FROM GEOMETRIC SEMANTICS TO ASYNCHRONOUS COMPUTABILITY

ÉRIC GOUBAULT

SAMUEL MIMRAM

CHRISTINE TASSON

École Polytechnique / PPS

DISC conference

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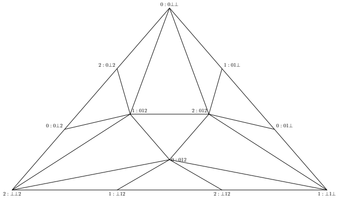
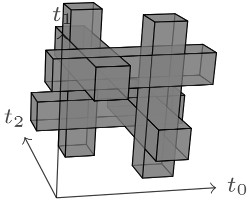
General idea

Given an asynchronous protocol, we relate two associated geometric constructions:





the **protocol complex** [Herlihy, . . .]

the **geometric semantics** [Goubault, . . .]

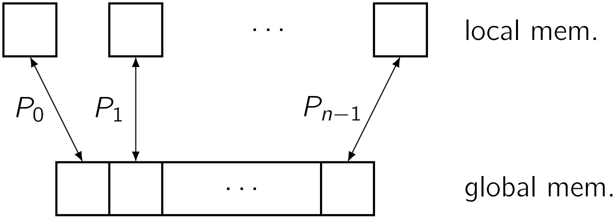
Aim: show impossibility results!
This work should help generalizing to more communication primitives.

ASYNCHRONOUS COMPUTABILITY

Asynchronous protocols

We consider here a model with n processes P_i:

- each process has a local memory cell
- there is a global memory with n cells



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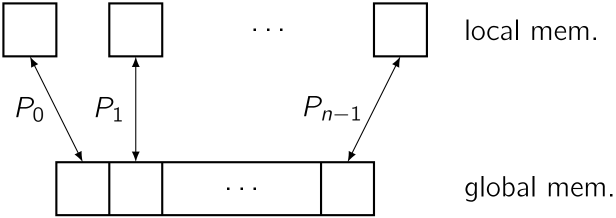

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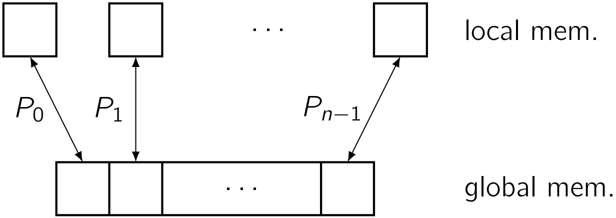

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A **protocol** = what processes compute depending on values.

Decision tasks

We write $\mathcal{V}$ for the set of values, where $\perp$ denotes a dead process.

The question is whether a protocol can solve a **task** $\Theta \subseteq \mathcal{V}^n \times \mathcal{V}^n$ (where $\mathcal{V}$ is the set of values) in the presence of faults.

We suppose here that the initial value of a process is its process number (for simplicity).

Example (*Consensus*)

All processes must end with the same value, which is among the input values of the alive processes, i.e.

$$\Theta \quad = \quad \left\{ \begin{array}{c} (01, 00) \\ (01, 11) \\ (0\perp, 0\perp) \\ (\perp 1, \perp 1) \end{array} \right\}$$

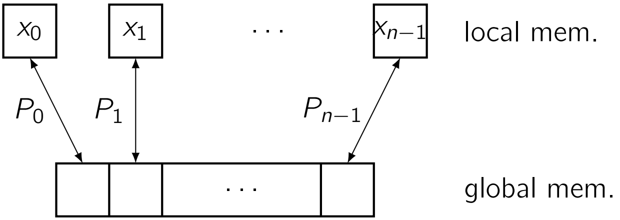
Execution traces

An **execution trace** is determined by a word in $\{u_i, s_i\}^*$.

