Globular weak ω-categories as models of a type theory

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Abstract

We study the dependent type theory CaTT, introduced by Finster and Mimram, which presents the theory of weak ω-categories, following the idea that type theories can be considered as presentations of generalized algebraic theories. Our main contribution is a formal proof that the models of this type theory correspond precisely to weak ω-categories, as defined by Maltsiniotis, by generalizing a definition proposed by Grothendieck for weak ω-groupoids: those are defined as suitable presheaves over a cat-coherator, which is a category encoding structure expected to be found in an ω-category. This comparison is established by proving the initiality conjecture for the type theory CaTT, in a way which suggests the possible generalization to a nerve theorem for a certain class of dependent type theories.

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Introduction

The notion of weak ω-category has emerged quite naturally by generalizing situations encountered in algebraic topology: it consists in an algebraic structure which comprises cells of various dimensions, which can be composed in various dimensions, and satisfy the expected laws. We are however interested in *weak* such structures here, which means that we want to encompass situations where the laws do not hold up to equality, but only up to higher-dimensional cells, which thus play the role of witnesses that those laws are satisfied. Those cells should themselves satisfy coherence laws, which should only hold up to higher cells, which should themselves satisfy coherence laws, and so on. Because of these towers of coherence cells, coming up with a suitable definition of weak ω-category is quite difficult. Historically, definitions of weak ω-groupoids (also called ∞-groupoids) were first proposed, such as Kan complexes [22]: those are weak ω-categories in which every cell is reversible, and are thus closer to spaces encountered in algebraic topology. Then, around the beginning of the century, various definitions for weak ω-categories have been proposed: we refer the reader to the surveys on the topic [25, 14] for a general presentation of those. The comparison between the proposals is still an ongoing research topic, and seems to be technically out of reach for now for some of them.

While originating from topology, unexpected connections were found with type theory: a series of works around 2010 revealed that the iterated identity types in Martin-Löf type theory endow each type with the structure of a weak ω-groupoid [27, 35, 1]. This key observation is in fact one of the motivations that lead to the development of homotopy type theory [34]. Based on this, and following Cartmell's insight that type theory could be used to formulate generalized algebraic theories [13], Brunerie managed to extract from the rules generating identity types, a definition of weak ω-groupoids [11], that he could show to be equivalent to a definition proposed by Grothendieck [19]. The novelty of this definition lies in the fact that it is itself formulated as a type theory.

Following Brunerie's approach, Finster and Mimram [16] gave a definition of weak ω-categories in the form of a type theory called CaTT. Their definition follows the lines of a generalization of Grothendieck's weak ω-groupoids to weak ω-categories, proposed by Maltsiniotis [28]. The goal of this article is to show that the type theory CaTT is equivalent to one of the definitions proposed by Maltsiniotis. Moreover Ara [2] has proved, up to a conjecture later proved by Bourke [10] that this definition is equivalent to a definition proposed by Leinster [26] following a method introduced by Batanin [3]. Our result completes this circle of ideas, establishing that these three definitions are three sides of the same story, expressed in different languages.

After brief general reminders about semantics of type theory in Section 1, we introduce a type theory for globular sets in Section 2, which serves both as a basis and as a baby version of our main proof. We then briefly present the Grothendieck-Maltsiniotis definition of weak ω-categories in Section 3, in order to recall the motivations for the introduction of the type theory CaTT, which is introduced in Section 4 along with some examples of derivations in this theory, and some of its properties. We then study in Section 5 the syntactic category of this theory and begin relating it to the Grothendieck-Maltsiniotis definition of weak ω-categories. Finally, in Section 6, we study the models of this type theory, and show that they are equivalent to the aforementioned definition of weak ω-categories. The reader who wishes to familiarize themselves with the type theory along the way may also experiment with the implementation [7].

1 Categorical semantics of type theory

We begin by recalling the categorical framework we use here to study type theory, together with an associated notion of model. We do not introduce any type theory just yet, but only the categorical description of type theories. We refer the reader to Section 2.2 for a presentation of the notion of type theory considered here. We denote **Cat** the category of categories, and $\underline{\mathbf{Cat}}$ the 2-category of categories. In general, we underline the 2-categories to distinguish them visually.

1.1 Categories with families.

The categorical models of type theory considered here are categories with families, which were introduced by Dybjer [15]. This particular choice has little impact on the developments performed here since most other notions of model, such as Cartmell's categories with attributes [13], are known to be equivalent to this one.

