A TYPE-THEORETICAL DEFINITION OF WEAK ω-CATEGORIES

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I have tried to pick a research subject mixing extensions of what you have seen in category theory and type theory.

Sorry for not being able to be present physically!

Toward weak ω-categories

The notion of category is very useful but should be generalized

- we would like to capture **higher-dimensional morphisms** (morphism between morphisms, etc.)
- we would like our structure to be **weak** (we want to ban strict equality!)

The resulting structure is quite difficult to define:
I will propose a *type-theoretic definition*.

This is joint work with Eric Finster and Thibaut Benjamin.

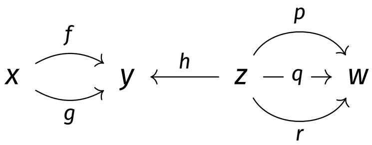
Graphs

A **graph** is a diagram

$$C_0 \underset{t}{\overset{s}{\leftleftarrows}} C_1$$

in **Set**.

For instance,



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in **Set**.

We write

$$f : x \to y$$

to indicate that we have $f \in C_1$ with $s(f) = x$ and $t(f) = y$.

Categories

A **category** is a graph

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Categories

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$$C_0 \underset{t}{\overset{s}{\leftleftarrows}} C_1$$

together with a notion of

- **identity**: for every object $x \in C_0$, we have $\mathrm{id}_x : x \to x$,
- **composition**: for every $f : x \to y$ and $g : y \to z$, we have $f * g : x \to z$,

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- **composition**: for every $f : x \to y$ and $g : y \to z$, we have $f * g : x \to z$,

such that

- composition is **associative**: $(f * g) * h = f * (g * h)$
- identities are **neutral**: $\mathrm{id} * f = f = f * \mathrm{id}$

Categories

This notion is pleasant because

- we can define the composition of n morphisms (with $n = 0, 1, 2, \ldots$), e.g.

$$f_1 * f_2 * f_3 * f_4 = f_1 * (f_2 * (f_3 * f_4))$$

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$$f_1 * f_2 * f_3 * f_4 = (f_1 * f_2 * f_3) * (\mathrm{id} * f_4)$$

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Otherwise said, all compositions are defined
and do not depend on the choice of bracketing!

Categorical concepts

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Two objects x, y are **isomorphic** when there are morphisms

$$f : x \to y \qquad\qquad g : y \to x$$

such that

$$f * g = \mathrm{id}_x \qquad\qquad g * f = \mathrm{id}_y$$

This definition makes sense in any category:

- in **Set**: isomorphism of sets,
- in **Grp**: isomorphism of groups,
- etc.

Categorical concepts

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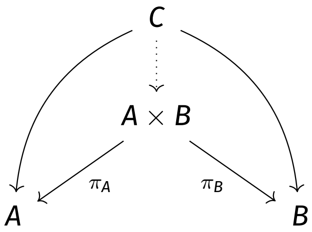
The **product** of two objects is defined by



Products in

- **Set**: cartesian product $A \times B$,
- **Vect**: direct sum $A \oplus B$,
- **Rel**: disjoint union $A \sqcup B$.

2-categorical concepts

An **equivalence** of categories C and D consists of two functors

$$F : C \to D \qquad G : D \to C$$

such that

$$F * G \cong \mathrm{Id}_C \qquad G * F \cong \mathrm{Id}_D$$

This definition makes sense in **Cat**. However, we cannot generalize it to other categories, why?

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There is no notion of “natural transformation” in general categories!

2-categorical concepts

An **adjunction** between categories C and D consists of two functors

$$F : C \to D \qquad\qquad G : D \to C$$

and two natural transformations

$$\eta : \mathrm{Id}_C \to F * G \qquad\qquad \varepsilon : G * F \to \mathrm{Id}_D$$

such that some conditions are satisfied.

This definition makes sense in **Cat**. However, we cannot generalize it to other categories, why?

There is no notion of “natural transformation” in general categories!

2-categories

In a category we have

$$x \qquad\qquad x \xrightarrow{f} y$$

objects morphisms

2-categories

In a 2-category we have

x	$x \xrightarrow{f} y$	$x \underset{g}{\overset{f}{\Downarrow \alpha}} y$
objects	morphisms	2-cells

2-categories

In a **2**-category we have

x	$x \xrightarrow{f} y$	$x \underset{g}{\overset{f}{\rightrightarrows}} y \quad (\Downarrow \alpha)$
0-cells	1-cells	2-cells

2-categories

In a 2-category we have

x	$x \xrightarrow{f} y$	$x \underset{g}{\overset{f}{\Downarrow \alpha}} y$
0-cells	1-cells	2-cells

The typical 2-category is **Cat**:

- 0-cells: categories
- 1-cells: functors
- 2-cells: natural transformations

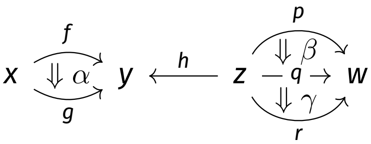
(but there are other examples)

2-graphs

A **2-graph** is a diagram

$$C_0 \underset{t_0}{\overset{s_0}{\leftleftarrows}} C_1 \underset{t_1}{\overset{s_1}{\leftleftarrows}} C_2$$

For instance



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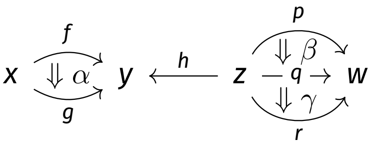

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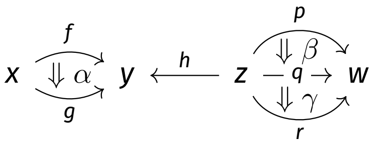

we have

$$s_0(s_1(\alpha)) = s_0(f) = x = s_0(g) = s_0(t_1(\alpha))$$

2-categories: structure

A **2-category** is a 2-graph

$$C_0 \underset{t_0}{\overset{s_0}{\longleftleftarrows}} C_1 \underset{t_1}{\overset{s_1}{\longleftleftarrows}} C_2$$

together with

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together with

- compositions and identities for 1-cells (morphisms)

$$x \xrightarrow{f} y \xrightarrow{g} z \quad \rightsquigarrow \quad x \xrightarrow{f *_0 g} z$$

$$x \quad \rightsquigarrow \quad x \xrightarrow{\mathrm{id}_x} x$$

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- we have two kinds of compositions for 2-cells:
  - horizontal composition:

$$x \underset{f'}{\overset{f}{\Downarrow \alpha}} y \underset{g'}{\overset{g}{\Downarrow \beta}} z \quad \rightsquigarrow \quad x \underset{f' *_0 g'}{\overset{f *_0 g}{\Downarrow \alpha *_0 \beta}} z$$

