Guaranteed Toplogical Methods for Dynamical Systems

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CEA, LIST

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People in the business

Errors are mine, ideas are not.

This is *not* my work:

- Rigorous geometrical methods for chaos: Marian Mrozek, Piotr Zgliczynski, . . .
- Taylor method for rigorous integration of flows: Martin Berz, . . .
- . . .

The big picture

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LMeASI should be able to help!

Part I

Dynamical systems

Dynamical systems

Suppose given

- a topological space X
- a *time domain* $\mathbb{T} \in \{\mathbb{R}, \mathbb{R}^+, \mathbb{Z}, \mathbb{N}\}$.

Definition

A **dynamical system** (or *flow*) is a continuous

$$\varphi \quad : \quad X \times \mathbb{T} \to X$$

such that

- $\varphi(x, 0) = x$
- $\varphi(\varphi(x, t_1), t_2) = \varphi(x, t_1 + t_2)$

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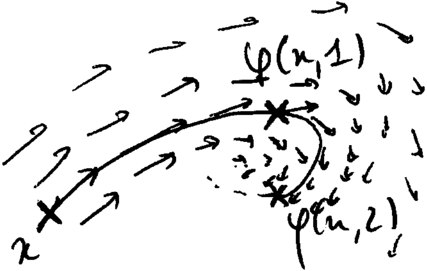
Remark

A discrete dynamical system is characterized by $f = x \mapsto \varphi(x, 1)$.

Flows vs. vector fields



Examples of dynamical systems

- Free fall

$$m\dot{v} = mg$$

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$$\varphi(v, t) \quad = \quad gt + v_0 \quad : \quad \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

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- [*insert your favorite physical system here*]

Limit behaviors

Theorem (Poincaré-Bendixon)

Given a differentiable real dynamical system defined on an open subset of the plane, then every non-empty compact ω-limit set of an orbit, which contains only finitely many fixed points, is either

- *a fixed point,*
- *a periodic orbit,*
- *or a connected set composed of a finite number of fixed points together with homoclinic and heteroclinic orbits connecting these.*

Moreover, there is at most one orbit connecting different fixed points in the same direction. However, there could be countably many homoclinic orbits connecting one fixed point.

Chaotic systems

Some dynamical systems exhibit much more complex limit behaviors. . .

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Definition

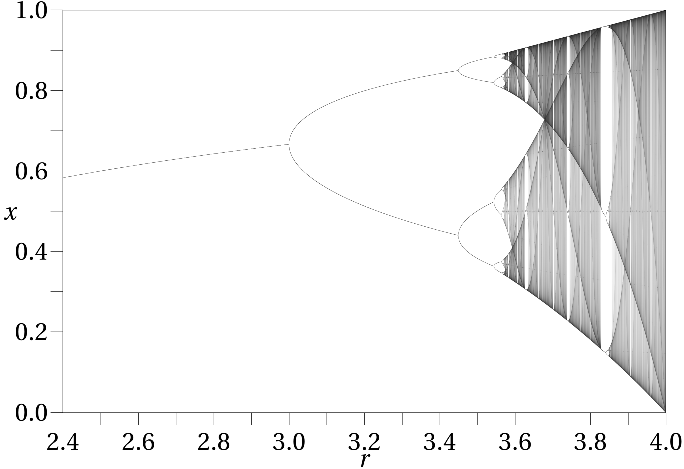
A dynamical system is **chaotic** if

1. it is sensitive to initial conditions
2. it exhibits topological mixing
3. it has a dense set of periodic orbits

Examples of dynamical systems – Chaotic

- The logistic map

$$x_{n+1} \quad = \quad rx_n(1 - x_n)$$



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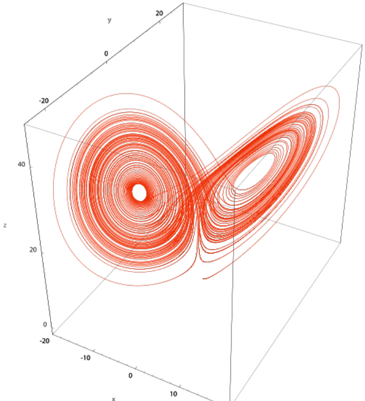
$$\left\{\begin{array}{rcl} \dot{x} & = & \sigma(y-x) \\ \dot{y} & = & x(\rho-z)-y \\ \dot{z} & = & xy-\beta z \end{array}\right.$$