We write **Fam** for the category of *families*, where an object is a family $(A_i)_{i\in I}$ consisting of sets A_i indexed by a set I, and a morphism $f : (A_i)_{i\in I} \to (B_j)_{j\in J}$ is a pair consisting of a function $f : I \to J$ and a family of functions $(f_i : A_i \to B_{f(i)})_{i\in I}$.

Suppose given a category $\mathcal{C}$ equipped with a functor $T : \mathcal{C}^{\mathrm{op}} \to \mathbf{Fam}$. Given an object Γ of $\mathcal{C}$, its image under T is a family denoted

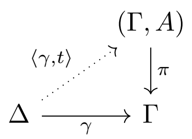
$$T\Gamma \quad = \quad \left(\mathrm{Tm}^{\Gamma}_{A}\right)_{A\in\mathrm{Ty}^{\Gamma}}$$

i.e., we write Ty^{Γ} for the indexing set and Tm^{Γ}_{A} for the elements of the family. Given a morphism $\gamma : \Delta \to \Gamma$ in $\mathcal{C}$ and an element $A \in \mathrm{Ty}^{\Gamma}$, we write $A[\gamma]$ for the object $\mathrm{Ty}^{\gamma}(A)$ of Ty^{Δ}. Similarly, given an element $t \in \mathrm{Tm}^{\Gamma}_{A}$, we write $t[\gamma]$ for the element $\mathrm{Tm}^{\gamma}_{A}(t)$ of $\mathrm{Tm}^{\Delta}_{A[\gamma]}$. With these notations, the functoriality of T is equivalent to the following equations

$$A[\gamma \circ \delta] = A[\gamma][\delta] \qquad\qquad t[\gamma \circ \delta] = t[\gamma][\delta]$$
$$A[\mathrm{id}] = A \qquad\qquad t[\mathrm{id}] = t$$

for composable morphisms of $\mathcal{C}$.

A *category with families* consists of a category $\mathcal{C}$ together with a functor $T : \mathcal{C}^{\mathrm{op}} \to \mathbf{Fam}$ as above, such that $\mathcal{C}$ has a terminal object, denoted $\varnothing$, and such that there is a *context comprehension* operation: given an object Γ and type $A \in \mathrm{Ty}^{\Gamma}$, there is an object (Γ, A), together with a projection morphism $\pi : (\Gamma, A) \to \Gamma$ and a term $p \in \mathrm{Tm}^{(\Gamma,A)}_{A[\pi]}$, such that for every morphism $\gamma : \Delta \to \Gamma$ in $\mathcal{C}$ together with a term $t \in \mathrm{Tm}^{\Delta}_{A[\gamma]}$, there exists a unique morphism $\langle\gamma, t\rangle : \Delta \to (\Gamma, A)$ such that $p[\langle\gamma, t\rangle] = t$ and the following diagram commutes:



In a category with families, the class of *display maps* is the smallest class of morphisms containing the projection morphisms $\pi : (\Gamma, A) \to \Gamma$ and closed under composition and identities.

A *morphism* between two categories with families $(\mathcal{C}, T)$ and $(\mathcal{C}', T')$, is a functor $F : \mathcal{C} \to \mathcal{C}'$ together with a natural transformation $\phi : T \to T' \circ F^{\mathrm{op}}$, such that F preserves the terminal object and the context comprehension operation on the nose. In this article we consider the category of categories with families that we denote **CwF**, as well as the 2-categories obtained by considering natural transformations between the morphisms of categories with families as functors, we denote it $\underline{\mathbf{CwF}}$.

Models of a category with families. We define a large category with families in a similar way, as a large category equipped with a functor into families of large sets indexed by a large set, and satisfying the exact same properties. Note that a category with families can be seen as a large category with families. There is a structure of a category with large families on the category **Set**, where, given a set X, Ty^X is the (large) set of all X-indexed families of sets $(Y_x)_{x\in X}$. Given such a family $Y = (Y_x)$, the set Tm_Y^X is the set of X-indexed families of elements $(y_x)_{x\in X}$ with $y_x \in Y_x$. For a map $f : X' \to X$ the action of f is given by $(Y[f])_x = Y_{f(x)}$ and $(y[f])_x = y_{f[x]}$. We define the category of *models* of a category with families $\mathcal{C}$, denoted $\mathbf{Mod}(\mathcal{C})$, to be the category whose objects are the morphisms of categories with families from $\mathcal{C}$ to **Set**. Explicitly we have

$$\mathbf{Mod}(\mathcal{C}) = \underline{\mathbf{CwF}}(\mathcal{C}, \mathbf{Set})$$

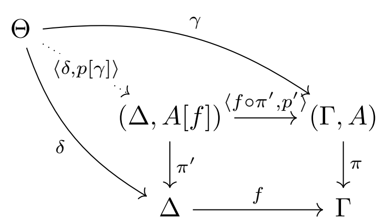
Pullbacks in a category with families. The structure of category with families enforces a compatibility condition between context comprehension and the action of morphisms on types, expressed by the following lemma: it states that all pullbacks along display maps exist and that they can be explicitly computed from the given structure.