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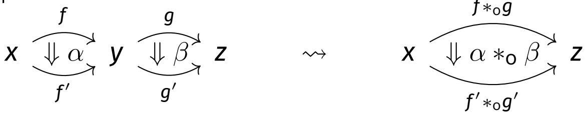
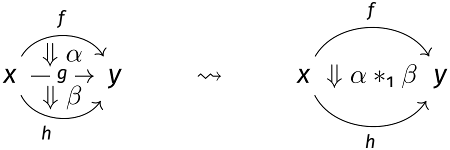


    - vertical composition:



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  - vertical composition: …
  - identities:

$$x \xrightarrow{f} y \qquad \rightsquigarrow \qquad x \underset{f}{\overset{f}{\Downarrow \mathrm{id}_f}} y$$

2-categories: axioms

There are axioms to be satisfied such as

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- composition of 1-cells is associative and unital

$$x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} w \qquad \rightsquigarrow \qquad (f *_0 g) *_0 h = f *_0 (g *_0 h)$$

$$x \xrightarrow{f} y \qquad \rightsquigarrow \qquad \mathrm{id}_x *_0 f = f = f *_0 \mathrm{id}_y$$

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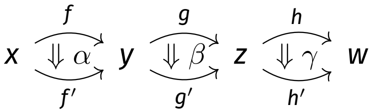
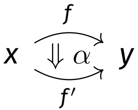


$\rightsquigarrow$ $$(\alpha *_0 \beta) *_0 \gamma = \alpha *_0 (\beta *_0 \gamma)$$



$\rightsquigarrow$ $$\mathrm{id}_{\mathrm{id}_x} *_0 \alpha = \alpha = \alpha *_0 \mathrm{id}_{\mathrm{id}_y}$$

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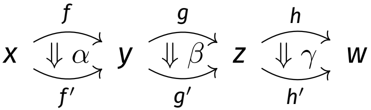

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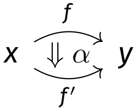

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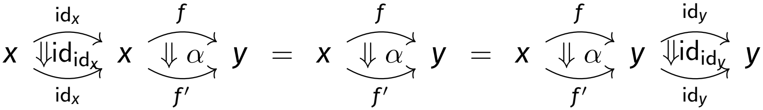

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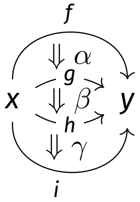
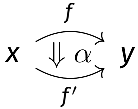
There are axioms to be satisfied such as

- composition of 1-cells is associative and unital
- horizontal composition of 2-cells is associative and unital
- vertical composition of 2-cells is associative and unital



$\rightsquigarrow \quad (\alpha *_1 \beta) *_1 \gamma = \alpha *_1 (\beta *_1 \gamma)$



$\rightsquigarrow \quad \mathrm{id}_f *_1 \alpha = \alpha = \alpha *_1 \mathrm{id}_g$

2-categories: axioms

There is still one axiom missing: can you spot which one?

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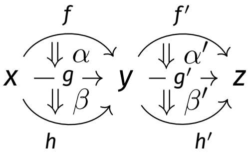

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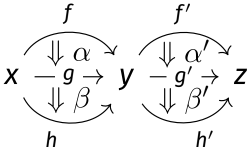

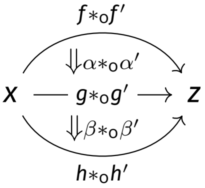

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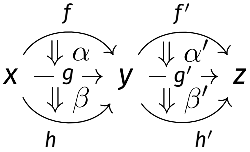

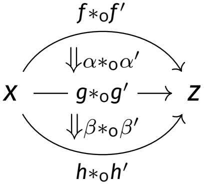

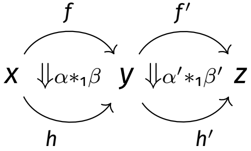

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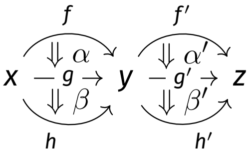


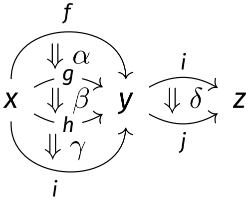
The **exchange law** should be satisfied:

$$(\alpha *_0 \alpha') *_1 (\beta *_0 \beta') = (\alpha *_1 \beta) *_0 (\alpha' *_1 \beta')$$

2-categories: coherence

It can be shown that given a *collection of composable arrows*, all the ways to compose them coincide.

For instance,



$$(\alpha *_1 (\beta *_1 \gamma)) *_0 \delta = (\alpha *_0 \mathrm{id}_i) *_1 (\beta *_0 \delta) *_1 (\gamma *_0 \mathrm{id}_j)$$

Adjunctions in 2-categories

An **adjunction** in a 2-category consists of

- two 0-cells x and y,
- two 1-cells $f : x \to y$ and $g : y \to x$,
- two 2-cells $\eta : \mathrm{id}_x \Rightarrow f *_0 g$ and $\varepsilon : g *_0 f \Rightarrow \mathrm{id}_y$
- such that …

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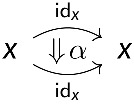
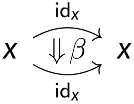
In particular, an adjunction in the 2-category **Cat** is an adjunction in the usual sense, but there are many other interesting examples!

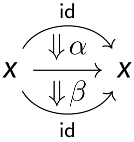
The Eckmann-Hilton observation

The exchange law has a surprising consequence: given two **2**-cells





with identity source and target **1**-cells, their vertical and horizontal composition coincide and are commutative:



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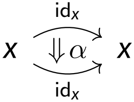

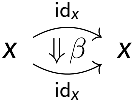

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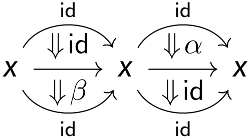

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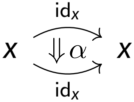

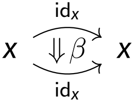

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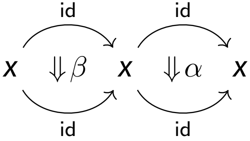

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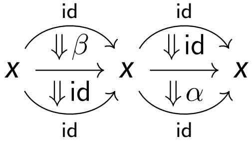

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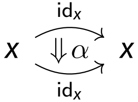

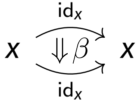

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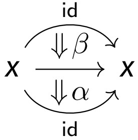


Generalizing 2-categories

In order to take in account more situations we can further generalize

- by increasing the dimension (easy)
- by weakening the axioms (hard)

n-categories

Of course, there is no reason to stop at dimension 2.