Remark

The effect of execution traces on memory is invariant under the smallest congruence $\approx$ such that

$$u_j u_i \approx u_i u_j \qquad s_j s_i \approx s_i s_j$$



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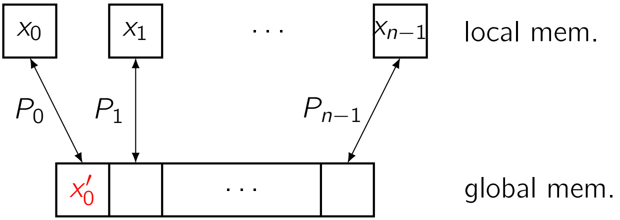

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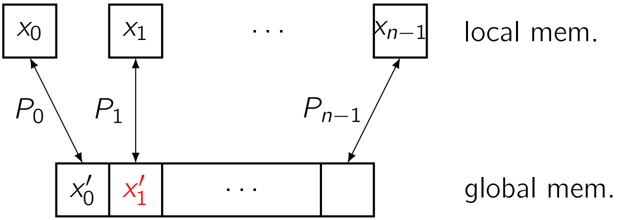

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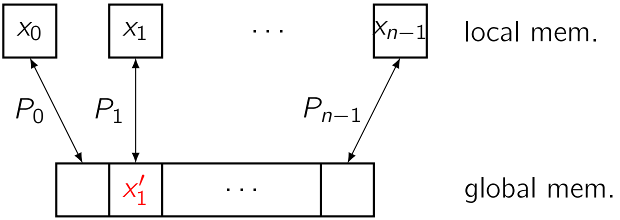

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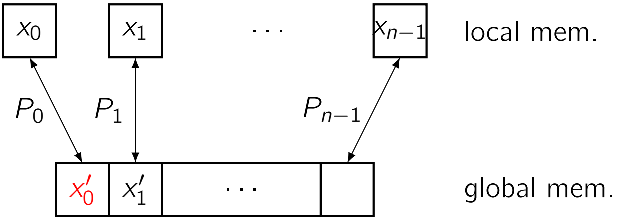

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GEOMETRIC SEMANTICS

Directed geometric semantics

The idea of geometric semantics is to formalize the dictionary:

$$\textbf{program} \qquad \Leftrightarrow \qquad \textbf{topological space}$$

so that we can import tools from (algebraic) topology in order to study concurrent programs.

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so that we can import tools from (algebraic) topology in order to study concurrent programs.

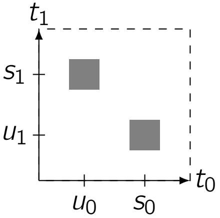
We actually need to use spaces equipped with a notion of direction in order to take in account irreversible time.

An example

Consider two processes executing one round of update/scan, i.e.

$$u_0.s_0 \quad \| \quad u_1.s_1$$

The geometric semantics of this program will be



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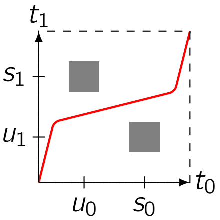

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directed path : $u_1 u_0 s_0 s_1$

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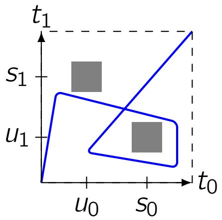

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non directed path : ???

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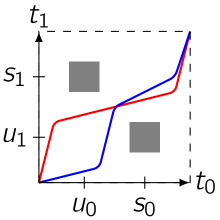

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homotopy between paths : $u_1 u_0 s_0 s_1 \approx u_0 u_1 s_0 s_1$

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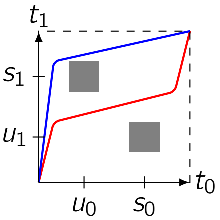

i.e. a square $[0,1] \times [0,1]$ minus two holes, which is directed componentwise.

some paths are not homotopic

More examples

This generalizes to *more rounds*:
consider two processes executing 2 and 4 rounds of update/scan,

$$u_0.s_0.u_0.s_0 \quad \| \quad u_1.s_1.u_1.s_1.u_1.s_1.u_1.s_1$$

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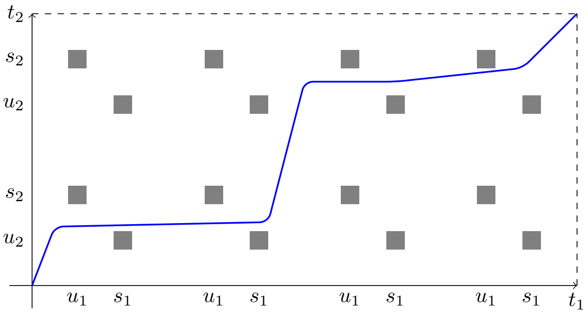
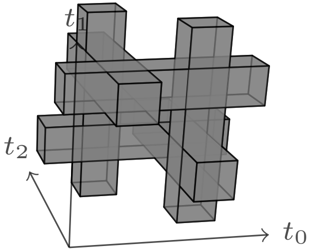