Lemma 1. *In a category with families $\mathcal{C}$, for every morphism $f : \Delta \to \Gamma$ in $\mathcal{C}$ and $A \in \mathrm{Ty}^\Gamma$, the square*

$$\begin{array}{ccc} (\Delta, A[f]) & \xrightarrow{\langle f\circ\pi', p'\rangle} & (\Gamma, A) \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ \Delta & \xrightarrow[f]{} & \Gamma \end{array}$$

is a pullback, where $\pi' : (\Delta, A[f]) \to \Delta$ and $p' \in \mathrm{Tm}_{A[f][\pi']}^{(\Delta,A[f])}$ are obtained by context comprehension.

Proof. Consider a diagram of the following form in $\mathcal{C}$, without the dotted arrow:



Given a term $p \in \mathrm{Tm}_{A[\pi]}^{(\Gamma,A)}$, we have $p[\gamma] \in \mathrm{Tm}_{A[\pi][\gamma]}^{\Theta} = \mathrm{Tm}_{A[f][\delta]}^{\Theta}$. By context extension, we obtain a map $\langle \delta, p[\gamma]\rangle : \Theta \to (\Delta, A[f])$ such that $\pi' \circ \langle \delta, p[\gamma]\rangle = \delta$ and $p'[\langle \delta, p[\gamma]\rangle] = p[\gamma]$. Since moreover $p' = p[\langle f \circ \pi', p'\rangle]$, the previous equality amounts to $p[\gamma] = p[\langle f \circ \pi', p'\rangle \circ \langle \delta, p[\gamma]\rangle]$. This condition is necessary for the upper triangle to commute. Hence the uniqueness of the map. We just have to show that this map makes the upper triangle commute. Notice that $\pi \circ \langle f \circ \pi', p'\rangle \circ \langle \delta, p[\gamma]\rangle = \pi \circ \gamma$, and $p[\gamma] = p[\langle f \circ \pi', p'\rangle \circ \langle \delta, p[\gamma]\rangle]$, by universal property of the extension for morphisms, this implies the commutativity of upper triangle. □

The structure of a category with families can be thought of as a way of ensuring that the pullbacks of the form of the above lemma exist, while also enforcing that they are split. This means that the choice of the pullbacks is such that taking a pullback along a composite morphism $g \circ f$ gives the same result as taking the pullback along f and then along g. In the formalism of categories with families, this means that we have $(\Gamma, A[\delta \circ \gamma]) = (\Gamma, A[\delta][\gamma])$. Since the structure of category with families provides these pullbacks, and since the morphisms of categories with families preserve this structure, these morphisms also preserve these pullbacks, as witnessed by the following result.

Lemma 2. *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories with families, together with a morphism $(F, \phi) : \mathcal{C} \to \mathcal{D}$. Then, for any object Γ in $\mathcal{C}$ together with an element $A \in \mathrm{Ty}^\Gamma$, and for any morphism $\gamma : \Delta \to \Gamma$ in $\mathcal{C}$, the following equation is satisfied:*

$$F(\Delta, A[\gamma]) = (F\Delta, (\phi_\Gamma A)[F\gamma])$$

Proof. By definition of a morphism of categories with families, we have

$$F(\Delta, A[\gamma]) = (F(\Delta), (\phi_\Delta(A[\gamma])))$$

and, by naturality of ϕ, the following square commutes

$$\begin{array}{ccc} Ty^\Gamma & \xrightarrow{\phi_\Gamma} & Ty^{F(\Gamma)} \\ {\scriptstyle \_[\gamma]}\downarrow & & \downarrow{\scriptstyle \_[F\gamma]} \\ Ty^\Delta & \xrightarrow{\phi_\Delta} & Ty^{F(\Delta)} \end{array}$$

This proves in particular the $\phi_\Delta(A[\gamma]) = (\phi_\Gamma A)[F\gamma]$, from which follows the desired equality. □

Lemma 1 allows us to understand this result as the fact that F preserves pullbacks along display maps. In fact, the following result shows that preserving these pullbacks is precisely the condition that a functor from a category with families to sets has to satisfy in order to be a model of the category with families.