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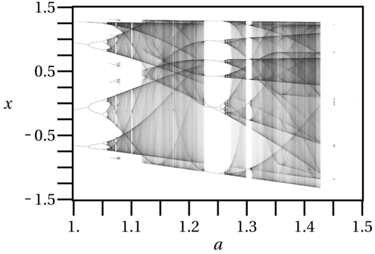
- Hénon map (simplification of Lorenz)

$$\begin{cases} x_{n+1} &= y_n + 1 - a x_n^2 \\ y_{n+1} &= b x_n \end{cases}$$



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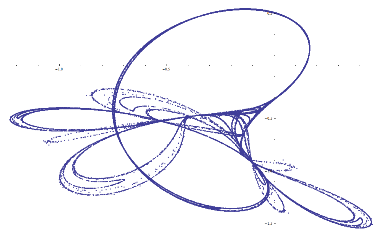
$$\begin{cases} x_{n+1} &= x_n^2 - y_n^2 + ax_n + by_n \\ y_{n+1} &= 2x_ny_n + cx_n + dy_n \end{cases}$$



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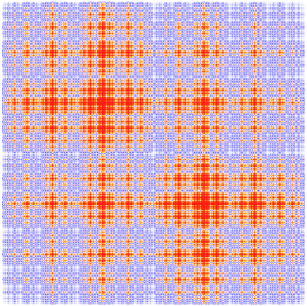

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- Wikipedia's `http://en.wikipedia.org/wiki/List_of_chaotic_maps`

Part II

Computing invariants of dynamical systems

Invariant sets

The **trajectory** of a point $x \in X$ is

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- **periodic**: $\exists t \in \mathbb{T}, \varphi(x, t) = x$
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Definition

Given $N \subseteq X$, its **invariant part** is

$$\mathrm{Inv}(N, \varphi) \quad = \quad \{x \in N \mid \varphi(x) \subseteq N\}$$

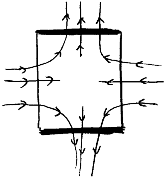
and N is **invariant** when $\mathrm{Inv}(N, \varphi) = N$.

We want to study the structure
of these invariant sets!

The kind of thing we will use

The **exit set** N^- of $N \subseteq X$ is

$$N^- \quad = \quad \{x \in N \mid \exists \varepsilon > 0, \forall 0 < t < \varepsilon, \varphi(x,t) \notin N\}$$



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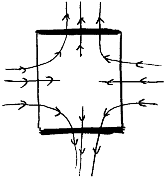

<u>“**Theorem**”</u> *(not the exact hypothesis but you get the idea)*
If N is connected and N^- is either empty or not connected then N admits a fixpoint.

Stationary points: a simple example

Suppose that $X = \mathbb{R}$, $\mathbb{T} = \mathbb{R}$ and consider a dynamical system

$$\varphi \quad : \quad X \times \mathbb{T} \to X$$

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Proposition

If we can find an interval $[a, b]$ *such that* $f(a) < 0$ *and* $f(b) > 0$ *then by the intermediate value theorem there exists a point* $s \in [a, b]$ *which is stationary:* $\dot{x} = f(x) = 0$*.*

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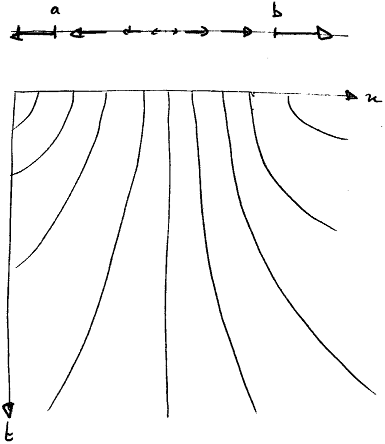

