# DELOOPING GENERATED GROUPS IN HOMOTOPY TYPE THEORY

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# Homotopy type theory

There are various levels of interpretation of logic:

- **-1.** types are booleans

$$A \vee (B \wedge C)$$

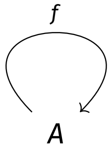
- **0.** types are sets

$$\mathbb{N} \to (\mathbb{N} \times \mathbb{Z})$$

- **$\infty$.** types are spaces

$$\Omega(\Sigma A * \Sigma B)$$

# A definition of the circle

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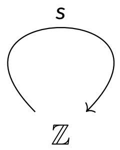


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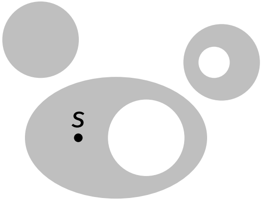


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$$\Sigma((A, f) : \mathcal{U}^{\circlearrowleft}).\|(\mathbb{Z}, s) = (A, f)\|_{-1}$$

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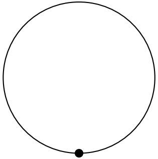

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$$\mathrm{S}^1 \quad := \quad \Sigma((A, f) : \mathcal{U}^{\circlearrowleft}).\|(\mathbb{Z}, s) = (A, f)\|_{-1}$$

is the **circle** [Bezem-Buchholtz-Grayson-Shulman'21]

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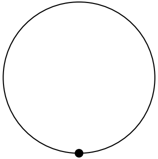

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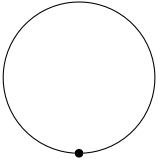

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Note that this only requires propositional truncation!

# Two approaches for formalizing groups

Suppose that we want to formalize **groups** in homotopy type theory.

There are two approaches

- **external approach**: use the traditional description in mathematics
  - a set $G$
  - an operation $m : G \to G \to G$
  - a unit $e : G$
  - satisfying the usual axioms

$$m(e, x) = x \qquad m(x, e) = x \qquad \cdots$$

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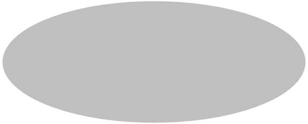
- **internal approach**: use the structure of types in order to define the notion

# Homotopy type theory

In homotopy type theory every type $A$ is a **space** (up to deformation).



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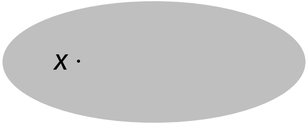

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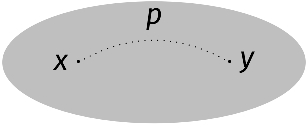


Given two elements $x, y : A$, we write

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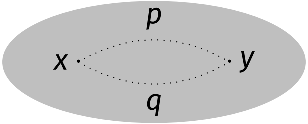


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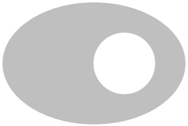
Note: for $p, q : x = y$ we can consider the type $p = q$.

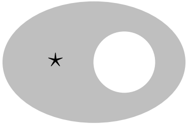
# Loop spaces

Suppose given a **space** $A$, i.e. a type.



# Loop spaces

Suppose given a space $A$ which is **pointed** by $\star : A$.



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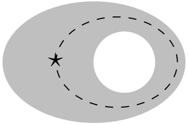


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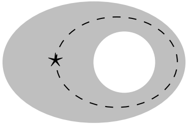

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It looks like a **group**

- we can concatenate paths,
- we can take path backwards,
- etc.

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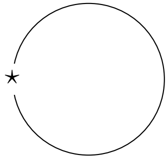


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For instance,

$$\Omega\, S^1 = \mathbb{Z}$$

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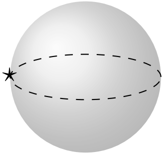


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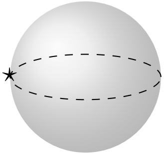


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This is the case when $A$ is a groupoid: there is at most one equality between paths.

# Delooping groups

We thus have

pointed
connected
groupoid $\xrightarrow{\Omega}$ groups

where

- isConnected$(A) := (x, y : A) \to \|x = y\|_{-1}$
- isGroupoid$(A) := (x, y : A) \to (p, q : x = y) \to (P, Q : p = q) \to (P = Q)$

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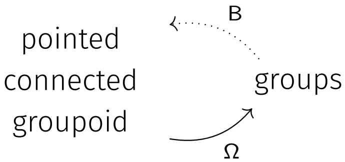


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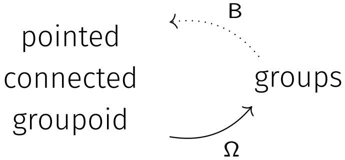


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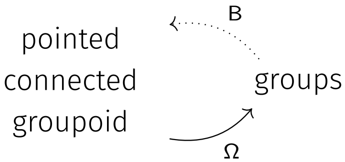


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In fact, $\mathrm{B}\,G$ always exists, is unique, and the above is an equivalence of types!

