THE HIGHER-DIMENSIONAL ALGEBRAIC STRUCTURE OF PARTIAL ORDERS

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CHoCoLa MEETING

10 MAY 2012

HIGHER-DIMENSIONAL REWRITING THEORY

Rewriting theory has proven to be very useful to study

- monoids (and groups)
- term algebras

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- monoids (and groups)
- term algebras
- ***n*-categories**

`It can be generalized`
`to higher dimensions!`

IN THIS TALK

I will be interested in what can be said about categories of

- relations
- partial orders
- increasing functions

The main result will be a *"coherence theorem for commutative monads"*.

Rewriting systems

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A **rewriting system** consists of

- a set of *terms* generated by a free construction:
  - free monoid: *string rewriting systems*
  - free term algebra: *term rewriting systems*
- a set of *rewriting rules*: $r : t \to u$

Example

$\Sigma = \{a, b\}$ $\quad$ $\text{terms} = \Sigma^*$ $\quad$ $\text{rules} = \{ba \to ab\}$

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A term t **rewrites** to a term t' when there exists

- a rule $r : u \to u'$
- a context C such that $t = C[u]$ and $t' = C[u']$

Example

$\Sigma = \{a, b\}$ $\qquad$ $\text{terms} = \Sigma^*$ $\qquad$ $\text{rules} = \{ba \to ab\}$

$$aabaab \xrightarrow{aarab} aaabab$$

CONVERGENT REWRITING SYSTEMS

- A rewriting system can be **terminating** when there is no infinite reduction path

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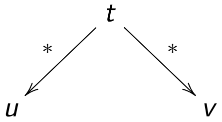

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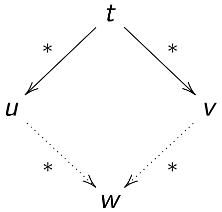

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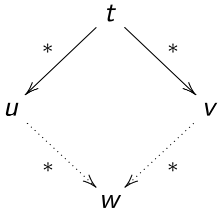

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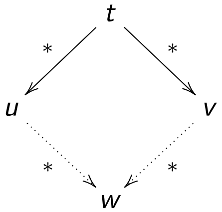

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In a convergent rewriting system, every term has a **normal form**: canonical representative of terms modulo rewriting.

Why
are those properties
interesting?

PRESENTATIONS OF MONOIDS

A **presentation**

$$\langle G \mid R \rangle$$

of a monoid M consists of

- a set G of *generators*
- a set $R \subseteq G^* \times G^*$ of *relations*

such that

$$M \quad \cong \quad G^* / \equiv_R$$

Example

- $\mathbb{N} \cong \langle a \mid \rangle$
- $\mathbb{N}/2\mathbb{N} \cong \langle a \mid aa = 1 \rangle$
- $\mathbb{N} \times \mathbb{N} \cong \langle a, b \mid ba = ab \rangle$
- $\mathfrak{S}_n \cong \langle \sigma_1, \ldots, \sigma_n \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1},\ \sigma_i^2 = 1,\ \sigma_i \sigma_j = \sigma_j \sigma_i \rangle$
- ...

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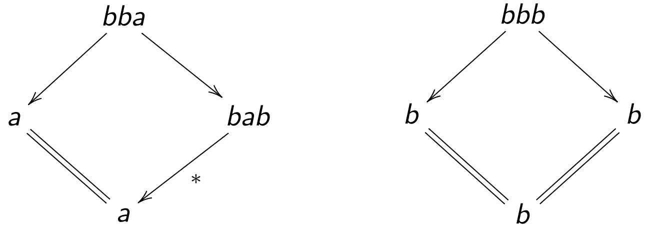
Critical pairs are joinable:



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Normal forms are:

$$a^n \qquad \text{and} \qquad a^n b$$

They are in bijection with $\mathbb{N} \times (\mathbb{N}/2\mathbb{N})$!

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Normal forms are:

a^n and $a^n b$

They are in bijection with $\mathbb{N} \times (\mathbb{N}/2\mathbb{N})$!

Remark: we actually only need normal forms

How do we generalize this
to present categories?

PRESENTING CATEGORIES

Presentation of a monoid $M \cong \langle G \mid R \rangle$:

$$G$$

PRESENTING CATEGORIES

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$$\begin{array}{c} G \\ \downarrow{\scriptstyle i} \\ G^* \end{array}$$

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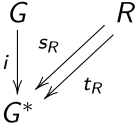

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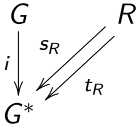

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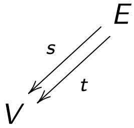

a graph

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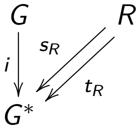

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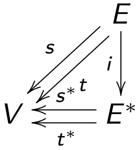

a free graph

PRESENTING CATEGORIES

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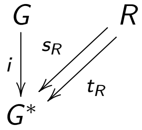

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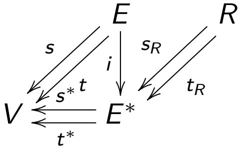


such that $s^* s_R = s^* t_R$ and $t^* s_R = t^* t_R$

a presentation of a category

$$\mathcal{C} \quad \cong \quad G^* / \equiv_R$$

We see a pattern emerge!

[Burroni93,Street76,Power90]

POLYGRAPHS

A 0-**polygraph**:

$$\Sigma_0^*$$

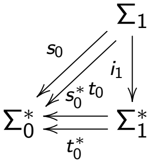
POLYGRAPHS

A 1-**polygraph**:

$$\Sigma_0^* \underset{t_0}{\overset{s_0}{\leftleftarrows}} \Sigma_1$$

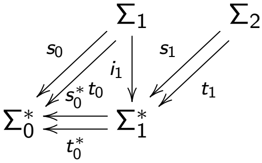
POLYGRAPHS

A 1-**polygraph** generates a category:



POLYGRAPHS

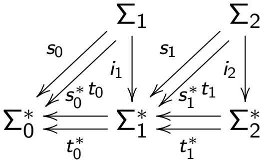
A 2-**polygraph**:



such that $s_0^* s_1 = s_0^* t_1$ and $t_0^* s_1 = t_0^* t_1$

POLYGRAPHS

A 2-**polygraph** generates a 2-category:



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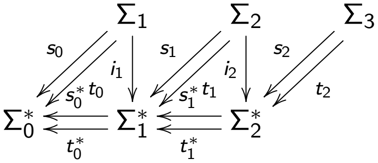

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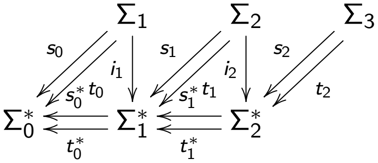
A 3-**polygraph** . . .