Lemma 3. *The models of a category with families $\mathcal{C}$ are in bijective correspondence with the functors $\mathcal{C} \to$ **Set** that preserve the terminal object and the pullbacks along display maps.*

Proof. By Lemma 2, the underlying functor of a morphism of categories with families preserves the pullbacks along display maps and, by definition, such a functor has to preserve the terminal object as well. So it suffices to prove that a functor $F : \mathcal{C} \to$ **Set** preserving the initial object and pullbacks along display maps gives rise to a unique model. Consider such a functor F, together with an object Γ in $\mathcal{C}$ and a type $A \in \mathrm{Ty}^\Gamma$. Suppose defined ϕ such that (F, ϕ) is a model of $\mathcal{C}$, then necessarily $F(\Gamma, A) = (F\Gamma, \phi_\Gamma A) = \bigsqcup_{x \in F\Gamma} (\phi_\Gamma(A))_x$ by definition of the context comprehension in **Set**. Thus $F(\Gamma, A)$ is uniquely determined by $F(\Gamma, A) = \bigsqcup_{x \in F\Gamma} (\phi_\Gamma(A))_x$. Similarly, for a term $t \in \mathrm{Tm}_A^\Gamma$, there is a morphism $\langle \mathrm{id}_\Gamma, t \rangle : \Gamma \to (\Gamma, A)$, so we have $F(\langle \mathrm{id}_\Gamma, t \rangle) = \langle \mathrm{id}_{F\Gamma}, \phi_{\Gamma,A}(t) \rangle$. By definition of the structure of category with families on **Set**, this completely characterize the map $F(\langle \mathrm{id}_\Gamma, t \rangle)$ as the one sending every element x of $F(\Gamma)$ onto $(\phi_{\Gamma,A}(t))_x$. Together with the equation $\phi_{\Gamma,A}(t) = p[F(\langle \mathrm{id}_\Gamma, t \rangle)]$, this completely characterizes the map $\phi_{\Gamma,A}$. Conversely, these assignments define a natural transformation ϕ, which make (F, ϕ) into a model of F. □

This condition relies on the specific structure of category with families of **Set**, and previous lemma does not extend as a characterization of morphisms between arbitrary categories with families. It also justifies retrospectively why it is not that important to be precise about size issues with **Set**, as one may as well ignore the structure of category with families on **Set** altogether, and define a model as a functor $\mathcal{C} \to$ **Set** that preserves the terminal object and the pullback along the display maps.

1.2 Contextual categories.

In order to carry on some inductive constructions on the syntax of a theory, and handle them in full generality, we also need introduce the notion of *contextual categories*, due to Cartmell [13], and studied by Streicher [32] and Voevodsky [36] under the name of *C-systems*. These equip categories with families with the extra structure required in order to perform those inductive constructions.

Definition 4. A *contextual category* consist in a category with families $\mathcal{C}$ together with a function ℓ associating to each object Γ of $\mathcal{C}$ a natural number $\ell(\Gamma)$, called its *length*, such that

- the terminal object $\varnothing$ is the unique object such that $\ell(\varnothing) = 0$,
- for every object Γ and type $A \in \mathrm{Ty}^{\Gamma}$, $\ell(\Gamma, A) = \ell(\Gamma) + 1$,
- for every object Γ such that $\ell(\Gamma) > 0$, there is a unique object Γ' together with a unique type $A \in \mathrm{Ty}^{\Gamma'}$ such that $\Gamma = (\Gamma', A)$.

Note that a contextual category is usually defined to be a category with attributes satisfying such properties. However, since categories with families and categories with attributes are equivalent, we will also refer to these as contextual categories. The notion of contextual category is not invariant under equivalences of categories and relies on a particular presentation of a category. Its use is justified by the fact that the syntax of a type theory gives a particular presentation of a category with families, which happens to be a contextual category.

Given a contextual category $\mathcal{C}$, an object Γ whose length is strictly positive decomposes in a unique way as Γ', A, and we simply write $\pi_\Gamma : \Gamma \to \Gamma'$ (or even π) instead of $\pi_{\Gamma',A}$. We also write x_Γ for the term $p_{\Gamma',A}$ in $\mathrm{Tm}^{\Gamma}_{A[\pi]}$, thought of as a variable. More generally, we declare that a term is a *variable* when it is of the form $x_\Gamma[\pi]$ where π is a display map. Note that in a contextual category, if $\pi : \Delta \to \Gamma$ is a display map, then necessarily $l(\Delta) > l(\Gamma)$. This implies that the variables of a non-empty context (Γ, A) are either $x_{(\Gamma,A)}$, or of the form $x[\pi_{(\Gamma,A)}]$ where x is a variable of Γ.