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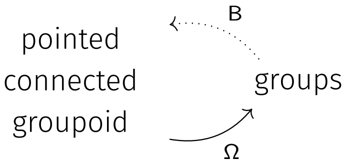


The above equivalence is useful, because we can manipulate groups as spaces, e.g. we can compute invariants such as cohomology

$$H_n(G) = \| \mathrm{B}\, G \to \mathrm{K}(\mathbb{Z}, n) \|_0$$

For those, we want simple descriptions of $\mathrm{B}\, G$!

# Delooping groups

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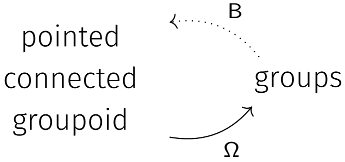


We can also easily generalize the notion of group:

$$\text{pointed connected space} \qquad = \qquad \infty\text{-group}$$

# Delooping groups

Given a group $G$, a **delooping** is a space $\mathrm{B}\,G$ such that $\Omega\,\mathrm{B}\,G = G$.

There are two known ways to construct deloopings

- torsors [Bezem-Buchholtz-Cagne-Dundas-Grayson, Wärn]
- higher-inductive types [Finster-Licata]

In this work:

**Our observation**
Both constructions can be much simplified when the group $G$ comes with a presentation (by generators and relations).

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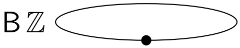
# Delooping with torsors

# Covering spaces

Consider $\mathsf{B}\,\mathbb{Z} = \mathsf{S}^1$:



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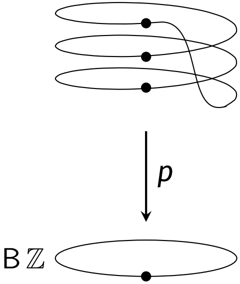
A **covering** is a space above $\mathrm{B}\,\mathbb{Z}$ which looks locally like a set (a partially unfolded variant of the space):



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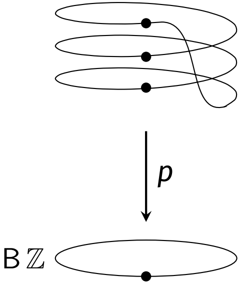


In this situation, we obtain an action on $\mathbb{Z}$ on the fiber of $p$ (all fibers are the same).

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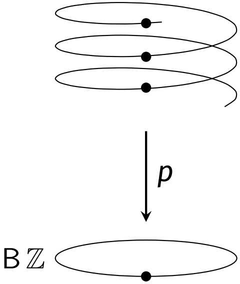


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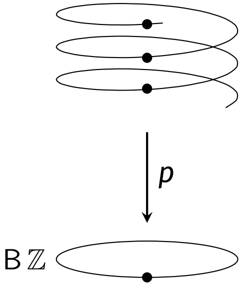


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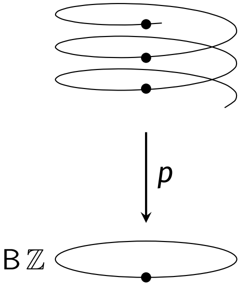


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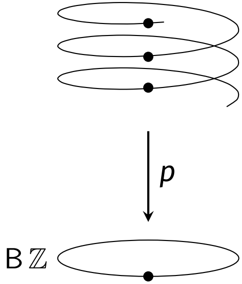


The automorphisms of the fiber are $\mathbb{Z}$.

# $G$-sets

Fix a group $G$ that we want to deloop.

**Definition**

A **$G$-set** is a set $A$ equipped with an **action**

$$\alpha : G \to A \to A$$

such that

$$\alpha(xy)(a) = \alpha(x)(\alpha(y)(a)) \qquad \alpha(1)(a) = a$$

The **domain** $\mathsf{dom}(\alpha)$ of the action is $A$.