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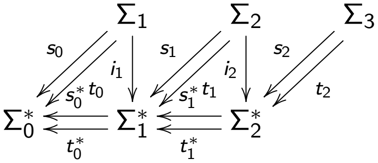

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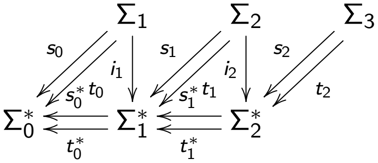

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- We write $\widetilde{\Sigma^*}$ for the 2-category obtained from Σ^* by identifying two 2-cells f and g for which there exists a 3-cell $\alpha : f \Rrightarrow g$

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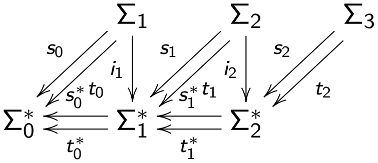

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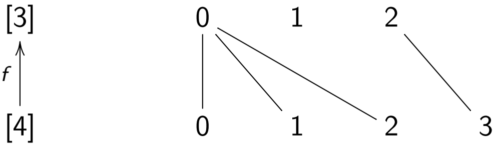
- We write $\widetilde{\Sigma^*}$ for the 2-category obtained from Σ^* by identifying two 2-cells f and g for which there exists a 3-cell $\alpha : f \Rrightarrow g$
- The 3-polygraph Σ **presents** a 2-category $\mathcal{C}$ when $\mathcal{C} \cong \widetilde{\Sigma^*}$

THE SIMPLICIAL CATEGORY

Consider the simplicial category Δ whose

- objects are natural integers $[n] = \{0, 1, \ldots, n-1\}$
- morphisms are increasing functions $f : [m] \to [n]$

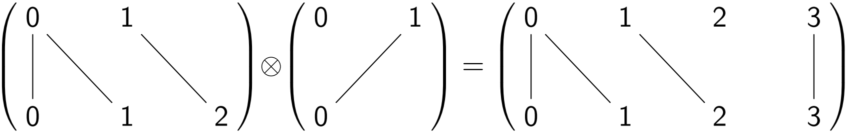
For instance $f : 4 \to 3$



THE SIMPLICIAL CATEGORY

The category Δ is monoidal with $[0]$ as unit and $\otimes$ defined

- on objects: $[m] \otimes [n] = [m+n]$
- on morphisms:



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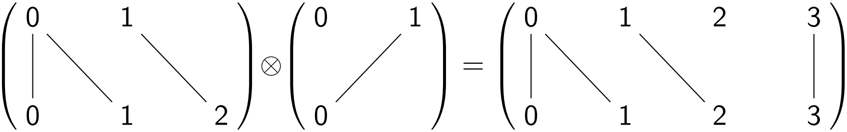

A monoidal category is the same as a 2-category with only one 0-cell so we can (hope to) present it with a 3-polygraph!
[MacLane,Burroni,Lafont]

PRESENTING THE SIMPLICIAL CATEGORY

We will show that the 2-category Δ is presented by the polygraph

$$\Sigma_0^*$$

whose generators are

- $\Sigma_0 = \{*\}$

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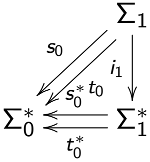

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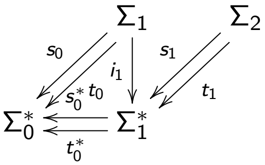

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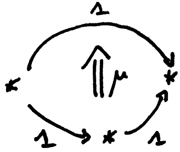

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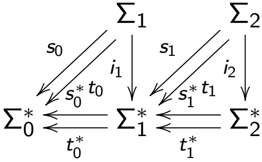

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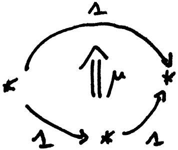

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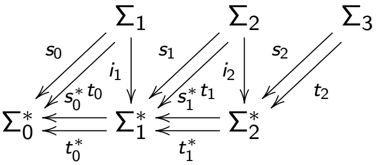

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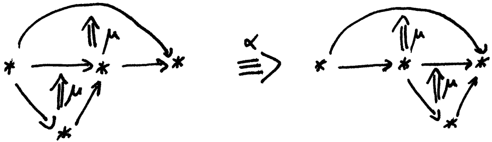
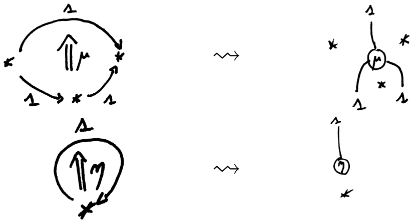
- $\Sigma_2 = \{\mu : (1 \otimes 1) \Rightarrow 1, \eta : 0 \Rightarrow 1\}$
- $\Sigma_3 = \left\{ \begin{array}{l} \alpha : \mu \circ (\mu \otimes 1) \Rrightarrow \mu \circ (1 \otimes \mu), \\ \lambda : \mu \circ (\eta \otimes 1) \Rrightarrow 1, \rho : \mu \circ (1 \otimes \eta) \Rrightarrow 1 \end{array} \right\}$



STRING DIAGRAMS

The 2-generators can be drawn as string diagrams:



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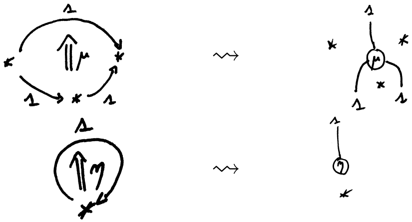
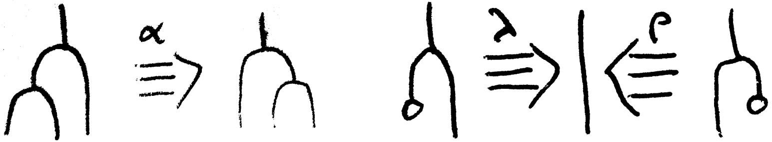


and the 3-generators become



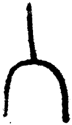
We recognize the laws for monoids!