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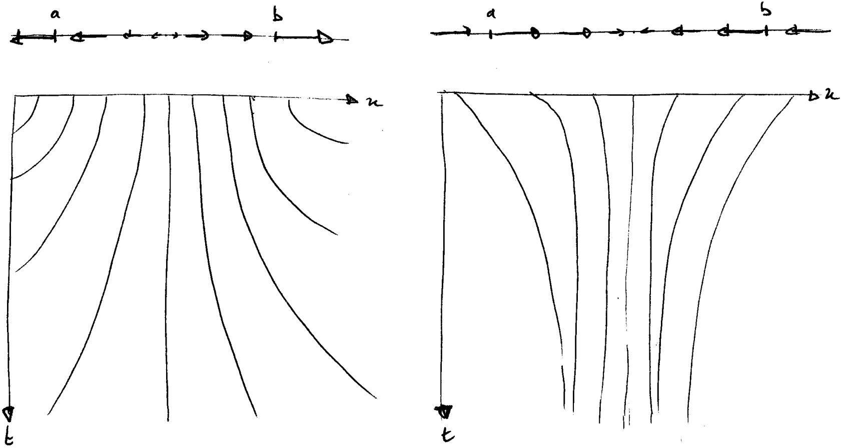

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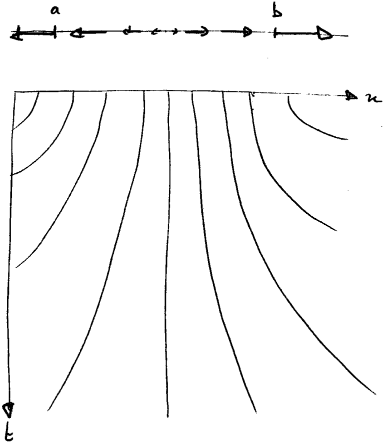

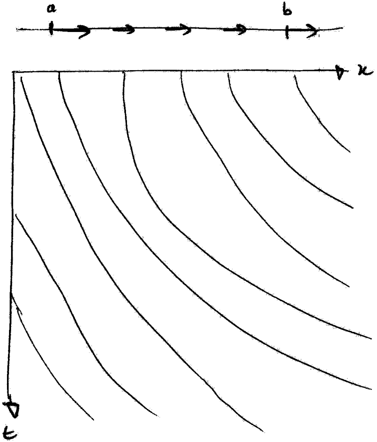

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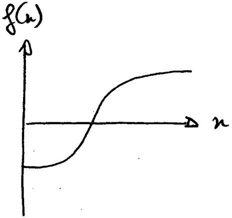
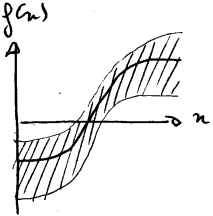
Remark

This methodology can be extended to guaranteed methods





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Proof.

By reduction ad absurdum.

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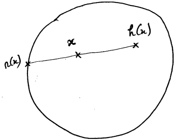

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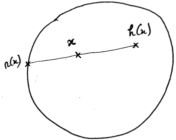

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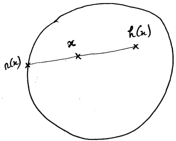

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This is the same kind of theorem!

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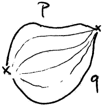

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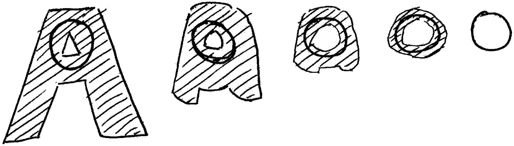
- $A \subseteq X$ is a **deformation retract** of X when $\mathrm{id}_X : X \to X$ is homotopic to a retraction $r : X \to A$ of X onto A (i.e. $r(X) = A$ and $r|_A = \mathrm{id}_A$).



The fundamental group

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Lemma

If $A \subseteq X$ is a deformation retract of X then $\pi_1(A, x_0) \cong \pi_1(X, x_0)$.

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- The **relative homology** $H(X, Y)$ of X wrt $Y \subseteq X$ is the homology of X where Y has been contracted to a point.

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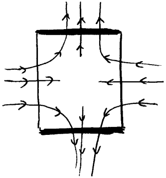
  ($\approx$ dimension of a vector space).
- The n-th **Betti number** $b_n = \mathrm{rank}(H_n(X))$ can be thought as the number of n-dimensional holes of X.
- The **relative homology** $H(X, Y)$ of X wrt $Y \subseteq X$ is the homology of X where Y has been contracted to a point.
- **Cohomology** $H^*(X)$ is defined in a "similar" (dual) way.