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In type theory:

$$\mathsf{Set}_G := \Sigma(A : \mathsf{Set}).\Sigma(\alpha : G \to A \to A).\ \mathsf{isAction}(\alpha)$$

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**Lemma**

*The type* $\Sigma(A : \mathsf{Set}).\Sigma(\alpha : G \to A \to A).\,\mathsf{isAction}(\alpha)$ *of* $G$*-sets is a groupoid.*

## $G$-sets

A **morphism** of $G$-sets

$$f : \alpha \to \beta$$

is a function

$$f : \mathsf{dom}(\alpha) \to \mathsf{dom}(\beta)$$

such that, for $x : G$ and $a : \mathsf{dom}\,\alpha$,

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**Lemma**

*Given two $G$-sets $\alpha$ and $\beta$, we have*

$$(\alpha = \beta) \qquad \simeq \qquad (\alpha \cong \beta)$$

**Proof.**

By univalence. □

# The principal $G$-set

The **principal $G$-set** $\mathrm{P}_G$ is the set $G$ equipped with the action

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*We have* $(\mathrm{P}_G = \mathrm{P}_G) = G$.

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*We have* $(\mathrm{P}_G \cong \mathrm{P}_G) \simeq G$.

**Proof.**

We define

$$\begin{array}{rcl} & (\mathrm{P}_G \simeq \mathrm{P}_G) \leftrightarrow G & \\ \phi : & f \mapsto f(1) & \\ & (y \mapsto yx) \leftarrow\!\shortmid x & : \psi \end{array}$$

$\square$

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**Lemma**

*We have* $(\mathsf{P}_G \cong \mathsf{P}_G) \simeq G$.

We have just shown:

$$\Omega\, \mathsf{Set}_G = G$$

with $\mathsf{Set}_G$ pointed by $\mathsf{P}_G$.

# Toward a delooping of $G$

We are tempted to define

$$\mathrm{B}\,G := \mathsf{Set}_G$$

so that

$$\Omega\,\mathrm{B}\,G = G$$

We have that $\mathsf{Set}_G$ is

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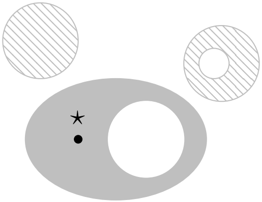
but not connected.

# Torsors




Given a type $A$ pointed by $\star$, the connected component of $\star$ is

$$\mathrm{Conn}(A) = \Sigma(x : A).\|x = \star\|_{-1}$$

which is pointed by $(\star, |\,\mathrm{refl}\,|_{-1})$

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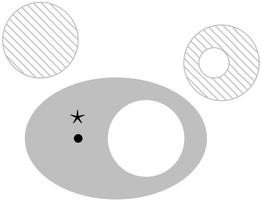

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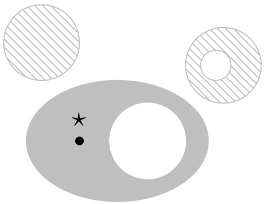

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**Definition / Theorem**

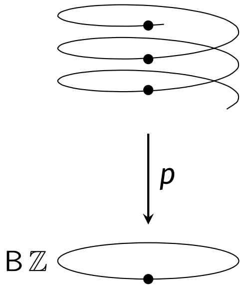
The type of **$G$-torsors** $\mathrm{Conn}(\mathrm{Set}_G)$ is a delooping of $G$.

# Torsors

An element in the type of torsors

$$\mathsf{Conn}(\mathsf{Set}_{\mathbb{Z}}) := \Sigma(A : \mathsf{Set}_{\mathbb{Z}}).\|A = \mathsf{P}_{\mathbb{Z}}\|_{-1}$$

is isomorphic to $\mathbb{Z}$ but not in a canonical way, i.e. "there is no 0":



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satisfying the usual relations, i.e. a family of functions

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$$\Sigma((A, f) : \mathsf{Set}^{\circlearrowleft}).\|(A, f) = (\mathbb{Z}, s)\|_{-1}$$

with $\mathsf{Set}^{\circlearrowleft}$ for the type of all endomorphisms

# Generated torsors

Given $X$ a set and $G$ a group, we say that a map

$$\gamma : X \to G$$

**generates** $G$ when

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The **principal $X$-torsor** is

$$\mathrm{P}_X := (G, x \mapsto a \mapsto \gamma(x)a)$$

# Generated torsors

**Theorem**
*When $X$ generates $G$, we have*

$$\operatorname{Comp} \mathrm{P}_X$$

*is a delooping of $G$.*

**Proof.**
The canonical map

$$\Omega\, \mathrm{P}_G \to \Omega\, \mathrm{P}_X$$

is an equivalence (where we obtain the inverse by generation). □

# Further simplifying

The delooping we constructed is

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The delooping we constructed is

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which can be simplified to

$$\Sigma(A : \mathcal{U}).\Sigma(f : X \to A \to A).\|(A, f) = \mathsf{P}_X\|_{-1}$$