PROVING THE PRESENTATION

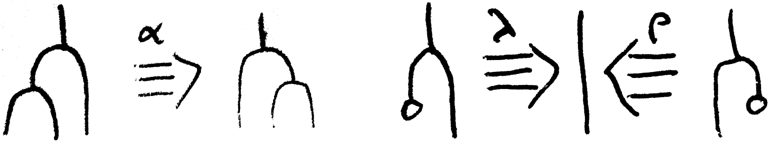
We have to prove that we have a presentation

$$\Delta \cong \widetilde{\Sigma^*}$$

which means that diagrams built from the 2-generators

$\mu =$  and $\eta =$ 

by composition and tensoring, considered modulo the relations

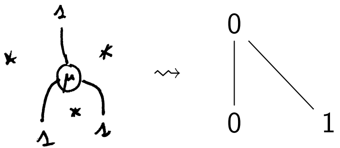


are in bijection with increasing functions.

PROVING THE PRESENTATION

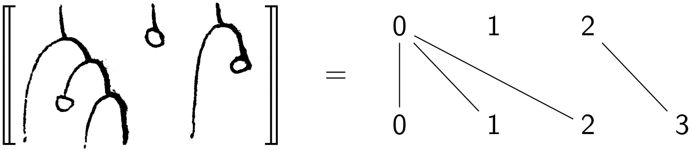
We have to prove that we have a presentation $\Delta \cong \widetilde{\Sigma^*}$.

- The generators can be interpreted as functions:



and

Thus inducing a functor $[\![-]\!] : \partial\Sigma^* \to \Delta$.



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- The generators can be interpreted as functions. Thus inducing a functor $[\![-]\!] : \partial\Sigma^* \to \Delta$.
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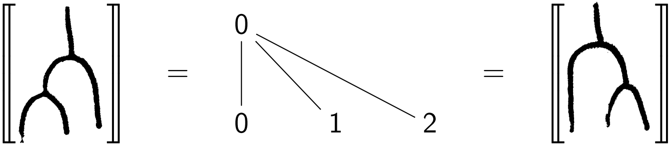

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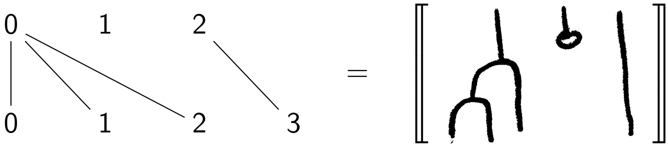

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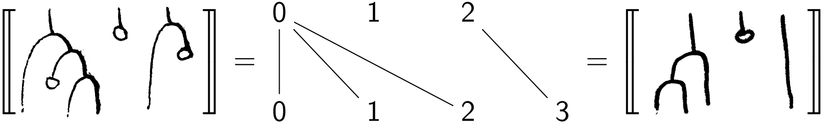
- The functor $[\![-]\!]$ is full.
- The 2-functor $[\![-]\!]$ is faithful (more difficult), i.e. $\widetilde{\Sigma^*} \cong \Delta$.

FAITHFULNESS

To show that the 2-functor $[\![-]\!] : \widetilde{\Sigma^*} \to \Delta$ is faithful, we have to show that if two diagrams get interpreted as the same function then they are equivalent modulo the 3-generators α, λ and ρ.



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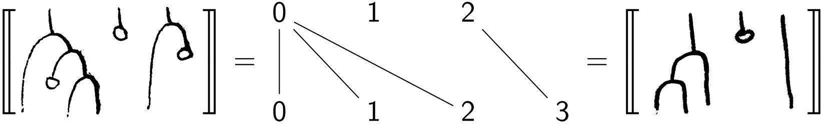

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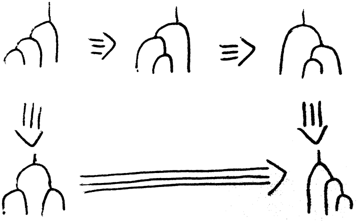

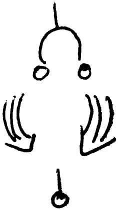

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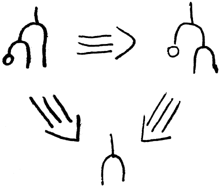

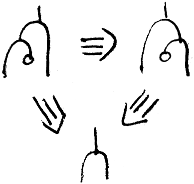

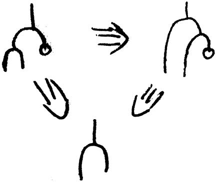

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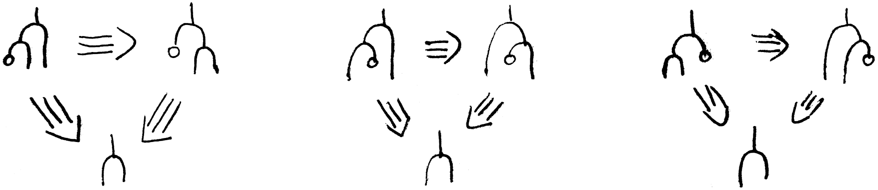

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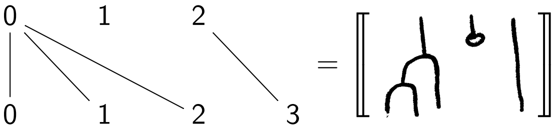

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- we have a presentation $\widetilde{\Sigma^*} \cong \Delta$
- i.e. diagrams built from μ and η modulo the relation generated by α, λ and ρ are in bijection with functions
- the category Σ is the theory for monoids.

Δ AS A THEORY FOR MONOIDS

Since we have described Δ by generators and relations we know that a strict monoidal functor $M : \Delta \to \mathcal{C}$ is uniquely determined by the images of the generators, which satisfy the relations:

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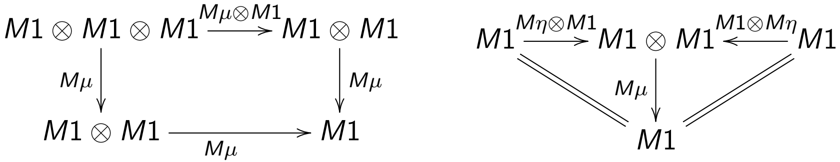

In other words, a monoidal functor $M : \Delta \to \mathcal{C}$ is a monoid in $\mathcal{C}$!

$$\mathbf{StrMonCat}(\Delta, \mathcal{C}) \cong \mathbf{Mon}(\mathcal{C})$$

Ex: in **Set**, **Cat**, ...