The following lemma shows that a map in a contextual category is entirely characterized by its action on the variables in its target context.

Lemma 5. *Consider two maps $\gamma, \delta : \Delta \to \Gamma$, in a contextual category, such that, for every variable x in Γ, $x[\gamma] = x[\delta]$. Then we have $\gamma = \delta$.*

Proof. We prove this result by induction on the length of the object Γ.

- Suppose that Γ is of length 0. Then necessarily, $\Gamma = \varnothing$ is the terminal object, and thus $\gamma = \delta$.
- Suppose that Γ is of length $l + 1$. Then it is of the form (Γ', A), and there is a map $\pi : \Gamma \to \Gamma'$. Suppose moreover that there are two maps $\gamma, \delta : \Delta \to \Gamma$, such that for every variable x of Γ, we have $x[\gamma] = x[\delta]$. Note that we necessarily have $\gamma = \langle \pi \circ \gamma, x_\Gamma[\gamma] \rangle$ and $\delta = \langle \pi \circ \delta, x_\Gamma[\delta] \rangle$, as it is the case for every maps. Then for the variable x_Γ, we have $x_\Gamma[\gamma] = x_\Gamma[\delta]$. Moreover, for every variable x of Γ', $x[\pi]$ is a variable of Γ, and thus $x[\pi][\gamma] = x[\pi][\delta]$, which proves $x[\pi \circ \gamma] = x[\pi \circ \delta]$, and, by induction hypothesis, $\pi \circ \gamma = \pi \circ \delta$. We thus have proved that $\langle \pi \circ \gamma, x_\Gamma[\gamma] \rangle = \langle \pi \circ \delta, x_\Gamma[\delta] \rangle$, i.e., $\gamma = \delta$. □

2 A type theory for globular sets

We first describe a type theory whose models are globular sets, on which we rely in order to introduce the type theory CaTT. It was previously considered by Brunerie [11] and Finster and Mimram [16], and we expand here on their work. This type theory is quite poor, as it has no term constructors (the only terms are variables): it will also serve as a simple setting in order to present the techniques and properties which will be generalized later on to the more complex type theory CaTT.

2.1 The category of globular sets.

Globular sets are a generalization of graphs, which comprise not only points and arrows, but also higher dimensional cells. Similarly to graphs, the category of globular sets can be defined as a presheaf category.

The category of globes. The *category of globes* $\mathcal{G}$ is the category whose objects are the natural numbers and morphisms are generated by

$$\sigma_i, \tau_i : i \to i+1$$

subject to following *coglobular relations*:

$$\sigma_{i+1} \circ \sigma_i = \tau_{i+1} \circ \sigma_i \qquad\qquad \sigma_{i+1} \circ \tau_i = \tau_{i+1} \circ \tau_i \tag{1}$$

The category of *globular sets* $\mathbf{GSet} = \widehat{\mathcal{G}}$ is the presheaf category over the category $\mathcal{G}$. Given a globular set G, we write G_n instead of $G\,n$. Equivalently, a globular set is a family of sets $(G_n)_{n\in\mathbb{N}}$ equipped with maps $s_i, t_i : G_{i+1} \to G_i$ satisfying the *globular relations*, dual to (1):

$$s_i \circ s_{i+1} \quad = \quad s_i \circ t_{i+1} \qquad\qquad t_i \circ s_{i+1} \quad = \quad t_i \circ t_{i+1} \tag{2}$$

An element of G_n is called a n-cell, and the maps s_i, t_i are called respectively the source and target. Often, the indices of the source and target maps can be inferred from the context and we therefore omit them and write s and t to simplify notations.

Given an object n, the associated representable $\mathrm{Y}(n)$ is called the *n-disk* and is usually written D^n. It can be explicitly described by

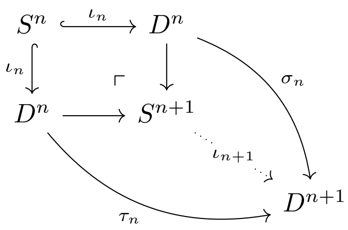
$$(D^n)_i = \begin{cases} \{*_0, *_1\} \text{ if } i < n \\ \{*\} \text{ if } i = n \\ \emptyset \text{ if } i > n \end{cases}$$

with $s(\_) = *_0$ and $t(\_) = *_1$. Throughout this paper, we use the Greek lower cases σ and τ to denote the morphisms in the category $\mathcal{G}$, or to denote the image of the morphisms in $\mathcal{G}$ via a functor $F : \mathcal{G} \to \mathcal{C}$, and we use their equivalent Latin lower cases s, t to denote the image of the morphisms in $\mathcal{G}$ via a functor $F : \mathcal{G}^{\mathrm{op}} \to \mathcal{C}$.