Ważezwski theorem

- The **exit set** N^- of $N \subseteq X$ is

$$N^- \quad = \quad \{x \in N \mid \exists \varepsilon > 0, \forall 0 < t < \varepsilon, \varphi(x,t) \notin N\}$$



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This gives invariant points in N for continuous dynamic systems.

The Conley index

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- generalizes this construction
- can be generalized to discrete dynamic systems

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`Let’s see the horseshoe map first!`

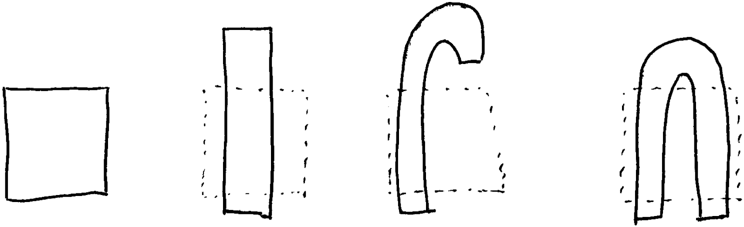
Part III

Chaos in the horseshoe map

The horseshoe map

Definition

The **horseshoe map** is the discrete dynamical system defined on a square as follows:



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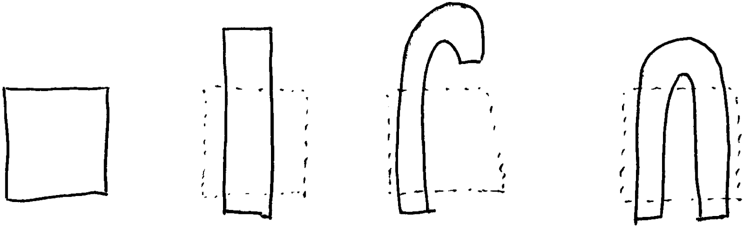
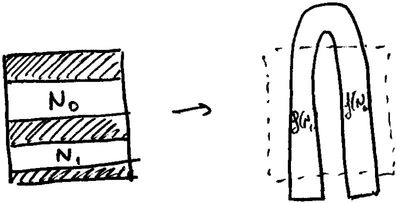


It can be extended to a dds on the whole plane $\mathbb{R}^2$ and we are interested in $\mathrm{Inv}(N, \varphi)$.

Invariant points and binary strings

Write $f = x \mapsto \varphi(x, 1)$ and $N_0 \uplus N_1 = f^{-1}(N)$:



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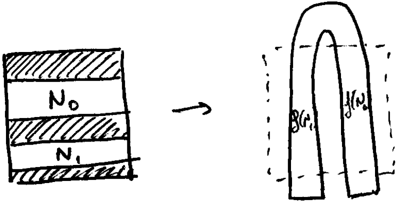

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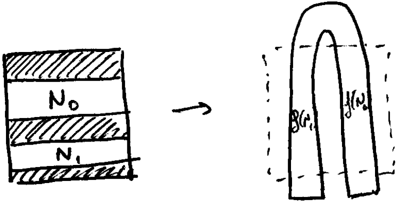

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- This map satisfies $\rho \circ f = \sigma \circ \rho$, where σ is the *shift map*

$$\begin{array}{ccccc} \sigma & : & \Sigma_2 & \to & \Sigma_2 \\ & & (n \mapsto s_n) & \mapsto & (n \mapsto s_{n+1}) \end{array}$$

Symbolic Dynamics – A chaotic map

The set Σ_2 admits a metric defined by

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Theorem

There exists an homeomorphism $\rho : \mathrm{Inv}(N, \varphi) \to \Sigma_2$ *(the important part is that* ρ *is a continuous surjection) such that* $\rho \circ f = \sigma \circ \rho$ *(it's called a topological conjugacy).*

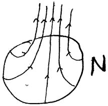
Part IV

The Conley index

The Conley index

- An **isolating neighborhood** N is a compact set such that $x \in \mathrm{bd}(N)$ implies $\varphi(x) \not\subseteq N$, i.e.

  $$\mathrm{Inv}(N, \varphi) \quad \subseteq \quad \mathrm{int}(N)$$



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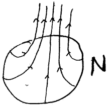

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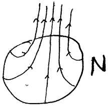

- An **isolated invariant set** S is a compact set such that $S = \mathrm{Inv}(N, \varphi)$ for some isolating neighborhood N.