AN IMPORTANT EXAMPLE: MONADS

Given a category $\mathcal{C}$, consider the 2-category with

- one 0-cell: $\mathcal{C}$
- 1-cells: endofunctors $\mathcal{C} \to \mathcal{C}$
- 2-cells: natural transformations

It's a 2-category with one 0-cell, i.e. a monoidal category.

Monoids in this category are precisely the monads on $\mathcal{C}$.

SOME REMARKS

[Lafont]

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- Actually, those higher-dimensional rewriting systems are much more complicated than usual (string/term) rewriting systems: a convergent rewriting system can have an infinite number of critical pairs!

Let's see some more examples.

MORE EXAMPLES OF PROS

Definition

A **PRO** is a monoidal category whose objects are integers and tensor product is given on objects by addition (e.g. Δ).

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A **PRO** is a monoidal category whose objects are integers and tensor product is given on objects by addition (e.g. Δ).

As for Δ, a presentation of a PRO necessarily have

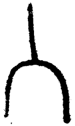
- $\Sigma_0 = \{*\}$: it is a 2-category with one 0-cell
- $\Sigma_1 = \{1\}$: the objects are $\Sigma_1^* \cong \mathbb{N}$
- it is thus enough to specify the 2-generators and the 3-generators (the relations)

A PRESENTATION OF Δ

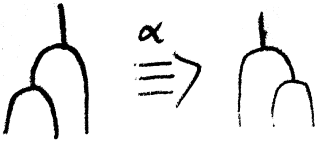
The simplicial category Δ admits a presentation with

- two 2-generators

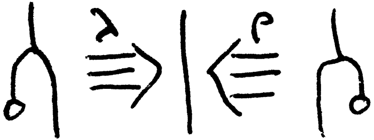
$\mu : 2 \to 1$ $\qquad\qquad$ $\eta : 0 \to 1$



- three relations (3-generators)



(associativity)



(unitality)

- Δ: theory of monoids

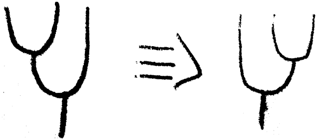
A PRESENTATION OF Δ^{op}

Dually, the category Δ^{op} admits a presentation with

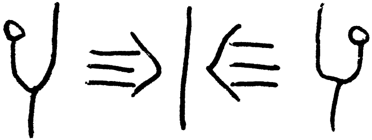
- two 2-generators

$\delta : 1 \to 2$ $\varepsilon : 1 \to 0$

- three relations (3-generators)



(coassociativity)



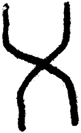
(counitality)

- Δ^{op}: theory of comonoids

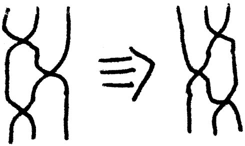
A PRESENTATION OF Bij

The PRO **Bij** with $\mathbb{N}$ as objects and bijections $f : [n] \to [n]$ as morphisms admits a presentation with

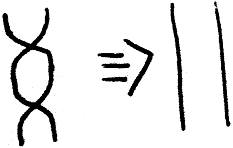
- one 2-generator $\gamma : 2 \to 2$



- two relations



(Yang-Baxter)



(involutivity)

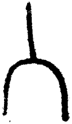
- it generalizes the usual presentation of the symmetric groups by products of transpositions

A PRESENTATION OF FinOrd

The PRO **FinOrd** with $\mathbb{N}$ as objects and functions $f : [m] \to [n]$ as morphisms admits a presentation with

- three 2-generators

$\mu : 2 \to 1$



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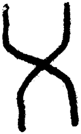

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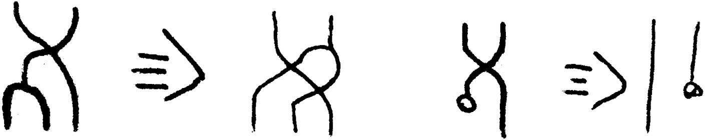
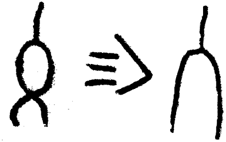
- relations expressing that
  - (μ, η) is a monoid + γ is a symmetry
  - compatibility between monoid and symmetry



  - commutativity of μ



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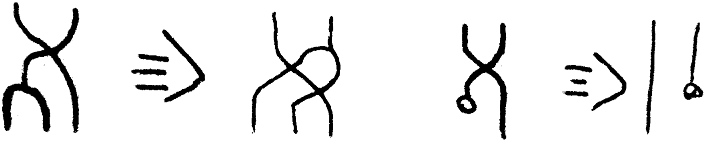

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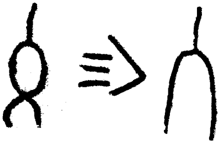

- **FinOrd** is thus the theory for commutative monoids

A PRESENTATION OF **MRel**

The PRO **MRel** with $\mathbb{N}$ as objects and $m \times n$ matrices with coefficients in $\mathbb{N}$ as morphisms $[m] \to [n]$.

For instance, a morphism $[3] \to [2]$:

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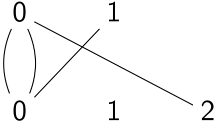

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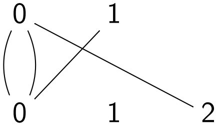

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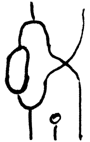

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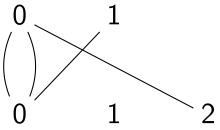

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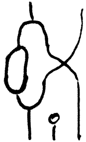

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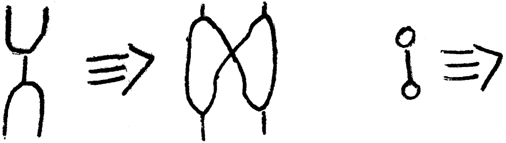
$\mu : 2 \to 1$ $\eta : 0 \to 1$ $\delta : 1 \to 2$ $\varepsilon : 1 \to 0$ $\gamma : 2 \to 2$

- relations
  - (μ, η, γ) is a commutative monoid
  - $(\delta, \varepsilon, \gamma)$ is cocommutative comonoid
  - bialgebra laws



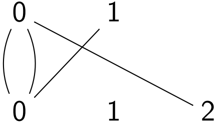
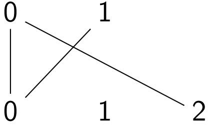
A PRESENTATION OF Rel

The PRO **Rel** with $\mathbb{N}$ as objects and relations $R \subseteq [m] \times [n]$ as morphisms $m \rightarrow n$.

