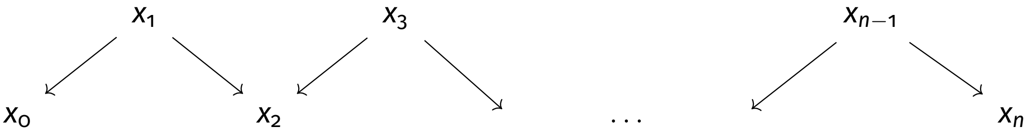
NB: we never suppose that our types are sets in polygraphs!

1-polygraphs: paths

A 1-polygraph is nothing but a type-theoretic graph.

We write $P^*(x, y)$ or $x \xrightarrow{*} y$ for the type of non-directed paths (composable sequences of possibly reversed 1-generators)



Tietze transformations for 1-polygraphs

The **Tietze transformations** for a 1-polygraph P are

(T0) given a type X and a function $\partial : X \to P'$, we define $P{\uparrow_0}\partial$ by

$$(P{\uparrow_0}\partial)' \equiv P' \sqcup X \qquad (P{\uparrow_0}\partial)(x,y) \equiv \begin{cases} P(x,y) & \text{if } x : P' \text{ and } y : P' \\ x = \partial(y) & \text{if } x : P' \text{ and } y : X \\ \bot & \text{otherwise} \end{cases}$$

$$y \quad \overset{(\text{T0})}{\rightsquigarrow} \quad y \xrightarrow{\tau_x} x$$

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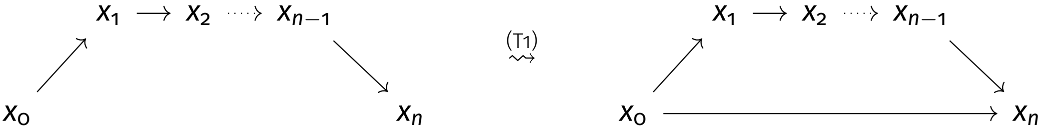
(T0) given a type X and a function $\partial : X \to P'$, we define $P{\uparrow_0}\partial$ by …

(T1) given a function $\partial : P' \times P' \to \mathcal{U}$ and a family of functions $\partial_{x,y} : \partial(x,y) \to (x \overset{*}{\to} y)$, we define $P{\uparrow_1}\partial$ by

$$(P{\uparrow_1}\partial)' \equiv P' \qquad\qquad (P{\uparrow_1}\partial)(x,y) \equiv P(x,y) \sqcup \partial(x,y)$$



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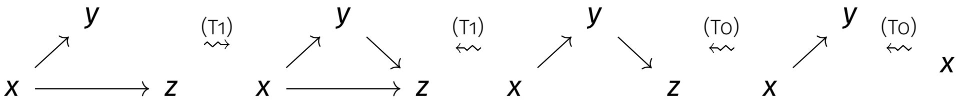
Two 1-polygraphs related by a Tietze transformations are **Tietze equivalent**.

Tietze transformations for 1-polygraphs

Example

We have the following series of Tietze transformations:



Tietze transformations: correctness

Theorem

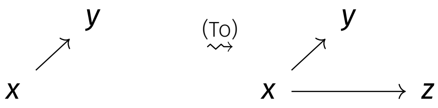
Given two Tietze equivalent 1-polygraphs P and Q we have $\|\overline{P}\|_0 = \|\overline{Q}\|_0$.

Proof.

We have to show that this is the case for all elementary Tietze transformations, which can be done by constructing an equivalence. □

Example

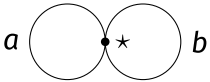
The following types are equivalent:



Tietze transformations: an application of correctness

Consider the space

$$X = S^2 \vee S^2 =$$



We want to show that this space has fundamental group $F_2 = \{a, b\}^*$.

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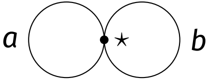

We want to show that this space has fundamental group $F_2 = \{a, b\}^*$.
We define a map

$$F : X \to \mathcal{U}$$
$$\star \mapsto F_2$$

and we want to show that its total space $\Sigma X.F$ is contractible.