Theorem

For every isolating neighborhood N of S there exists an isolating block $S \subseteq M \subseteq N$ and $H^*(M, M^-)$ only depends on S (or N), where H^* denotes the Alexander-Spanier cohomology (with coefficients in $\mathbb{Q}$).

Generalizing to discrete systems

Suppose given a dds φ, and write $f = x \mapsto \varphi(x, 1)$.

Definition

An **index pair** (P_1, P_2) of an isolated invariant set S is a pair of compact sets such that

$$\begin{aligned} f(P_2) \cap P_1 &\subseteq P_2 \\ P_1 \cap \mathrm{cl}(f(P_1) \setminus P_1) &\subseteq P_2 \\ S = \mathrm{Inv}(\mathrm{cl}(P_1 \setminus P_2), f) &\subseteq \mathrm{int}(P_1 \setminus P_2) \end{aligned}$$

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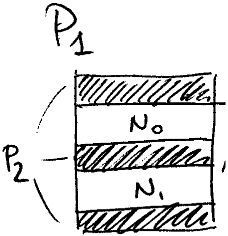

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Problem: $H^*(P_1, P_2)$ is *not* an invariant...

Index pairs

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Definition

An **index quadruple** $(P_1, P_2, \overline{P_1}, \overline{P_2})$ consists of

- an index pair (P_1, P_2)
- we have

$$P_1 \cup f(P_1) \subseteq \overline{P_1} \qquad\qquad P_2 \cup f(P_2) \subseteq \overline{P_2}$$

- the inclusion $\iota : (P_1, P_2) \hookrightarrow (\overline{P_1}, \overline{P_2})$ is an excision:

$$\iota^* \quad : \quad H^*(P_1, P_2) \overset{\sim}{\hookrightarrow} H^*(\overline{P_1}, \overline{P_2})$$

The discrete Conley index

Theorem

For every isolating neighborhood N of f there exists an index quadruple such that $\operatorname{Inv}(N, f) \subseteq P_1 \subseteq \overline{P_1} \subseteq N$ and the Conley index of f in N is

$$\operatorname{Con}(N, f) \quad = \quad L(H^*(P_1, P_2), I_P)$$

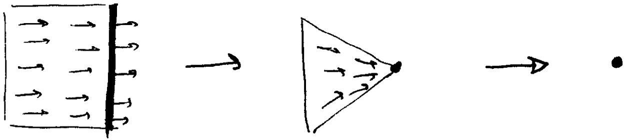
with $I_P = f^* \circ (\iota^*)^{-1} : H^*(P_1, P_2) \to H^*(P_1, P_2)$ where

$$\begin{tikzcd}(P_1, P_2) \arrow[r, "f"] & (\overline{P_1}, \overline{P_2}) & (P_1, P_2) \arrow[l, "\iota"']\end{tikzcd}$$

and this does not depend on the choice of the index quadruple.

Example: simple Conley indexes

- $\mathrm{Con}(N, f) = 0$



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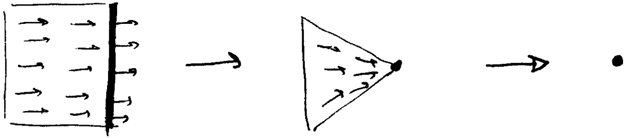
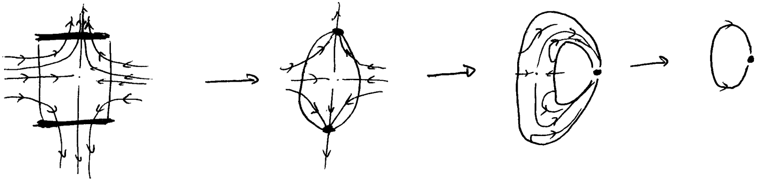