More examples

This generalizes to *more processes*:
consider three processes executing one round of update/scan,

$$u_0.s_0 \quad \| \quad u_1.s_1 \quad \| \quad u_2.s_2$$

The geometric semantics of this program will be



NB: we will illustrate in dimension 2, where things are simpler

Formally,

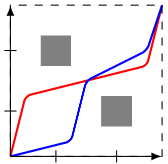
Definition

A **pospace** $(X, \leq)$ consists of a topological space X equipped with a partial order $\leq \subseteq X \times X$, which is closed.

A **dipath** p is a continuous non-decreasing map $p : [0, 1] \to X$.

A **dihomotopy** H from a path p to a path q is a continuous map $H : [0, 1] \times [0, 1] \to X$ such that

- $H(0, t) = p(t)$ for every t
- $H(1, t) = q(t)$ for every t
- $t \mapsto H(s, t)$ is a dipath for every s
- $t \mapsto H(0, t)$ and $t \mapsto H(1, t)$ are constant



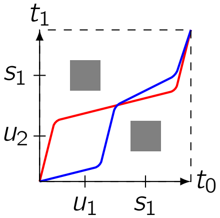
Directed paths vs traces

Theorem

Fixing a number of rounds for each process, there is a bijection between

(i) *directed paths up to directed homotopy in the geometric semantics*

(iii) *execution traces up to* $\approx$



$\Leftrightarrow$ $\quad$ $u_1\,u_0\,s_0\,s_1 \approx u_0\,u_1\,s_0\,s_1$

Directed paths vs traces

Theorem

Fixing a number of rounds for each process, there is a bijection between

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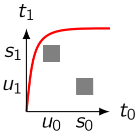
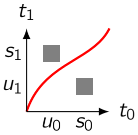
(ii) *colored interval orders*

(iii) *execution traces up to* $\approx$



$[u_0, s_0] \succ [u_1, s_1]$



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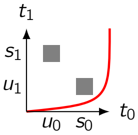

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Definition

A family (I_j) of intervals $I_j = [a_j, b_j]$ of $\mathbb{R}$ is partially ordered by $I_j \prec I_k$ whenever $x < y$ for every $x \in I_j$ and $y \in I_k$, e.g.

$$[1,3] \prec [5,6] \qquad\qquad [1,3] \parallel [2,6]$$

We write $I_j \parallel I_k$ when two elements are not comparable.
A poset isomorphic to such a poset is called an **interval order**.

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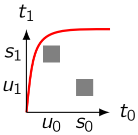
Here, we consider a colored version, where elements are of the form (i, k) with $0 \le i < n$ a process number, such that two elements with the same labels are comparable.



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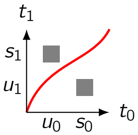

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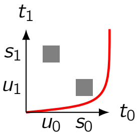

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THE PROTOCOL COMPLEX

The protocol complex

The **protocol complex** is a simplicial complex associated to a protocol, introduced by Herlihy et al. as a central tool in order to characterize tasks which are solvable.

We show here how to reconstruct it from the geometric semantics.

Generic protocols

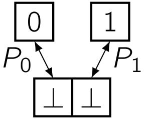
Given a protocol solving a given task, we can without loss of generality suppose that it is

- *full-information*:
  u_i writes the exact local memory in the global one
- *generic*:
  s_i adds the contents of global memory to local one

(except at the end where a choice is made).

Example

With two processes, we have for instance



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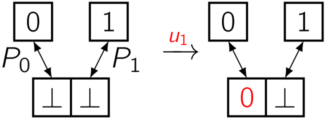

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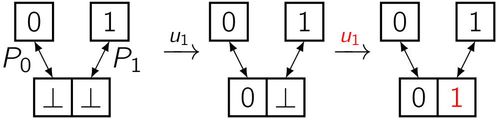

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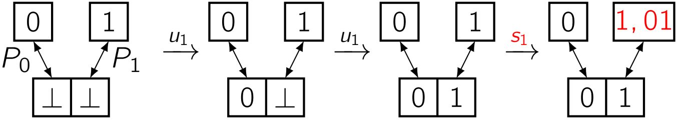

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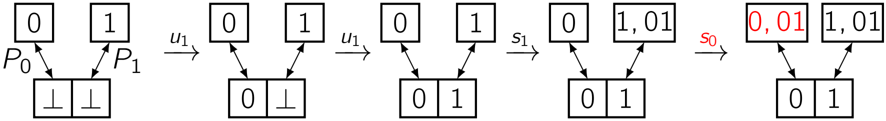

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Definition

The local memory of a process is called its **view**.
Two (or more) views are **coherent** when they can occur at the same time in some execution.