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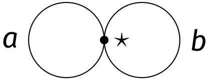

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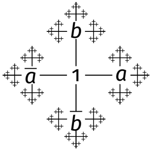

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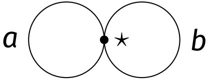

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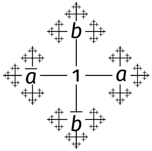

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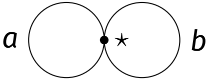
$$X = S^2 \vee S^2 =$$ 

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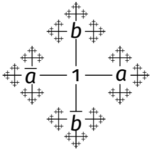
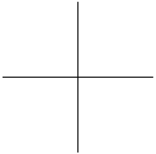


$$= \text{colim}\left(\cdot \hookrightarrow \right.$$  $$\left. \right)$$

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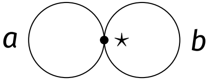

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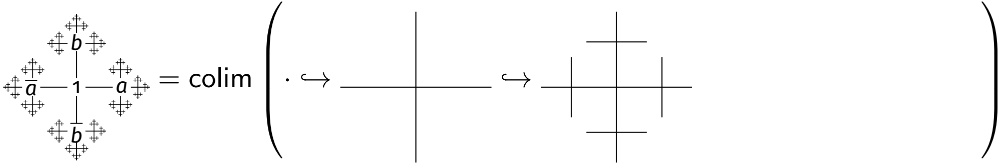

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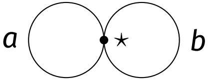
$$X = S^2 \vee S^2 = $$ 

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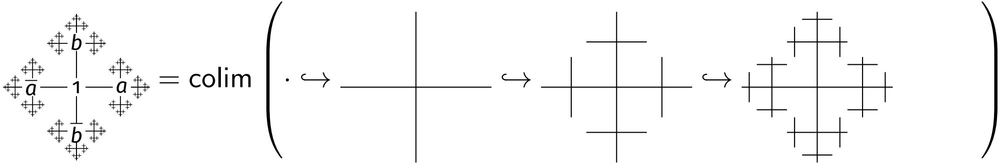

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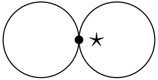
$X = S^2 \vee S^2 =$ a  b

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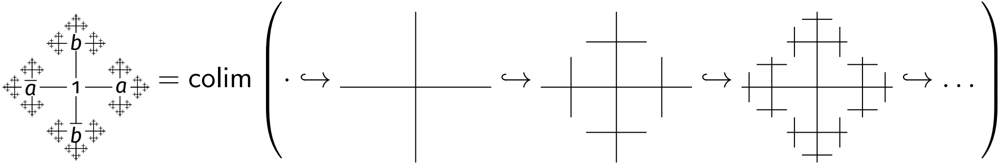


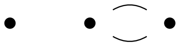
Tietze transformations: completeness

Theorem
Two 1-polygraphs P and Q with $\|\overline{P}\|_0 = \|\overline{Q}\|_0$ are Tietze equivalent.

Supposing that our polygraphs have *sets* of 0- and 1-generators (which is the interesting case), we have

$$(P', P) \sim (\|\overline{P}\|_0, \emptyset) \sim (\|\overline{Q}\|_0, \emptyset) \sim (Q', Q)$$

For instance,



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Limitation: *this approach will not generalize to higher-dimensional polygraphs!*

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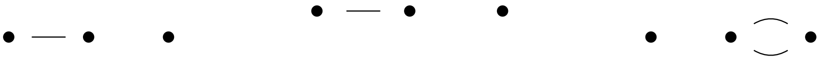

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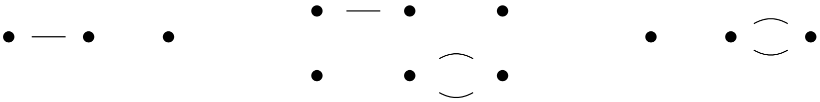

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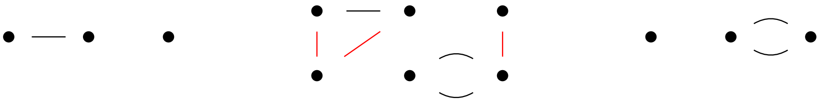

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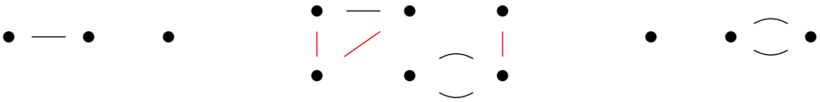

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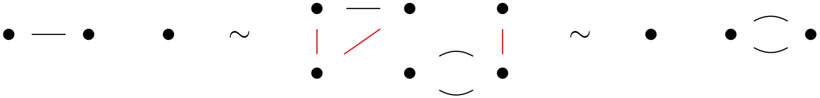

Part IV

Coherent Tietze transformations

Questions ?