A PRESENTATION OF **Rel**

The PRO **Rel** with $\mathbb{N}$ as objects and relations $R \subseteq [m] \times [n]$ as morphisms $m \to n$.

It can be seen as a quotient of **MRel**:

$\begin{pmatrix} 2 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} =$  $\approx$  $= \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}$

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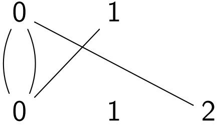
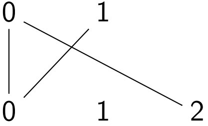
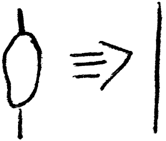
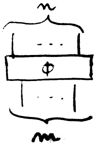
$\begin{pmatrix} 2 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} =$  $\approx$  $= \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}$

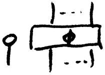
It admits the same presentation as **MRel** with the following extra relation:

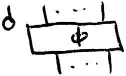


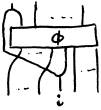
Rel: theory for qualitative bialgebras

THE PROOF

Given a morphism $\phi =$  $: m \to n$ we define

$E\phi =$  $: m+1 \to n$

$H\phi =$  $: m \to n+1$

$W_i\phi =$  $: m \to n$

$Z =$ $0 \to 0$

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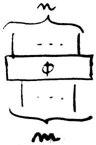
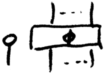
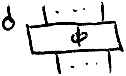
Given a morphism $\phi =$  $: m \to n$ we define

$E\phi =$  $: m+1 \to n$

(add a line)

$H\phi =$  $: m \to n+1$

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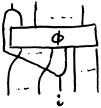
$W_i\phi =$  $: m \to n$

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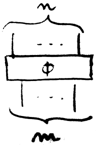
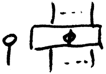
Given a morphism $\phi =$  $: m \to n$ we define

$E\phi =$  $: m+1 \to n$
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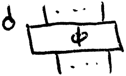
$H\phi =$  $: m \to n+1$
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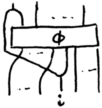
$W_i\phi =$  $: m \to n$
(add a link)

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Lemma

Every diagram is equivalent (modulo the relations) to a composite of those morphisms (called ***pre-canonical forms****).*

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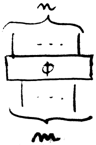
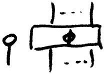
Given a morphism $\phi =$  $: m \to n$ we define

$E\phi =$  $: m+1 \to n$

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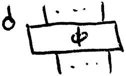
$H\phi =$  $: m \to n+1$

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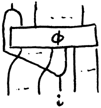
$W_i\phi =$  $: m \to n$

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Lemma

$W_i W_j \phi = W_j W_i \phi \qquad EH\phi = HE\phi \qquad EW_i\phi = W_{i+1}E\phi$

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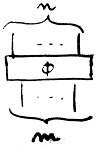
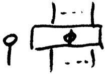
Given a morphism $\phi =$  $: m \to n$ we define

$E\phi =$  $: m+1 \to n$

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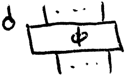
$H\phi =$  $: m \to n+1$

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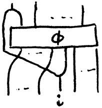
$W_i\phi =$  $: m \to n$

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Lemma

$$W_i W_j \Rrightarrow W_j W_i \quad (i < j) \qquad EH \Rrightarrow HE \qquad EW_i \Rrightarrow W_{i+1} E$$

normal forms are in bijection with multirelations.

Now begins the novel part:
partial orders

THE CATEGORY OF FINITE POSETS

We write **FinPOSet** for the PRO whose

- objects are integers
- a morphism $f : [m] \to [n]$ is a finite poset $(f, \leq_f)$ with m chosen minimal elements and n chosen maximal elements (both sets being distinct)

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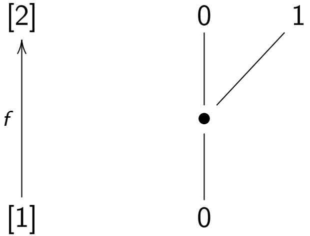
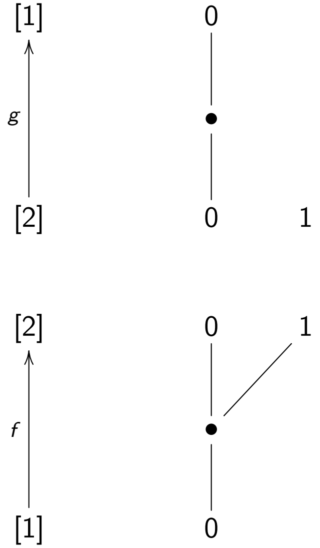
For instance:



COMPOSITION



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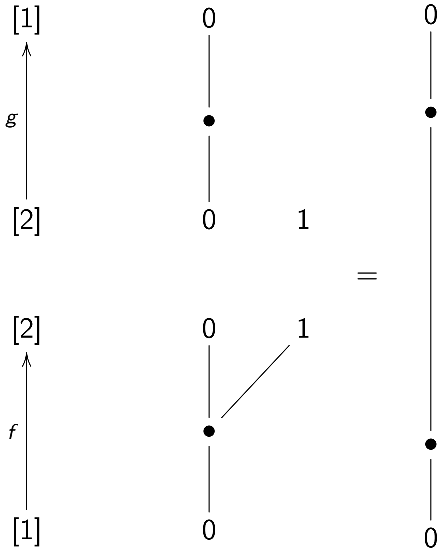

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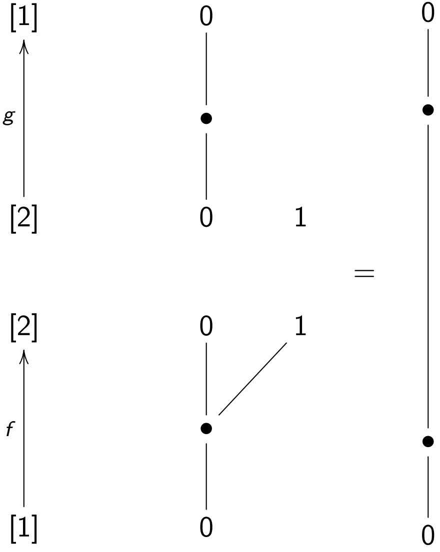

(and tensor product is juxtaposition as usual)

RELATIONS IN FinPOSet

An element of a poset is *internal* when it is not in the source or the target.