- $\mathrm{Con}(N, f) = \mathbb{Q}$



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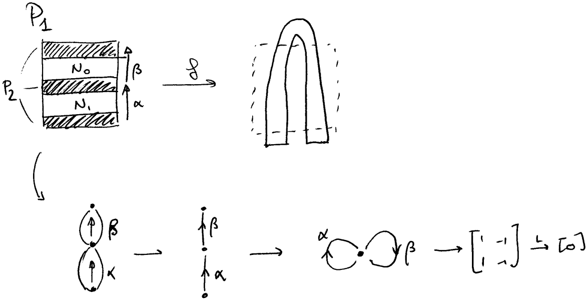
We set $P_1 = N$, $P_2 = N \setminus (N_1 \uplus N_2)$ and $\overline{P_i} = P_i \cup f(P_i)$:



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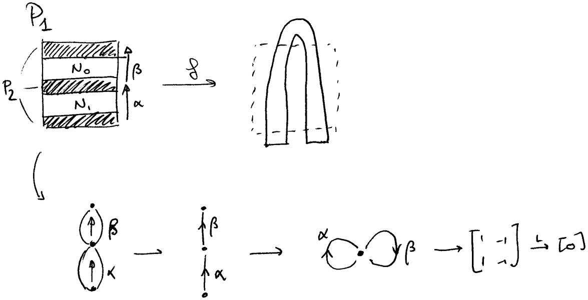

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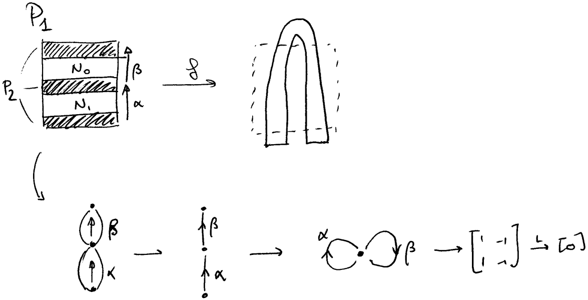

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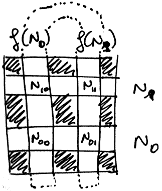
We have $A^2 = 0$ and therefore $\operatorname{gker} A = \mathbb{Q}$ and $\operatorname{Con}(N, f) = 0$.

Chaos with Conley

Theorem

If $N = N_0 \cup N_1$ *is an isolating neighborhood with* $N_0 \cap N_1 = \emptyset$*. If for* $i \in \{0, 1\}$,

$$\mathrm{Con}(N_i, f)_n = \begin{cases} (\mathbb{Q}, \mathrm{id}) & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$



and the map parts of

$$\mathrm{Con}(N_{00,01,11}, f) \quad \text{and} \quad \mathrm{Con}(N_{00,10,11}, f)$$

are different from the identity then there exists a continuous surjection $\rho : \mathrm{Inv}(N, f) \to \{0, 1\}^{\mathbb{Z}}$ *such that*

$$\rho \circ f^d = \sigma \circ \rho$$

for some $d \in \mathbb{N}$.

Part V

Guaranteed methods

Abstract interpretation

Suppose given a Galois connection

$$\begin{tikzcd}\mathcal{P}(\mathbb{R}^n) \arrow[r, bend left, "\alpha"] \arrow[r, phantom, "\perp"] & D \arrow[l, bend left, "\gamma"]\end{tikzcd}$$

Typical example: the elements of $D = \mathcal{P}(\mathcal{K}_{\mathbb{R}^n})$ are sets of cubes.

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Every map

$$f : X \to Y$$

can be approximated as a map

$$F : X \to \mathcal{P}(\mathcal{K}_Y)$$

such that

$$\forall x \in X, \quad \alpha \circ f(x) \leq F(x)$$

and previous computations can be done on approximated maps.

Guaranteed computations on dds

Given a dds $f : X \to X$, we “replace” f by an approximation $F : X \to \mathcal{P}(\mathcal{K}_X)$ in the computations (previous definitions are adapted to the approximated case).

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If N is an isolating neighborhood of F and (P_1, P_2) is an index pair for F in N, then for every function f approximated by F, $\gamma(N)$ is an isolating neighborhood for f and $(\gamma(P_1), \gamma(P_2))$ is an index pair for f.

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Theorem

...similarly for index quadruples...

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We also have to compute the homology of a map to compute the Conley index!