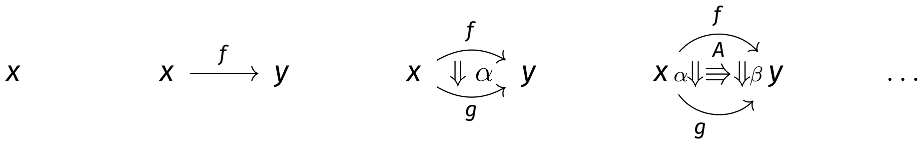
An ***n*-graph**, or **globular set**, is

$$C_0 \underset{t_0}{\overset{s_0}{\leftleftarrows}} C_1 \underset{t_1}{\overset{s_1}{\leftleftarrows}} C_2 \underset{t_2}{\overset{s_2}{\leftleftarrows}} \cdots \underset{t_{n-1}}{\overset{s_{n-1}}{\leftleftarrows}} C_{n-1} \underset{t_n}{\overset{s_n}{\leftleftarrows}} C_n$$

such that

$$s_i \circ s_{i+1} = s_i \circ t_{i+1} \qquad\qquad t_i \circ s_{i+1} = t_i \circ t_{i+1}$$

We now have



0-cells 1-cells 2-cells 3-cells ...

n-categories

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We can also define the notion of **ω-category** by letting n go to ∞.

An important point: I could write the definition of ω-categories in one page.

14.2.1 Definition. A *strict ω-category* is given by a globular set C together with a family of partial binary composition operations $(*_i)_{i \in \mathbb{N}}$ and identity operations $(1^i)_{i \in \mathbb{N} \setminus \{0\}}$ subject to the following conditions:

- if $0 \leqslant i < k$ and x, y are k-cells such that $t_i(x) = s_i(y)$ (in which case we say that x and y are *i-composable*) there is a k-cell $x *_i y$,
- if $k > 0$ and x is a $(k-1)$-cell, there is a k-cell 1^k_x, and more generally, if $i \geq 0$ and x is an i-cell, we may define recursively on $k > i$ a k-cell 1^k_x by $1^k_x = 1^k_{1^{k-1}_x}$.

Compositions and units are subject to:

1. positional conditions prescribing the source and target of composites and units, namely
   - if $0 \leqslant i < j$, then $s_j(x *_i y) = s_j(x) *_i s_j(y)$ and $t_j(x *_i y) = t_j(x) *_i t_j(y)$,
   $$s_j(x *_i y) = s_j(x) *_i s_j(y) \quad \text{and} \quad t_j(x *_i y) = t_j(x) *_i t_j(y),$$
   - if $0 \leqslant j \leqslant i$, then
   $$s_j(x *_i y) = s_j(x) \quad \text{and} \quad t_j(x *_i y) = t_j(y),$$
   - if $0 \leqslant i < k$ and x is an i-cell, then
   $$s_i(1^k_x) = x = t_i(1^k_x),$$
2. computational conditions of
   - associativity: if $i < k$ and x, y, z are k-cells such that $t_i(x) = s_i(y)$ and $t_i(y) = s_i(z)$, then
   $$(x *_i y) *_i z = x *_i (y *_i z),$$
   - neutrality of units: if $0 \leqslant i < k$ and x is a k-cell, then
   $$1^k_{s_i(x)} *_i x = x *_i 1^k_{t_i(x)} = x,$$
   - exchange: if $i < j < k$ and x, y, z, v are k-cells such that $t_j(x) = s_j(y)$, $t_j(z) = s_j(v)$ and $t_i(x) = s_i(z)$, then also $t_j(y) = s_j(v)$, and
   $$(x *_j y) *_i (z *_j v) = (x *_i z) *_j (y *_i v),$$
   - compatibility of units: if $0 \leqslant i < j < k$ and x, y are i-composable j-cells, then
   $$1^k_{x *_i y} = 1^k_x *_i 1^k_y.$$

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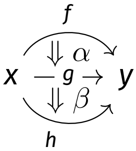
Weak n-categories

The notion of higher-dimensional category we obtain is very nice,
but there are important examples which are not n-categories,
and the problems already show up for $n = 2$.

Weak 2-categories

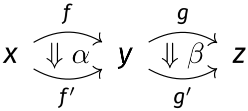
In 2-categories, we have the intuition that

- vertical composition



  corresponds to sequential composition of morphisms

- horizontal composition



  corresponds to putting morphisms in “parallel”

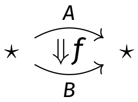
Set as a 2-category

We thus expect that we can see **Set** as a 2-category in the following way:

- there is one 0-cell $\star$
- the 1-cells are sets

$$\star \xrightarrow{A} \star$$

- the 2-cells are functions



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so that

- vertical composition $\star \overset{A}{\underset{C}{\rightrightarrows}} \star$ with $\star \xrightarrow{A} \star \;\Downarrow f\; \star \xrightarrow{B} \star \;\Downarrow g\; \star \xrightarrow{C} \star$ is the usual composition of functions,

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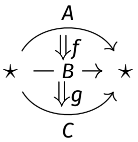
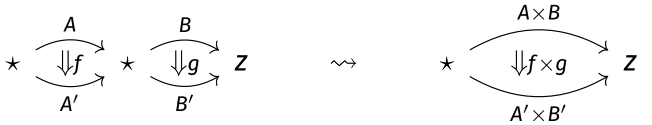
- vertical composition 

is the usual composition of functions,
- horizontal composition correspond to taking cartesian products:



Set as a 2-category

Most of the axioms of 2-categories are satisfied excepting for associativity and unitality of 0-cells: given

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the two possible compositions do *not* coincide:

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For instance, in OCaml we do not have

$$\texttt{(int * int) * int} = \texttt{int * (int * int)}$$

This can be observed by typing

```
compare ((3,4),5) (3,(4,5));;
Error: This expression has type int but an expression was expected of type
         int * int
```

Set as a 2-category

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What we however have is that

$$(A \times B) \times C \cong A \times (B \times C)$$

Bicategories

The notion of **bicategory** is defined almost as for 2-categories, excepting that we replace the requirement that composition of 1-cells is associative and unital by