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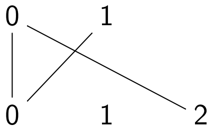
A relation can be seen as a poset with no internal elements: we have a faithful embedding **Rel** $\hookrightarrow$ **FinPOSet**.



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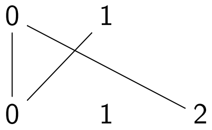


So, it makes sense to build a presentation extending the presentation for **Rel**.

A PRESENTATION FOR FinPOSet

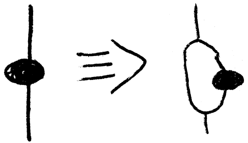
Theorem

The category **FinPOSet** *is presented by the 3-polygraph with*

- *six 2-generators*

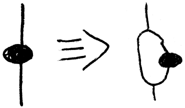
$\mu : 2 \to 1 \quad \eta : 0 \to 1 \quad \delta : 1 \to 2 \quad \varepsilon : 1 \to 0 \quad \gamma : 2 \to 2 \quad \sigma : 1 \to 1$

- *relations*
  - $(\mu, \eta, \delta, \varepsilon, \gamma)$ *is a qualitative bialgebra (as for* **Rel***)*
  - *dependencies are transitive*



ABOUT THE PROOF

Notice that it cannot be done using a canonical rewriting system:

-  does not terminate

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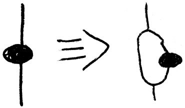
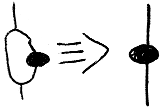
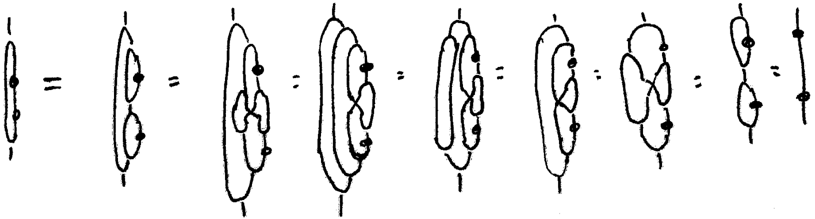
-  does not terminate
-  does not allow to derive



What about presenting
increasing functions between posets?

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We extend this
to better understand
commutative monads.

MONADS

Definition

A **monad** T on a category $\mathcal{C}$ is an endofunctor $T : \mathcal{C} \to \mathcal{C}$ together with two natural transformations

$$\mu : TT \Rightarrow T \qquad\qquad \eta : \mathrm{Id} \Rightarrow T$$

such that

$$\begin{array}{ccc} TTT & \xrightarrow{\mu_T} & TT \\ {\scriptstyle T\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ TT & \xrightarrow[\mu]{} & T \end{array} \qquad\qquad \begin{array}{ccccc} T & \xrightarrow{\eta_T} & TT & \xleftarrow{T\eta} & T \\ & \searrow\!\!\!\!\!= & \downarrow{\scriptstyle \mu} & =\!\!\!\!\!\swarrow & \\ & & T & & \end{array}$$

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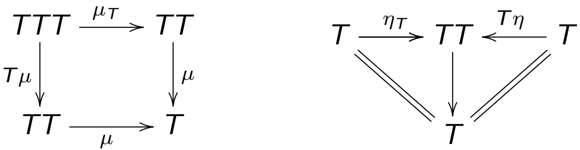

Example

The *stream monad* $TA = A^R$ with

$$\eta_A : A \to TA, \quad a \mapsto \lambda t.a \qquad\qquad \mu_A : TTA \to TA, \quad s \mapsto \lambda t.stt$$

STRONG MONADS

Definition

A **strength** for a monad T on a monoidal category $\mathcal{C}$ is a natural transformation

$$\tau_{A,B} : A \otimes TB \to T(A \otimes B)$$

such that

$$\begin{array}{ccccc}
(A \otimes B) \otimes TC & & \xrightarrow{\tau_{A \otimes B, C}} & & T((A \otimes B) \otimes C) \\
{\scriptstyle \alpha_{A,B,TC}} \downarrow & & & & \downarrow {\scriptstyle T\alpha_{A,B,C}} \\
A \otimes (B \otimes TC) & \xrightarrow[A \otimes \tau_{B,C}]{} & A \otimes T(B \otimes C) & \xrightarrow[\tau_{A, B \otimes C}]{} & T(A \otimes (B \otimes C))
\end{array}$$

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\downarrow{\scriptstyle A \otimes \mu_B} & & & & \downarrow{\scriptstyle \mu_{A \otimes B}} \\
A \otimes TB & & \xrightarrow[\tau_{A,B}]{\qquad\qquad\qquad} & & T(A \otimes B)
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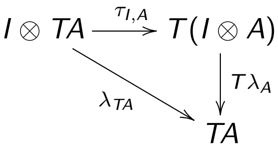

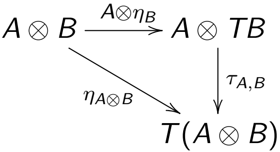

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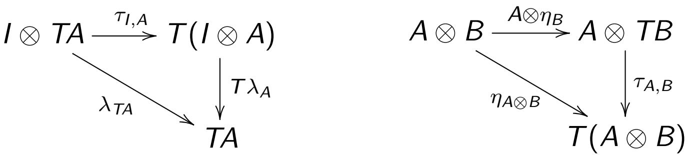

Definition

A **costrength** $\tau_{A,B} : TA \otimes B \to T(A \otimes B)$ is defined dually.