The protocol complex

Definition

Given a number of rounds, the protocol complex is the simplicial complex such that

- vertices are the possible views,
- two vertices are linked by an edge when they are coherent,
- three vertices bound a triangle when they are coherent,
- etc.

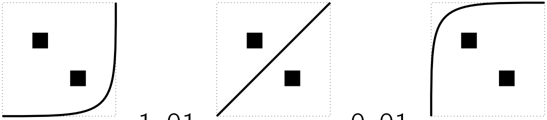
Example

With two processes, after 0 and 1 rounds, the complexes are

- $0 \text{ —— } 1$
- $0, 0\bot \text{ —— } 1, 01 \text{ —— } 0, 01 \text{ —— } 1, \bot 1$

From geometry to the complex

One can notice in the last example that edges are in bijection with directed paths up to homotopy (and with interval orders):

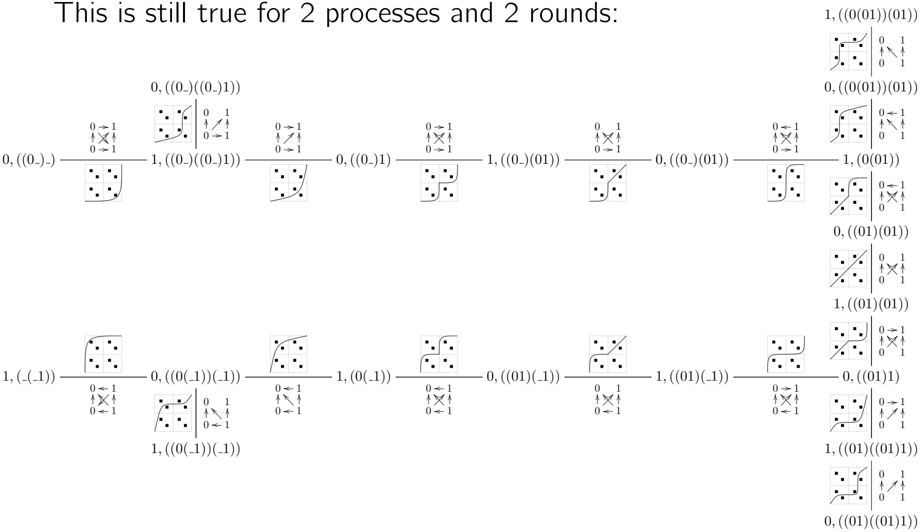


$$0,0\bot \xrightarrow[0\prec 1]{} 1,01 \xrightarrow[0\quad 1]{} 0,01 \xrightarrow[0\succ 1]{} 1,\bot 1$$

(more generally maximal simplices are in bijection with maximal directed paths up to homotopy).

From geometry to the complex

This is still true for 2 processes and 2 rounds:



From interval orders to the complex

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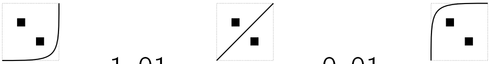
Proposition

Given a colored interval order $(X, \preceq)$, the **view** V_i^k of the i-th process at round k is given by its restriction to the k-th scan of the i-th process

$$V_i^k \quad = \quad \{(j,l) \mid (i,k) \parallel (j,l) \text{ or } (j,l) \prec (i,k)\}$$



$$0,0\bot \xrightarrow[0\prec 1]{\hspace{3em}} 1,01 \xrightarrow[0\quad 1]{\hspace{3em}} 0,01 \xrightarrow[0\succ 1]{\hspace{3em}} 1,\bot 1$$