- **weak associativity**: given

  $$x \xrightarrow{a} y \xrightarrow{b} z \xrightarrow{c} w$$

  there is an invertible 2-cell, the **associator**,

  $$\alpha_{a,b,c} : (a *_0 b) *_0 c \Rightarrow a *_0 (b *_0 c)$$

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- **weak unitality**: given

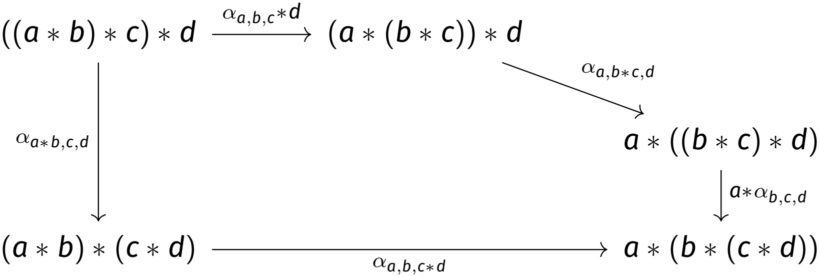
  $$x \xrightarrow{a} y$$

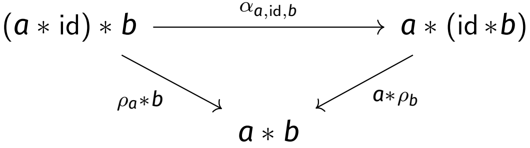
  there are invertible 2-cells, the **left** and **right unitors**,

  $$\lambda_a : \mathrm{id}_x *_0 a \Rightarrow a \qquad \rho_a : a *_0 \mathrm{id}_y \Rightarrow a$$

Bicategories: axioms

We also need to ensure that those satisfy suitable axioms, the **pentagon** and the **triangle**:





Bicategories: coherence

This notion is pleasant because

Theorem (Mac Lane's coherence theorem)
Any two ways of composing 1*-cells are isomorphic and there is one such structural isomorphism.*

For instance,

$$f_1 * (f_2 * (f_3 * f_4)) \cong (f_1 * f_2 * f_3) * (\mathrm{id} * f_4)$$

Bicategories vs 2-categories

The morale of this is that **equality is evil**.

We do not want axioms such as associativity

$$(f * g) * h = f * (g * h)$$

we rather higher cells which are witnesses for associativity

$$\alpha_{f,g,h} : (f * g) * h \stackrel{\simeq}{\Rightarrow} f * (g * h)$$

Bicategories vs 2-categories

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Any bicategory is equivalent to a 2-category.

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Theorem (Mac Lane's coherence theorem v2)
Any bicategory is equivalent to a 2-category.

But this will not generalize to higher dimensions:

Observation (Gordon,Power,Street'95)
Not every tricategory is equivalent to a 3-category.

Tricategories

Defining **tricategories** can be done starting from the definition of 3-categories and

1. replacing all equalities between 0-, 1- and 2- cells by 1-, 2- and 3- cells,
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For instance, we replace associativity of composition between 1-cells

$$(f *_0 g) *_0 h = f *_0 (g *_0 h)$$

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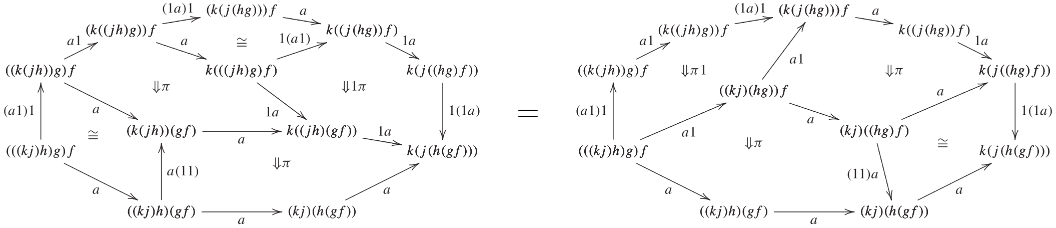
but by "invertible", we mean here that $\alpha_{f,g,h}$ should be an equivalence:

$$\eta : \mathrm{Id} \Rrightarrow \alpha_{f,g,h} *_1 \overline{\alpha}_{f,g,h} \qquad\qquad \varepsilon : \overline{\alpha}_{f,g,h} *_1 \alpha_{f,g,h} \Rrightarrow \mathrm{Id}$$

and so on...

Tricategories

The definition of tricategories takes roughly 4 pages with axioms such as



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Instead of explicitly trying to give the axioms of weak higher categories, one should try to find a systematic way of generating those..

If we go all the way, we obtain **weak ω-categories** aka **(∞,∞)-categories**.

In those, we never have axioms, only higher cells. This can be thought of as very constructive definition: we want to have witnesses for all the laws.

Weak higher categories are closely related to geometry.

Reversible cells

Suppose that we have managed to define the notion of weak ω-category.

An n-cell $f : x \to y$ is **reversible** when it is weakly invertible, this means that there exists $\bar{f} : y \to y$ such that

$$f * \bar{f} = \mathrm{id} \qquad \bar{f} * f = \mathrm{id}$$

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which are isomorphisms.

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NB: this is a <u>co</u>inductive definition!

∞-groupoids

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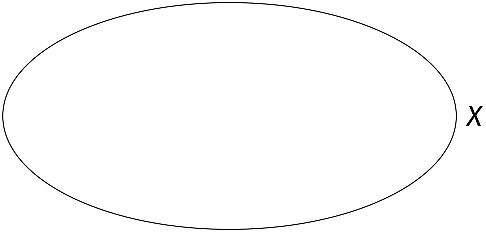

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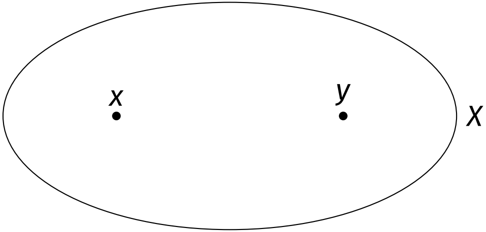

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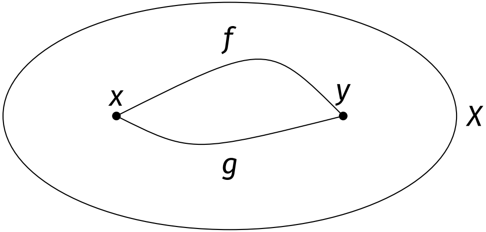

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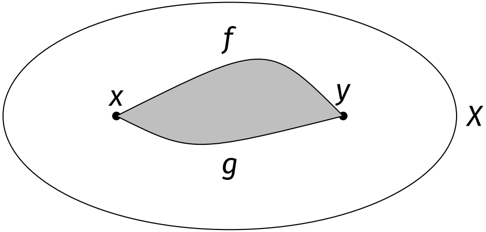

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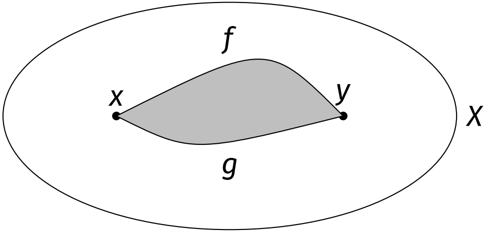
- etc.



We have identities, compositions, inverses, etc...