The n-sphere globular set. Given $n \in \mathbb{N}$, the *n-sphere* S^n is the globular set, equipped with an inclusion $\iota^n : S^n \hookrightarrow D^n$, defined by

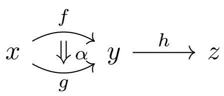
- $S^{-1} = \emptyset$ is the initial object, and $\emptyset \hookrightarrow D^1$ is the unique arrow,

– S^{n+1} and ι^{n+1} are obtained by the pushout



This definition is well defined since, as a presheaf category, the category of globular sets is cocomplete (and the colimits are computed pointwise).

Finite globular sets. A globular set G is *finite* if it can be obtained as a finite colimit of representable objects. It can be shown that this is the case precisely when the set $\bigsqcup_{i\in\mathbb{N}} G_i$ is finite, because all representables themselves satisfy this property. We write **FinGSet** for the full subcategory of **GSet** whose objects are the finite presheaves. We sometimes call a finite globular set a *diagram*, and describe it using a diagrammatic notation. For instance, the diagram



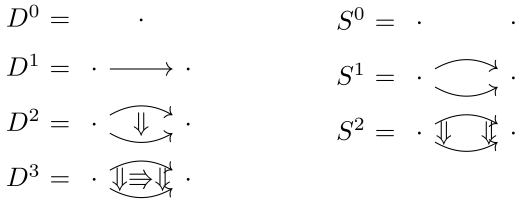
denotes the finite globular set G, whose only non-empty cell sets are

$$G_0 = \{x, y, z\} \qquad G_1 = \{f, g, h\} \qquad G_2 = \{\alpha\}$$

and whose the sources and targets are defined by

$$\begin{array}{llll} s(f) = x & s(g) = x & s(h) = y & s(\alpha) = f \\ t(f) = y & t(g) = y & t(h) = z & t(\alpha) = g \end{array}$$

Disks and spheres are finite globular sets. In small dimensions, they can be depicted as



By definition, **FinGSet** is the free cocompletion of G under all finite colimits (see Section 2.4).

2.2 The theory GSeTT.

In this section, we introduce our notation for the type theories we consider, and describe a particular type theory describing globular sets. The precise relation between this type theory and the category of globular sets is detailed in Section 2.3 and Section 2.4.

Signature. We consider a countably infinite set whose elements are called *variables*. A *term* is this theory is simply a variable (later on, we consider theories where terms are more general than just variables). A *type* is defined inductively to be either

$$\star \qquad \text{or} \qquad t \underset{A}{\to} u$$

where A is a type and t, u are terms. A *context* is a list

$$(x_1 : A_1, \ldots, x_n : A_n)$$

of variables $x_1, \ldots, x_n$ together with types $A_1, \ldots, A_n$, the empty context is denoted $\varnothing$. A *substitution* is a list

$$\langle x_1 \mapsto t_1, \ldots, x_n \mapsto t_n \rangle$$

of variables $x_1, \ldots, x_n$ together with terms $t_1, \ldots, t_n$. From now on, we use the following naming conventions

variables : $x, y, \ldots$	terms : $t, u, \ldots$	types : $A, B, \ldots$
contexts : $\Gamma, \Delta, \ldots$	substitutions : $\gamma, \delta, \ldots$	

Judgments. The theory consists in four different kinds of *judgments*, for which we give the notations, along with the intuitive meaning:

$\Gamma \vdash$: the context Γ is well-formed
$\Gamma \vdash A$: the type A is well-formed in the context Γ
$\Gamma \vdash t : A$: the term t has type A in context Γ
$\Delta \vdash \gamma : \Gamma$: the substitution γ goes from the context Δ to the context Γ

Most of the time, when we refer to a context, a type, a term or a substitution, we implicitly mean such an object satisfying the adequate judgment. To emphasize this convention we add the adjective *raw* to designate an object as given by the signature, without supposing that a corresponding judgment is derivable, and we state that a property is *syntactic* when it holds for raw expressions.