Example

The stream monad is strong with

$$\begin{array}{rcrcl} \tau_{A,B} & : & A \times TB & \to & T(A \times B) \\ & & (a, s) & \mapsto & \lambda t.(a, st) \end{array}$$

STRONG MONADS

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$$\begin{array}{ccccc} A \otimes TTB & \xrightarrow{\tau_{A,TB}} & T(A \otimes TB) & \xrightarrow{T\tau_{A,B}} & TT(A \otimes B) \\ {\scriptstyle A\otimes\mu_B}\downarrow & & & & \downarrow{\scriptstyle \mu_{A\otimes B}} \\ A \otimes TB & & \xrightarrow[\tau_{A,B}]{} & & T(A \otimes B) \end{array}$$

means

$$\lambda t.(\lambda t_1 t_2.(a, st_1 t_2))tt \qquad = \qquad \lambda t.(a, (\lambda t'.st't')t)$$

COMMUTATIVE MONADS

Definition

A **commutative** monad $T : \mathcal{C} \to \mathcal{C}$ is a monad together with a strength and a costrength

$$\tau_{A,B} : A \otimes TB \to T(A \otimes B) \qquad \upsilon_{A,B} : TA \otimes B \to T(A \otimes B)$$

such that

$$\begin{array}{ccccc}
 & T(A \otimes TB) & \xrightarrow{T\tau_{A,B}} & TT(A \otimes B) & \\
{\scriptstyle \upsilon_{A,TB}} \nearrow & & & & \searrow {\scriptstyle \mu_{A\otimes B}} \\
TA \otimes TB & & & & T(A \otimes B) \\
{\scriptstyle \tau_{TA,B}} \searrow & & & & \nearrow {\scriptstyle \mu_{A\otimes B}} \\
 & T(TA \otimes B) & \xrightarrow[T\upsilon_{A,B}]{} & TT(A \otimes B) &
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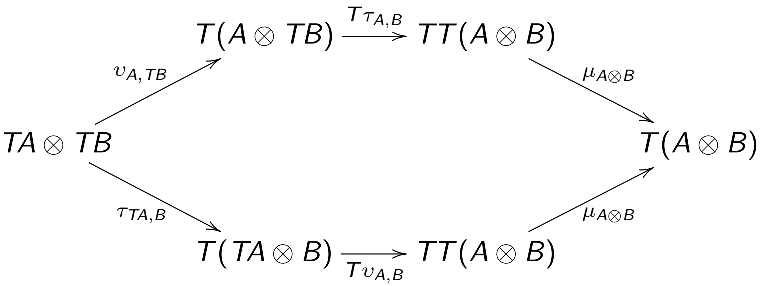
The stream monad:



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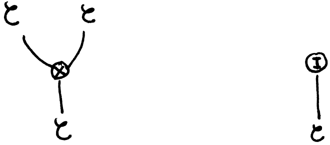
$$\lambda t.(\lambda t_1 t_2.(s_1 t_1, s_2 t_2))tt \quad = \quad \lambda t.(\lambda t_2 t_1.(s_1 t_1, s_2 t_2))tt$$

IN STRING DIAGRAMS

We can try to draw these laws using string diagrams:

- a monoidal category is a (pseudo-)monoid in **Cat**:

$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ $\qquad$ $I : 1 \to \mathcal{C}$



satisfying associativity and unitality

(actually up to iso)

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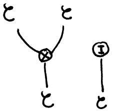
- a monoidal category is a (pseudo-)monoid in **Cat**: 

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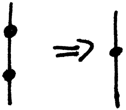
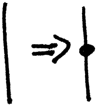
μ : 

η : 

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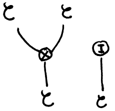
- a monoidal category is a (pseudo-)monoid in **Cat**: 
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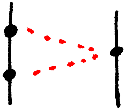
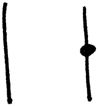
μ : 

η : 

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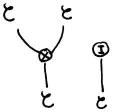
- a monoidal category is a (pseudo-)monoid in **Cat**: 

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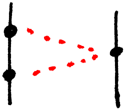
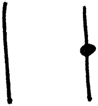
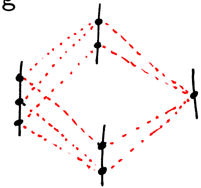
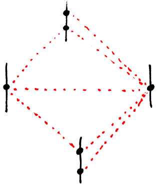
μ : 

η : 

satisfying

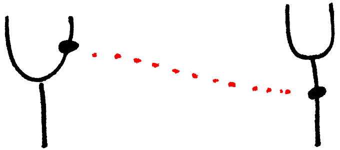




(these define exactly functions between totally ordered sets)

IN STRING DIAGRAMS

- the strength $\tau_{A,B} : A \otimes TB \to T(A \otimes B)$



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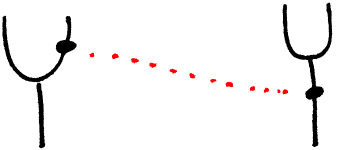
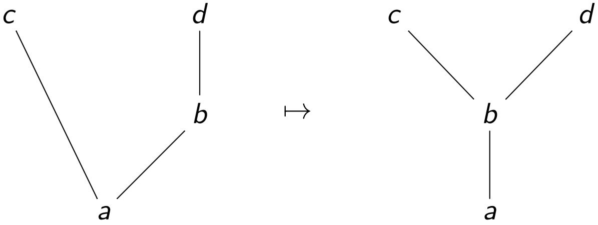


looks like an increasing function between posets:

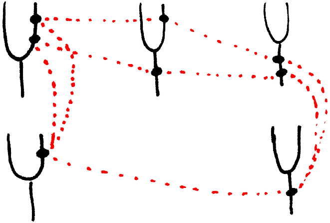


IN STRING DIAGRAMS

and actually all the laws of commutative monads are compatible with this interpretation:

$$\begin{array}{ccccc} A\otimes TTB & \xrightarrow{\tau_{A,TB}} & T(A\otimes TB) & \xrightarrow{T\tau_{A,B}} & TT(A\otimes B) \\ \Big\downarrow{\scriptstyle A\otimes\mu_B} & & & & \Big\downarrow{\scriptstyle \mu_{A\otimes B}} \\ A\otimes TB & & \xrightarrow[\tau_{A,B}]{\qquad\qquad\qquad} & & T(A\otimes B) \end{array}$$

becomes



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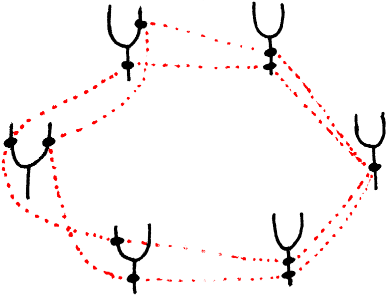