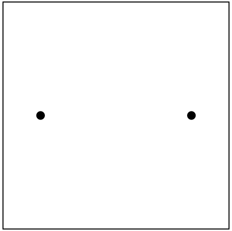
Weak ∞-groupoids

- In 1983, Alexander Grothendieck gave a definition of ∞-groupoids and with the conjecture that those are "the same as" spaces.
- In 2007, Maltsiniotis proposed a modified definition for weak ω-categories.
- In 2017, Finster and I managed to encode this definition as a type system.

[there are many more works than this on the subject…]

The braiding

With the geometric point of view, we can provide an explanation why tricategories are not equivalent to 3-categories:



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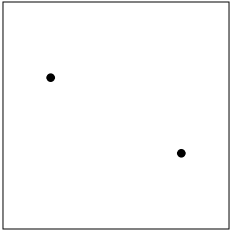

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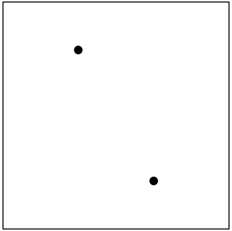

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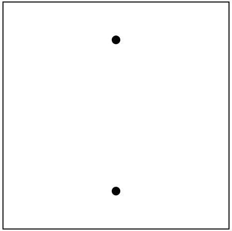

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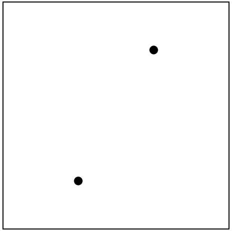

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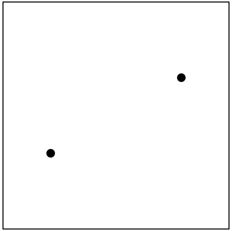

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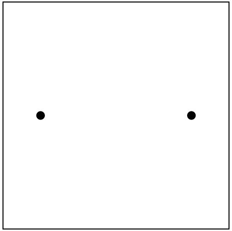

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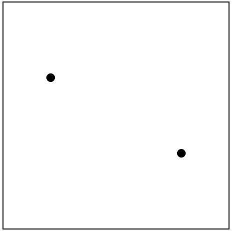

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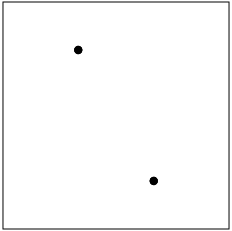

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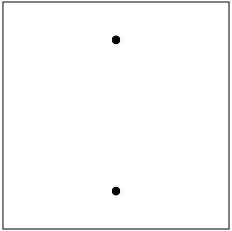

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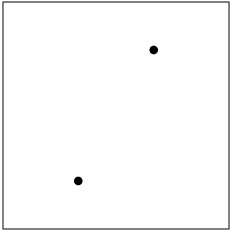

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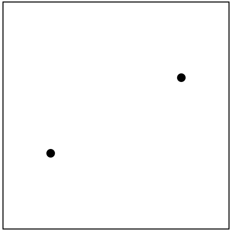

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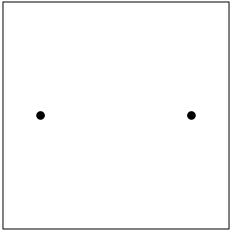

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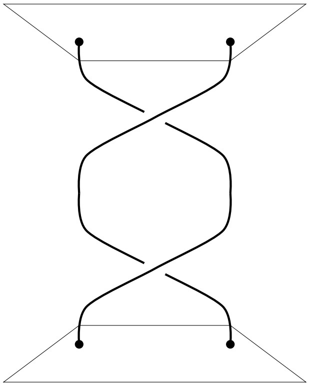

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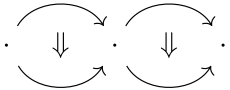

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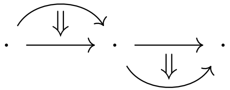

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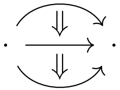

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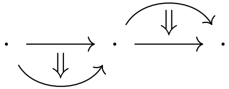

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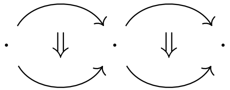

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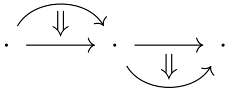

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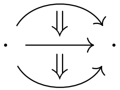

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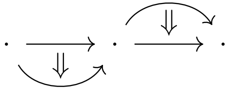

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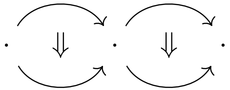

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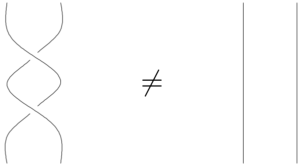

Why is this useful

- We have a **simple definition**
  (no advanced categorical concepts, a few inference rules)
- We have a **syntax**
  (we can reason by induction, etc.)
- We have **tools**
  (we can have the machine check our terms)
- A step toward **directed homotopy type theory**?
  (we are still far from handling variance, univalence, etc.)

A TYPE-THEORETIC DEFINITION OF CATEGORIES

As a first example, we are going to give a type theory for 1-categories.

This theory will be defined from three things:

- **terms**, which will comprise variables
  (those will correspond to objects and morphisms of our category)
- **types**, which can depend on terms
- **contexts**, which are lists of pairs of variables and types

$$\Gamma \qquad = \qquad x_1 : A_1, \ldots, x_n : A_n$$

Judgments in type-theory

- Γ is a well-formed context:

$$\Gamma \vdash$$

- A is a well-formed type in context Γ:

$$\Gamma \vdash A$$

- t is a term of type A in context Γ:

$$\Gamma \vdash t : A$$

- t and u are equal terms of type A in context Γ:

$$\Gamma \vdash t = u : A$$

A type-theoretic definition of categories

Cartmell, 1984:

- type constructors:

$$\frac{\Gamma \vdash}{\Gamma \vdash \star} \qquad\qquad \frac{\Gamma \vdash x : \star \qquad \Gamma \vdash y : \star}{\Gamma \vdash x \to y}$$

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- term constructors:

$$\frac{}{x : \star \vdash \mathrm{id}(x) : x \to x}$$

$$\frac{}{x : \star, y : \star, f : x \to y, z : \star, g : y \to z \vdash \mathrm{comp}(f, g) : x \to z}$$

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- plus “standard rules” (contexts, weakening, substitutions, ...)

Models of the type theory

A **model** of the type theory consists in interpreting

- closed types as sets,
- closed terms as elements of their type,

in such a way that axioms are satisfied.

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- a set $[\![\star]\!]$
- for each $x, y \in [\![\star]\!]$, a set $[\![\rightarrow]\!]_{x,y}$
- for each $x \in [\![\star]\!]$, an element $[\![\mathrm{id}]\!]_x \in [\![\rightarrow]\!]_{x,x}$
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- ...

In other words, a model of the type theory is precisely a **category** (and a morphism is a functor).