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- $F : X \to \mathcal{P}(\mathcal{K}_X)$ is **acyclic-valued** if for every $x \in X$, $\gamma(F(x))$ is acyclic.
- $F : X \to \mathcal{P}(\mathcal{K}_X)$ is **lower-continuous** if

$$\forall x \in X, \quad F(x) = \bigcap \{F(Q) \mid x \in Q \in \mathcal{K}_X\}$$

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Theorem

If F is lower-continuous and acyclic-valued, then for every chain map f approximated by F we have

$$H_*(f) \quad = \quad H_*(F)$$

Guaranteed homology of a map

A function $F : X \to Y$ can be represented by its *graph*

$$G(F) \quad = \quad \{(x, y) \in X \times Y \mid y \in F(x)\}$$

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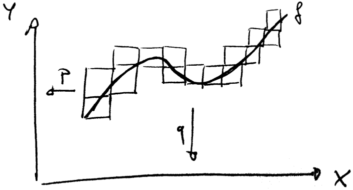

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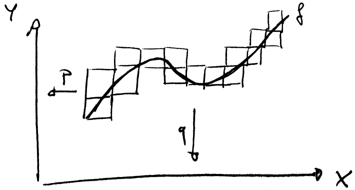

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Theorem

$$H_*(f) \quad = \quad H_*(q)H_*(p)^{-1}.$$

Cubical homology

Images are approximated by finite sets of cubes, one can devise very fast methods for computing the (cubical) homology. . .

Part VI

Theorems

A few more definitions

Given a continuous dynamical system φ, we define the following.

- The **return time** $t_{\varphi,A} : A \to \mathbb{R}^+$ of φ in $A \subseteq X$ is

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- $A \subseteq X$ is a **Poincaré section** when $P_{\varphi,A}$ is continuous and not empty.
- Given a boolean matrix A of size $n \times n$, we define

$$\Sigma(A) \quad = \quad \{s \in \{0,\dots,n-1\}^{\mathbb{Z}} \mid \forall i \in \mathbb{Z}, A(s_i, s_{i+1}) = 1\}$$

  i.e. the paths in the graph defined by A.

The kind of theorems we get

Theorem

*Consider the **Lorenz equations** and the plane $P = \{(x,y,z) \mid z = 27\}$. For all parameter values in a sufficiently small neighborhood of $(\sigma, \rho, \beta) = (28, 10, 8/3)$ there exists a Poincaré section $N \subseteq P$ such that the associated Poincaré map g is Lipschitz and well defined. Furthermore, for*

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

there is a continuous surjection $\rho : \mathrm{Inv}(N,g) \to \Sigma(A)$ such that $\rho \circ g = \sigma \circ \rho$. In particular $h(\mathrm{Inv}(N,g)) \geq 0.48$. Moreover, for every $\alpha \in \Sigma(A)$ which is periodic there exists an $x \in \mathrm{Inv}(N,g)$ on a periodic trajectory such that $\rho(x) = \alpha$.

The kind of theorems we get

Theorem

Consider the ***Hénon map*** $h : \mathbb{R}^2 \to \mathbb{R}^2$ *given by the formula* $h(x, y) = (1 + y/5 - ax^2, 5bx)$ *at the classical parameter values* $a = 1.4$ *and* $b = 0.2$*. The discrete dynamical system induced by the Hénon map admits an invariant set* S *semiconjugate with a subshift of finite type on* 8 *symbols and topological entropy* $h = 0.28$ *Moreover, if*

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

then for each periodic sequence $\theta \in \Sigma(A)$ *with period* p *the set* $\rho^{-1}(\theta)$ *contains a periodic orbit with period* p*. In particular* $h(S) \geq 0.28$*.*

Part VII

Improving abstract interpretation

Alternatives to cubical sets

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- Taylor models

$$\begin{aligned} f(x) \quad &= \quad \sum_{k=0}^{n} \frac{f^{(k)}(x - x_0)}{k!}(x - x_0)^k + \frac{f^{(n+1)}(\zeta)}{(n+1)!}(x - x_0)^{n+1} \\ &= \quad P_n^f(x - x_0) + I_n^f \end{aligned}$$

Thanks!

Questions?