# Examples

- a delooping of $\mathbb{Z}$ is

$$\mathrm{B}\,\mathbb{Z} = \mathrm{S}^1 = \Sigma((A,f) : \mathcal{U}^{\circlearrowleft}).\|(A,f) = (\mathbb{Z}, s)\|_{-1}$$

- a delooping of $\mathbb{Z}_n$ is

$$\mathrm{B}\,\mathbb{Z}_n = \Sigma((A,f) : \mathcal{U}^{\circlearrowleft}).\|(A,f) = (\mathbb{Z}_n, s)\|_{-1}$$

- a delooping of the *dihedral group* $D_n$ is

$$\Sigma(A : \mathcal{U}).\Sigma(f, g : A \to A).\|(A,f,g) = (D_n, s, r)\|_{-1}$$

# Delooping presented groups

# Presented groups

Suppose that we have a group $G$ with a **presentation** $\langle X \mid R \rangle$ with $R \subseteq X^* \times X^*$:

$$G = X^* / \sim_R$$

Examples

- $\mathbb{Z} = \langle s \mid \ \rangle$
- $\mathbb{Z}_n = \langle s \mid s^n = 1 \rangle$
- $D_n = \langle r, s \mid r^n = 1, s^2 = 1, sr = r^{n-1}s \rangle$

# Presented groups

Suppose that we have a group $G$ with a **presentation** $\langle X \mid R \rangle$ with $R \subseteq X^* \times X^*$:

$$G = X^* / \sim_R$$

Lemma

*Any group $G$ admits a **standard presentation** with*

$$\langle G \mid \{ab = a \times b \mid a, b \in G\} \rangle$$

Example

$\mathbb{Z}_3 = \langle 0, 1, 2 \mid 00 = 0, 01 = 1, 02 = 2, 10 = 1, 11 = 2, 12 = 0, 20 = 2, 21 = 0, 22 = 1 \rangle$

# Presented groups

Suppose that we have a group $G$ with a **presentation** $\langle X \mid R \rangle$ with $R \subseteq X^* \times X^*$:

$$G = X^* / \sim_R$$

**Theorem**

*A delooping of $G$ is the higher inductive $\mathrm{B}\,G$ type generated by*

- $\star : \mathrm{B}\,G$
- $[a] : \star = \star$ *for* $a : X$
- $[u] = [v]$ *for* $(u, v) \in R$
- $\mathrm{isGroupoid}(\mathrm{B}\,G)$

NB: starting from the standard presentation, we recover the delooping of [Finster-Licata].

# Freeness of the presentation

We have the following inductive types:

Delooping of $X^*$:

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- $[a] : \star = \star$ for $a : X$

Delooping of $G$:

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Can we quantify the difference between the two?

There is a canonical inclusion

$$\mathrm{B}\,X^* \xrightarrow{\quad f \quad} \mathrm{B}\,G$$

# Cayley graphs

We define

$$C \qquad = \qquad \ker f \qquad = \qquad \Sigma(x : \mathrm{B}\, X^*).(fx = \star)$$

so that we have a fiber sequence

$$C \longrightarrow \mathrm{B}\, X^* \xrightarrow{f} \mathrm{B}\, G$$

**Theorem**

*The type $C$ is the **Cayley graph** of $G$ with respect to $X$.*

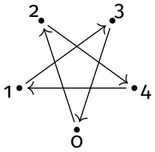
## Cayley graphs

Given a group $G$ with generating set $X$, the **Cayley graph** is the type $C$ generated by

- vertex : $G \to C$
- edge : $(a : G)(x : X) \to \text{vertex}\, a = \text{vertex}(ax)$

### Example

The type associated to $\mathbb{Z}_5$ with $X = \{2\}$ is



## Cayley graphs

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We have

$$C := \Sigma(x : \mathrm{B}\,X^*).(f(x) = \star)$$

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Therefore, by flattening, we have a coequalizer

$$\Sigma(x : X).(f(\star) = \star) \rightrightarrows \Sigma(x : \mathbf{1}).(f(\star) = \star) \dashrightarrow \Sigma(\mathrm{B}\,X^*).(f(x) = \star)$$

□

# Cayley graphs

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Therefore, by flattening, we have a coequalizer

$$X \times G \rightrightarrows G \dashrightarrow C$$

□

# Future work

- use this to develop synthetic group/homotopy theory
- develop the theory of polygraphs [Kraus-von Raumer]:
  Delooping of $G$:
  - $\star$
  - $[a] : \star = \star$ for $a : X$
  - $[u] = [v]$ for $(u, v) \in R$
  - $\mathsf{isGroupoid}(\mathrm{B}\,G)$
- develop higher-dimensional Cayley graphs

Questions ?