Going higher

We could gradually implement weak *n*-categories:

- bicategories
- tricategories
- tetracategories
- pentacategories
- …

The problem is that

- the number of axioms is exploding
- nobody knows the definition excepting in low dimensions
- we would like to have a “uniform” definition

Unbiased definition

Since the composition is associative for categories, the composite of any diagram like

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \ldots \xrightarrow{f_n} x_n$$

is uniquely defined.

So, instead of having a binary composition and identities, we could have a more general rule

$$\frac{}{x_0 : \star, x_1 : \star, f_1 : x_0 \to x_1, \ldots, x_n : \star, f_n : x_{n-1} \to x_n \vdash \mathsf{comp}(f_1, \ldots, f_n) : x_0 \to x_n}$$

Unbiased definition

We can axiomatize categories with n-ary composition.

- This is very redundant, for instance

$$\mathrm{comp}(\mathrm{comp}(f,g),h) = \mathrm{comp}(f,g,h) = \mathrm{comp}(f,\mathrm{comp}(g,h))$$

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- We have to characterize what we want to compose exactly. For instance, should be able to compose

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  but not

$$x \underset{g}{\overset{f}{\rightleftarrows}} y \qquad z \qquad\qquad \text{or} \qquad\qquad x \xrightarrow{f} y \xleftarrow{g} z$$

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- However, this generalizes nicely in higher dimensions!

A TYPE-THEORETIC DEFINITION OF GLOBULAR SETS

Graphs

Recall that we defined a **graph** to be a diagram

$$C_0 \underset{t}{\overset{s}{\leftleftarrows}} C_1$$

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Alternatively, we can define a **graph** as

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- for every $x, y \in C_0$, a set C_y^x.

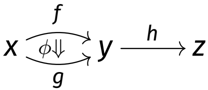
The n-graphs, aka **globular sets**, can be defined in the same way.

Globular sets

Definition
A **globular set** consists of

- a set G, and
- for every $x, y \in G$, a globular set G^x_y.

Example



corresponds to

$$G = \{x, y, z\} \qquad G^x_y = \{f, g\} \qquad (G^x_y)^f_g = \{\phi\} \qquad ((G^x_y)^f_g)^\phi_\phi = \emptyset \qquad \ldots$$

Globular sets

Definition
A **globular set** consists of

- a set G, and
- for every $x, y \in G$, a globular set G^x_y.

Alternatively, this can be defined as

- a sequence of sets G_n of n-cells for $n \in \mathbb{N}$,
- with source and target maps

$$s_n, t_n : G_{n+1} \to G_n$$

  satisfying suitable axioms.

Globular sets

Proposition
Globular sets are precisely the models of the type theory

$$\frac{\Gamma \vdash}{\Gamma \vdash \star} \qquad \frac{\Gamma \vdash t : A \qquad \Gamma \vdash u : A}{\Gamma \vdash t \underset{A}{\to} u} \qquad \ldots$$

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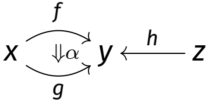
Remark
A finite globular set



can be encoded as a context

$$x : \star, y : \star, z : \star, f : x \underset{\star}{\to} y, g : x \underset{\star}{\to} y, h : z \underset{\star}{\to} y, \alpha : f \underset{x \underset{\star}{\to} y}{\to} g$$

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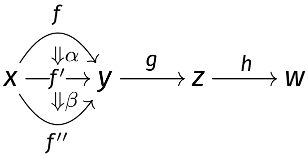
Proposition
The syntactic category (of contexts and substitutions) of this type theory is the opposite of the category of finite globular sets.

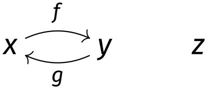
PASTING SCHEMES

Pasting schemes

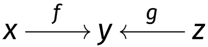
We now want to define **pasting schemes** which are diagrams for which we expect to have a composition. For instance,



is a pasting scheme, but not



or



Disks

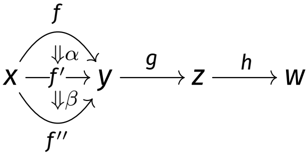
Given $n \in \mathbb{N}$, the ***n*-disk** D_n is the globular set corresponding to a general n-cell:

D_0 D_1 D_2 D_3

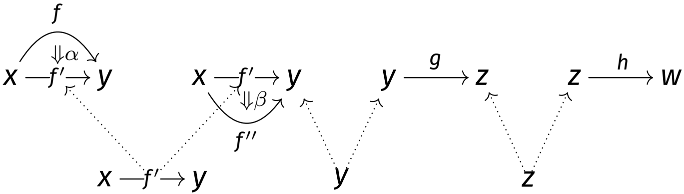
Those are basic building blocks of globular sets: any globular set can be obtained by gluing such disks.

Pasting schemes

A **pasting scheme** is a globular set



- *Grothendieck*: which can be obtained as a particular colimit of disks



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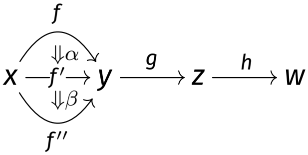
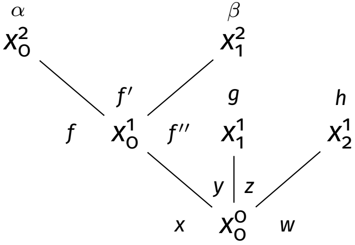


- *Batanin*: which is described by a particular tree



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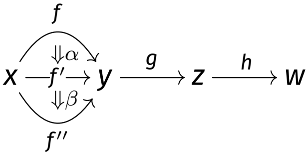


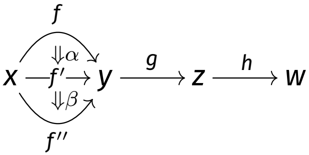
- *Finster-Mimram*: which is “totally ordered”

Order relation

We can define a preorder $\triangleleft$ on the cells of a globular set by

$$\mathrm{source}(x) \triangleleft x \qquad \text{and} \qquad x \triangleleft \mathrm{target}(x)$$

For the globular set



we have

$$x \triangleleft f \triangleleft \alpha \triangleleft f' \triangleleft \beta \triangleleft f'' \triangleleft y \triangleleft g \triangleleft z \triangleleft h \triangleleft w$$

Characterization of pasting schemes

Theorem
A globular set is a **pasting scheme** *if and only if it is*

- *non-empty,*
- *finite, and*
- *the relation $\triangleleft$ is a total order.*

Construction of pasting schemes

A *pointed globular set* is a globular set with a distinguished cell.