Syntactic properties. Given a raw term t (resp. a raw type A, a raw context Γ, a raw substitution γ), we define the set of its *free variables* $\mathrm{Var}(t)$ (resp. $\mathrm{Var}(A)$, $\mathrm{Var}(\Gamma)$, $\mathrm{Var}(\gamma)$) by induction as follows

$$\begin{array}{rll}
\text{on terms:} & \mathrm{Var}(x) = \{x\} & \\
\text{on types:} & \mathrm{Var}(\star) = \emptyset & \mathrm{Var}(t \underset{A}{\to} u) = \mathrm{Var}(A) \cup \mathrm{Var}(t) \cup \mathrm{Var}(u) \\
\text{on contexts:} & \mathrm{Var}(\varnothing) = \emptyset & \mathrm{Var}(\Gamma, x : A) = \{x\} \cup \mathrm{Var}(\Gamma) \\
\text{on substitutions:} & \mathrm{Var}(\langle\rangle) = \emptyset & \mathrm{Var}(\langle \gamma, x \mapsto t \rangle) = \mathrm{Var}(t) \cup \mathrm{Var}(\gamma)
\end{array}$$

Given a raw type A in this theory, we define its *dimension* $\dim(A)$ by induction by

$$\dim(\star) = -1 \qquad\qquad \dim(t \underset{A}{\to} u) = \dim(A) + 1$$

The choice of starting at -1 is dictated here by the correspondence established in Lemma 16. We define the dimension of a context $\Gamma = (x_i : A_i)_{1 \leq i \leq n}$ to be

$$\dim(\Gamma) = \max_{1 \leq i \leq n} \dim(A_i)$$

and the dimension of a term t in the context Γ, when the judgment $\Gamma \vdash t : A$ holds to be

$$\dim(t) = \dim(A) + 1$$

Action of substitutions, composition, identity. Given a raw substitution γ, we define its action $t[\gamma]$ on a raw term t, its action $A[\gamma]$ on a raw type A, and is composition $\delta \circ \gamma$ with another raw substitution δ by

$$t[\langle\rangle] = t \qquad t[\langle \gamma, x \mapsto u \rangle] = \begin{cases} u & \text{if } t = x \\ t[\gamma] & \text{otherwise} \end{cases}$$

$$\star[\gamma] = \star \qquad (t \underset{A}{\to} u)[\gamma] = (t[\gamma]) \xrightarrow[(A[\gamma])]{} (u[\gamma])$$

$$\langle\rangle \circ \gamma = \langle\rangle \qquad \langle \delta, x \mapsto t \rangle \circ \gamma = \langle \delta \circ \gamma, x \mapsto t[\gamma] \rangle$$

We also define a special raw substitution associated to a raw context Γ, that we call the *identity substitution* id_Γ, by induction by

$$\mathrm{id}_\varnothing = \langle\rangle \qquad \mathrm{id}_{\Gamma, x:A} = \langle \mathrm{id}_\Gamma, x \mapsto x \rangle$$

Typing rules. The inference rules for the theory GSeTT are given in figure 1. We say that a context (resp. a type, term substitution) is derivable if there is a derivation tree leading to its well-formedness judgment.

For contexts:

$$\frac{}{\varnothing \vdash}(\text{EC}) \qquad \frac{\Gamma \vdash A}{\Gamma, x : A \vdash}(\text{CE}) \quad \text{when } x \notin \mathrm{Var}(\Gamma)$$

For types:

$$\frac{\Gamma \vdash}{\Gamma \vdash \star}(\star\text{-INTRO}) \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash t : A \qquad \Gamma \vdash u : A}{\Gamma \vdash t \underset{A}{\to} u}(\to\text{-INTRO})$$

For terms:

$$\frac{\Gamma \vdash \qquad (x : A) \in \Gamma}{\Gamma \vdash x : A}(\text{VAR})$$

For substitutions:

$$\frac{\Delta \vdash}{\Delta \vdash \langle\rangle : \varnothing}(\text{ES}) \qquad \frac{\Delta \vdash \gamma : \Gamma \qquad \Gamma, x : A \vdash \qquad \Delta \vdash t : A[\gamma]}{\Delta \vdash \langle \gamma, x \mapsto t \rangle : (\Gamma, x : A)}(\text{SE})$$

Figure 1: Derivation rules of the theory GSeTT

We have defined the sets of free variables as a syntactic function on terms and types, thus independent of the judgments, but we are often rather interested in the variables of a typed term together. To express this, we write $\mathrm{Var}(t : A)$ for the union $\mathrm{Var}(t) \cup \mathrm{Var}(A)$, with the implicit convention that in the current context Γ, the judgment $\Gamma \vdash t : A$ is derivable.

The first few results that we mention about the theory GSeTT are proved by induction on the rules of the theory. These induction are typically tedious and uninformative, and we refer the reader to our Agda formalisation [5] for the details.