MAKING THIS PRECISE

We define the PRO **PTrees** as the monoidal subcategory of **FinPOSet** whose morphisms $m \to n$ are posets with m minimal and n maximal chosen elements which are *planar forests*:

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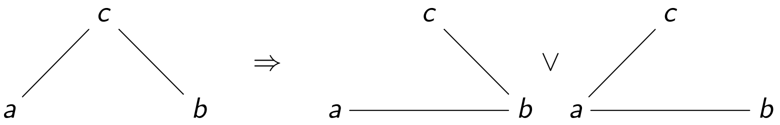

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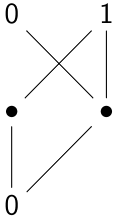
- *planar* means that it can be drawn without crossings:



is forbidden

A PRESENTATION OF PTrees

Proposition

The PRO **PTrees** *is presented by the 3-polygraph with*

- *three 2-generators*

- *three relations*

MAKING THIS PRECISE

We define the monoidal 2-category **IncPTrees** by considering **PTrees** (planar forests) together with increasing functions between them, which preserve the number of trees.

Theorem

The category **IncPTrees** *is presented by the 3-polygraph with*

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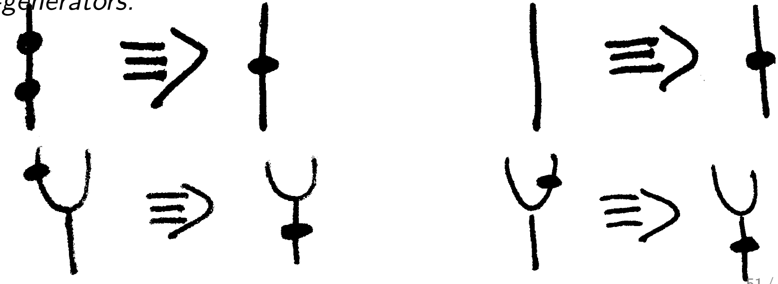

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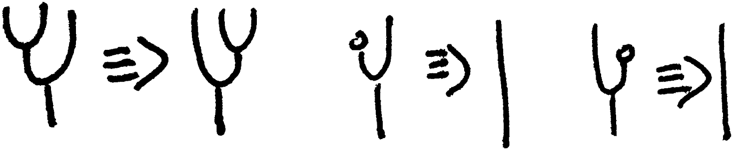

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The category **IncPTrees** *is presented by the 3-polygraph with*

- *three 2-generators:* 
- *3-generators*
- *relations: the axioms of commutative monads*

MAKING THIS PRECISE

Theorem

A strong monoidal functor $\mathbf{IncPTrees} \to \mathbf{Cat}$
is the same as
a category together with a commutative monad

THE COHERENCE THEOREM

Theorem (MacLane)

In a monoidal category, "all diagrams" commute.

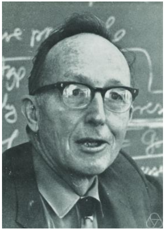
$$((A \otimes I) \otimes B) \otimes C \xrightarrow{(\rho_A \otimes B) \otimes C} (A \otimes B) \otimes C$$

$$(A \otimes (I \otimes B)) \otimes C \xrightarrow{\alpha^{-1}_{A,I,B} \otimes C} ((A \otimes I) \otimes B) \otimes C$$

$$(A \otimes B) \otimes C \xrightarrow{\alpha_{A,B,C}} A \otimes (B \otimes C)$$

$$(A \otimes (I \otimes B)) \otimes C \xrightarrow{\alpha_{A,I \otimes B,C}} A \otimes ((I \otimes B) \otimes C)$$

$$A \otimes ((I \otimes B) \otimes C) \xrightarrow{A \otimes (\lambda_B \otimes C)} A \otimes (B \otimes C)$$



COHERENCE THEOREM FOR COMMUTATIVE MONADS

Theorem

Given a monoidal category $\mathcal{C}$ with a strong monad there are as many canonical morphisms in $\mathcal{C}(A, B)$ as there are functions from A to B seen as posets:

$$T(TTA \otimes (TI \otimes B)) \quad \longrightarrow \quad T(A \otimes TB)$$

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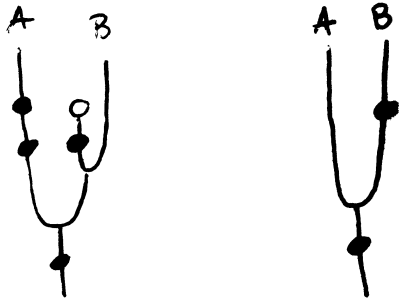

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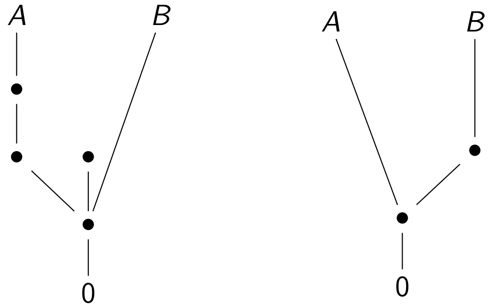

TOWARDS MONADIC COERCIONS?

In a programming language, if $s : T\mathbb{N}$ is a stream of integers, one would like to automatically make sense of programs such as

$$s : T\mathbb{N} \qquad \vdash \qquad 3 + s : T\mathbb{N}$$

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A monad is characterized by:

- its **return** (or unit): $\rho_A : A \to TA$
- its **bind**: $\beta_A : (A \to TB) \to (TA \to TB)$

We would like to implicitly use those as coercions, but it would have to be done in a coherent way!

CONCLUSION

We have shown that
higher-dimensional rewriting methods
can be helpful to
better understand algebraic structures.

But lots remains to be done...