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- *we start from a* 0*-cell* x
- *we can add a new* $(n{+}1)$*-cell and its new target,*
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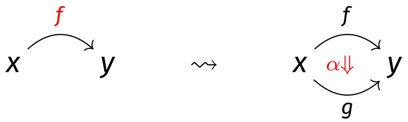

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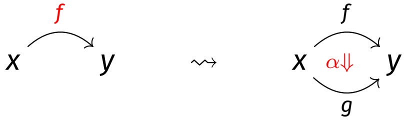

- *or the distinguished cell becomes the target of the previous one*

Construction of pasting schemes

The construction of the pasting scheme

x

corresponds to its order

x

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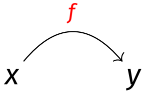
The construction of the pasting scheme



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$$x \quad \triangleleft \quad f$$

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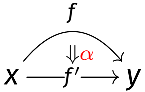

corresponds to its order

$$x \quad \triangleleft \quad f \quad \triangleleft \quad \alpha$$

Construction of pasting schemes

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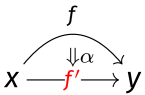

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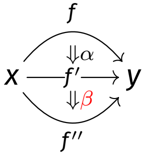

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Construction of pasting schemes

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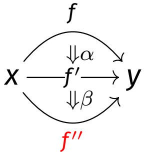

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$$x \quad \triangleleft \quad f \quad \triangleleft \quad \alpha \quad \triangleleft \quad f' \quad \triangleleft \quad \beta \quad \triangleleft \quad f''$$

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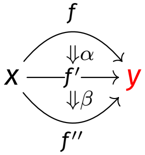

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$$x \lhd f \lhd \alpha \lhd f' \lhd \beta \lhd f'' \lhd y$$

Construction of pasting schemes

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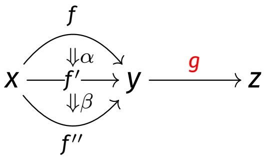
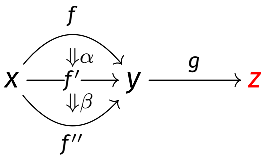


corresponds to its order

$$x \quad \triangleleft \quad f \quad \triangleleft \quad \alpha \quad \triangleleft \quad f' \quad \triangleleft \quad \beta \quad \triangleleft \quad f'' \quad \triangleleft \quad y \quad \triangleleft \quad g$$

Construction of pasting schemes

The construction of the pasting scheme



corresponds to its order

$$x \lhd f \lhd \alpha \lhd f' \lhd \beta \lhd f'' \lhd y \lhd g \lhd z$$

Construction of pasting schemes

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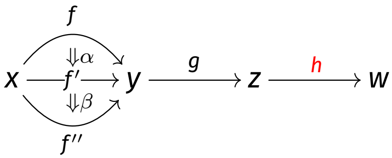

corresponds to its order

$$x \quad \lhd \quad f \quad \lhd \quad \alpha \quad \lhd \quad f' \quad \lhd \quad \beta \quad \lhd \quad f'' \quad \lhd \quad y \quad \lhd \quad g \quad \lhd \quad z \quad \lhd \quad h$$

Construction of pasting schemes

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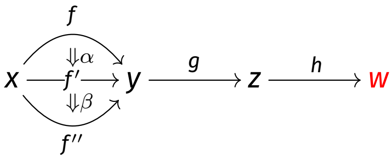
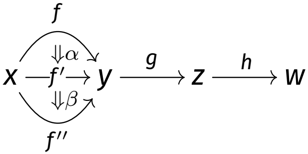


corresponds to its order

$$x \triangleleft f \triangleleft \alpha \triangleleft f' \triangleleft \beta \triangleleft f'' \triangleleft y \triangleleft g \triangleleft z \triangleleft h \triangleleft w$$

Type-theoretic pasting schemes

Now, recall that a pasting scheme



can be seen as a context

$$x : \star, y : \star, f : x \to y, f' : x \to y,$$
$$\alpha : f \to f', f'' : x \to y, \beta : f' \to f'',$$
$$z : \star, g : y \to z, w : \star, h : z \to w$$

Type-theoretic pasting schemes

A context Γ (seen as a globular set) is a **pasting scheme** iff

$$\Gamma \vdash_{\mathsf{ps}}$$

is derivable with the rules

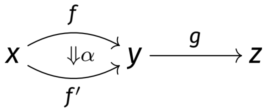
$$\frac{}{x : \star \vdash_{\mathsf{ps}} x : \star} \qquad \frac{\Gamma \vdash_{\mathsf{ps}} x : \star}{\Gamma \vdash_{\mathsf{ps}}}$$

$$\frac{\Gamma \vdash_{\mathsf{ps}} x : A}{\Gamma, y : A, f : x \underset{A}{\to} y \vdash_{\mathsf{ps}} f : x \underset{A}{\to} y} \qquad \frac{\Gamma \vdash_{\mathsf{ps}} f : x \underset{A}{\to} y}{\Gamma \vdash_{\mathsf{ps}} y : A}$$

Type-theoretic pasting schemes

Note that with those rules

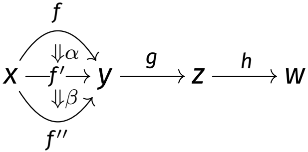
- the order of cells matters:



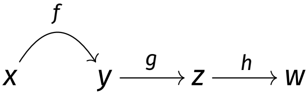
- because of this we can easily check
- proofs are canonical

Source and targets

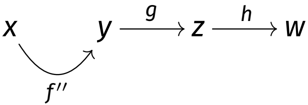
A pasting scheme Γ has



- a **source** $\partial^{-}(\Gamma)$:



- a **target** $\partial^{+}(\Gamma)$:



both of which can be defined by induction on contexts.

A TYPE-THEORETIC DEFINITION OF ω-CATEGORIES

Type-theoretic ω-categories

We expect that in an ω-category every pasting scheme has a composite:

$$\frac{\Gamma \vdash_{\mathsf{ps}} \qquad \Gamma \vdash A}{\Gamma \vdash \mathsf{coh}_{\Gamma,A} : A}$$

Type-theoretic ω-categories

We expect that in an ω-category every pasting scheme has a composite:

$$\frac{\Gamma \vdash_{\mathsf{ps}} \qquad \Gamma \vdash A}{\Gamma \vdash \mathsf{coh}_{\Gamma,A} : A}$$

You can derive expected operations, such as composition:

$$x : \star, y : \star, f : x \underset{\star}{\to} y, z : \star, g : y \underset{\star}{\to} z \vdash \mathsf{coh} : x \underset{\star}{\to} z$$

Type-theoretic ω-categories

We expect that in an ω-category every pasting scheme has a composite:

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However, you can derive too much:

$$x : \star, y : \star, f : x \underset{\star}{\to} y \vdash \mathsf{coh} : y \underset{\star}{\to} x$$