The interval order complex

Definition

The **interval order complex** is the simplicial complex whose

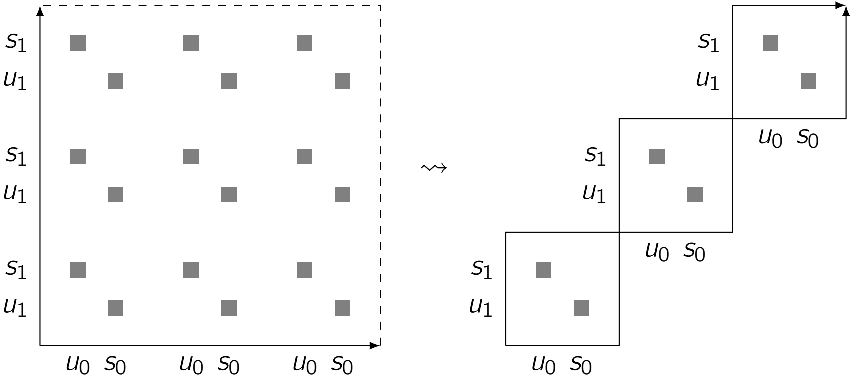
- *vertices* are $((i,k), V_i^k)$ where i stands for the i-th process, k for the round number and V_i^k for an interval order such that for all $(j,l) \in V_i^k$, either $(i,k) \parallel (j,l)$ or $(j,l) \prec (i,k)$,
- *maximal simplices* are $\left\{((0,r_0), V_0^{r_0}), \ldots, ((n,r_n), V_n^{r_n})\right\}$ such that there is an interval order $(X_{(r)}^n, \prec)$ whose restriction to (i, r_i) is $V_i^{r_i}$.

Theorem

The interval order complex is isomorphic to the protocol complex.

The layered protocol complex

Generally, one considers executions which are **layered**: all processes must have finished round n (or died) before process can start round $n+1$.



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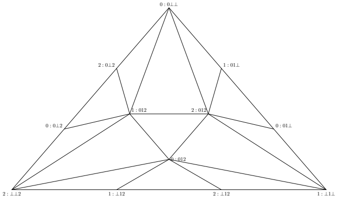
Proposition

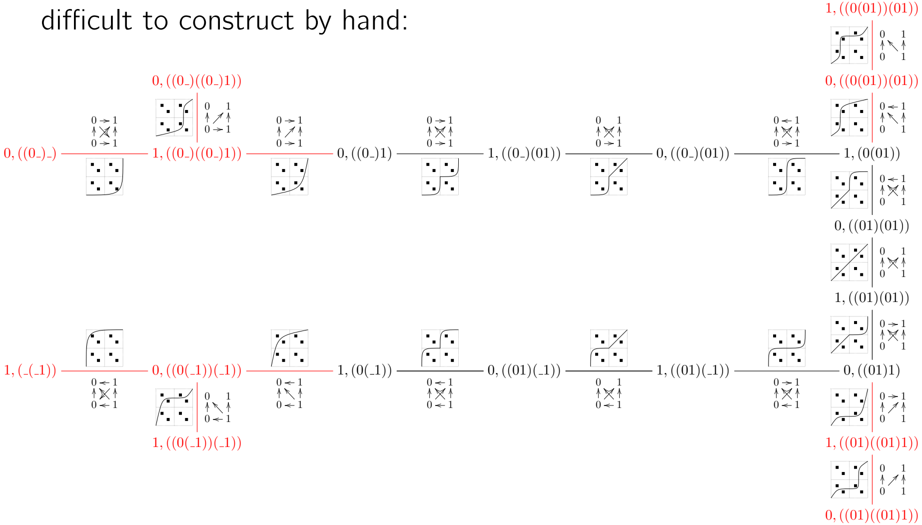
Layered immediate snapshot executions correspond to the interval orders such that $J \prec K$ and $I \parallel J$ implies $I \prec K$.

Moreover, we can recover the fact that layered protocol complexes are iterated chromatic subdivisions of the standard simplex.



The layered protocol complex

Here, we can compute non-layered protocols, which would be difficult to construct by hand:



Conclusion

We have linked geometric semantics and asynchronous computability.

The geometric semantics of many more primitives than update/scan is known (e.g. test/set, compare/swap, etc.) the next step is to try to start from the geometric semantics in order to invent the corresponding “protocol complex”
(NB: interval orders were not really crucial in this work).