We have in fact a definition of ω-**groupoids**

Type-theoretic ω-categories

We need to take care of side-conditions and in fact split the rule in two:

- operations:

$$\frac{\Gamma \vdash_{\mathsf{ps}} \qquad \Gamma \vdash t \underset{A}{\to} u \qquad \partial^-(\Gamma) \vdash t : A \qquad \partial^+(\Gamma) \vdash u : A}{\Gamma \vdash \mathsf{coh}_{\Gamma, t \underset{A}{\to} u} : t \underset{A}{\to} u}$$

  whenever

$$FV(t) = FV(\partial^-(\Gamma)) \qquad \text{and} \qquad FV(u) = FV(\partial^+(\Gamma))$$

- coherences:

$$\frac{\Gamma \vdash_{\mathsf{ps}} \qquad \Gamma \vdash A}{\Gamma \vdash \mathsf{coh}_{\Gamma, A} : A}$$

  whenever

$$FV(A) = FV(\Gamma)$$

Type-theoretic ω-categories

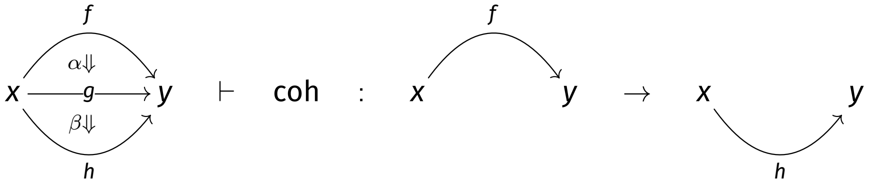
Definition
An ω-**category** is a model of this type theory.

Type-theoretic ω-categories

Definition
An ω-**category** is a model of this type theory.

Conjecture
This definition coincides with Grothendieck-Maltsiniotis'.

A typical example of **operation** is *composition*



(this coherence is noted "**comp**" in the following).

Type-theoretic ω-categories

A typical example of **coherence** is *associativity*

$$x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} w$$

$$\vdash$$

$$\mathsf{coh} \quad : \quad x \xrightarrow{\mathsf{comp}(\mathsf{comp}(f,g),h)} w \quad \to \quad x \xrightarrow{\mathsf{comp}(f,\mathsf{comp}(g,h))} w$$

Coherences are reversible

Note that if we derive a coherence

$$\frac{\Gamma \vdash_{\mathsf{ps}} \qquad \Gamma \vdash A}{\Gamma \vdash \mathsf{coh}_{\Gamma,A} : A} \qquad \text{with} \qquad FV(A) = FV(\Gamma)$$

where

$$A \quad = \quad t \to u \; ,$$

there is also one with

$$A \quad = \quad u \to t \; .$$

Coherences are reversible

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where

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$$A \quad = \quad u \to t \,.$$

Definition
An n-cell $f : x \to y$ is **reversible** when there exists

- an n-cell $g : y \to x$ and
- reversible $(n+1)$-cells

Implementation(s)

There are currently three implementations:

- `https://github.com/ericfinster/catt`
  - follows closely the rules of the article
- `https://github.com/smimram/catt`
  - has support for implicit arguments and various small extensions
  - has a web interface
- `https://github.com/ThiBen/catt`
  - best of both worlds
  - many more extensions

In practice,

- you simply enter a list of coherences (there is no reduction, etc.),
- if the program does not complain then they are valid operations in weak ω-categories.

“Demo”

- identity 1-cells

```
coh id (x : *) : * | x -> x ;
```

"Demo"

- identity 1-cells

  ```
  coh id (x : *) : * | x -> x ;
  ```

- composition of 1-cells:

  ```
  coh comp (x : *) (y : *) (f : * | x -> y)
           (z : *) (g : * | y -> z)
           : * | x -> z ;
  ```

“Demo”

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coh id (x : *) : * | x -> x ;
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- composition of 1-cells:

```
coh comp (x : *) (y : *) (f : * | x -> y)
         (z : *) (g : * | y -> z)
         : * | x -> z ;
```

- associativity of composition of 1-cells:

```
coh assoc
    (x : *) (y : *) (f : * | x -> y) (z : *)
    (g : * | y -> z) (w : *) (h : * | z -> w)
    : * | x -> w
        | comp x z (comp x y f z g) w h ->
          comp x y f w (comp y z g w h) ;
```

“Demo”

- identity 1-cells

```
coh id (x : *) : * | x -> x ;
```

- composition of 1-cells:

```
coh comp (x : *) (y : *) (f : * | x -> y)
         (z : *) (g : * | y -> z)
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    : * | x -> w
        | comp x z (comp x y f z g) w h ->
          comp x y f w (comp y z g w h) ;
```
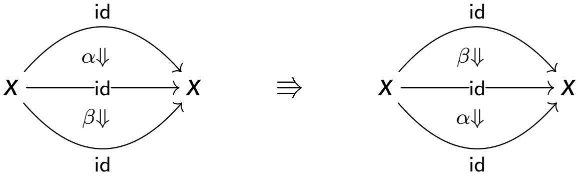

- ...

“Demo”

Only defining the Eckmann-Hilton morphism takes 300 lines



because you have to

- define usual operations and coherences,
- explicitly insert and remove identities,
- take care of bracketing of composites

```
let eh (X : Hom) (x : X) (a : id x -> id x) (b : id x -> id x)
     : (comp' a b -> comp' b a) =
    comp11 (comp' (unitl'- a) (unitr'- b)) (assoc3 _ _ _ _)
    (compl2r' _ _ (unitlr x) _) (compl2' _ _ (comp3 (assoc- _ _ _) (comp'
    (compl' _ (assoc- _ _ _)) (complr' _ (ich b a) _)
    (complr' _ (compr' (comp (unitr- _) (compl' _ (unitr+-- _))) _) _)
    (comp (complr' _ (assoc3 _ _ _ _) _) (compl' _ (assoc4 _ _ _ _ _)))
    (comp' (unitlr- x) (compl' _ (compl' _ (comp' (unitrl- x) (compl' _ (u
    (assoc3- _ _ _ _)
    (comp' (unitr' b) (unitl' a))
```

“Demo”

- no inverses:

```
coh inv (x : *) (y : *) (f : * | x -> y)
        : * | y -> x ;
```

produces

```
Checking coherence: inv
Valid tree context
Src/Tgt check forced
Source context: (x : *)
Target context: (y : *)
Failure: Source is not algebraic for y : *
```

CONCLUSION

Current work

Many things remain to be done:

- understand more exotic features
  (implicit arguments, reduction, etc.)
- some work has been started by Finster and Vicary to make associativity and unitality implicit
  thanks to this they have been able to construct the syllepsis 5-cell
- we should study the relationship with homotopy type theory