Lemma 6. *The following properties can be shown and are useful for later proofs*

– *if* $\Gamma \vdash A$ *then* $\Gamma \vdash$,

– *if* $\Gamma \vdash t : A$ *then* $\Gamma \vdash A$,

– *if* $\Delta \vdash \gamma : \Gamma$ *then* $\Delta \vdash$ *and* $\Gamma \vdash$,

– *if* $\Gamma \vdash x \underset{A}{\to} y$ *then* $\Gamma \vdash x : A$ *and* $\Gamma \vdash y : A$,

– *if* $\Gamma \vdash A$, *then* $\mathrm{Var}(A) \subseteq \mathrm{Var}(\Gamma)$,

– *if* $\Gamma \vdash t : A$ *then* $\mathrm{Var}(t : A) \subseteq \mathrm{Var}(\Gamma)$.

Lemma 7. *A term admits at most one type in a given context: if both* $\Gamma \vdash t : A$ *and* $\Gamma \vdash t : B$ *are derivable then* $A = B$.

Lemma 8. *A given judgment admits at most one derivation.*

Notational conventions. In a type $t \underset{A}{\to} u$, the type A is the common type of both t and u and is sometimes left implicit. Similarly, when a substitution $\gamma = \langle x_i \mapsto t_i \rangle_{1 \leq i \leq n}$ is such that the judgment $\Delta \vdash \gamma : \Gamma$ holds for some context $\Gamma = (y_i : A_i)_{1 \leq i \leq m}$, then necessarily $m = n$ and $x_i = y_i$ for $1 \leq i \leq n$. For this reason, when the context Γ is fixed, we may leave the variables $x_1, \ldots, x_n$ implicit and simply write

$$\gamma = \langle t_1, \ldots, t_n \rangle = \langle t_i \rangle_{1 \leq i \leq n}$$

Finally, following Lemma 8, we sometimes abusively assimilate a derivation with the judgment it derives.

2.3 The syntactic category of GSeTT.

Our main tool to study the semantics of a type theory is a category we associate to it, called its syntactic category. We define it in the special case of the theory GSeTT, and state some results which ensure that it is well-defined. We then study the structures present in this category, and illustrate how one can use those in order to study the semantics of the theory. The construction will be analogous later on in the case of the type theory CaTT, albeit more technically involved.

Admissibility of the action of the substitutions. When introducing the type theory, we have defined the actions of substitution, their compositions and the identity substitution syntactically, by induction on the signature. By induction over the rules of the theory, we can check that these operations preserves the derivability of the judgments. Again, this properties have been formally verified in Agda [5].

Proposition 9. *The following rules are admissible*

$$\frac{\Gamma \vdash A \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash A[\gamma]} \qquad \frac{\Gamma \vdash t : A \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash t[\gamma] : A[\gamma]}$$

$$\frac{\Gamma \vdash \theta : \Theta \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \theta \circ \gamma : \Theta} \qquad \frac{\Gamma \vdash}{\Gamma \vdash \mathrm{id}_\Gamma : \Gamma}$$

The syntactic category. The last two statements of Proposition 9 ensure that the composition of substitution and the identity substitution preserve derivability and thus can be lifted as operations on derivable objects. We keep the same notation for these operations.

Proposition 10. *The following equalities hold:*

$$\mathrm{id}_\Gamma \circ \gamma = \gamma \qquad \gamma \circ \mathrm{id}_\Delta = \gamma \qquad \gamma \circ (\delta \circ \theta) = (\gamma \circ \delta) \circ \theta$$

Note that we assume here that all the objects we manipulate are derivable, even if we leave their derivation implicit, in particular, although the second equation holds syntactically, it is not the case for the first equation which only holds for a derivable substitution $\Delta \vdash \gamma : \Gamma$, nor for the last equation which only holds for three a derivable substitutions γ, δ and θ which are composable.

The last two results of Proposition 9 as well as Proposition 10 ensure that we can build a category $\mathcal{S}_{\mathsf{GSeTT}}$, called the *syntactic category* of the theory GSeTT, whose objects are the contexts Γ such that $\Gamma \vdash$ and morphisms $\Delta \to \Gamma$ are the substitutions $\Delta \vdash \gamma : \Gamma$. The first two statements of Proposition 9 can then be read as the fact that it acts on derivable types and terms:

Proposition 11. *The composition of substitutions and the identity substitution are compatible with the action of the substitution on types and terms. More precisely, the following equations hold, for derivable objects:*

$$A[\mathrm{id}_\Gamma] = A \qquad\qquad A[\gamma \circ \delta] = A[\gamma][\delta]$$
$$t[\mathrm{id}_\Gamma] = t \qquad\qquad t[\gamma \circ \delta] = t[\gamma][\delta]$$

Propositions 9, 10 and 12 can be summarized into the following proposition, which is crucial for studying the semantics of type theories:

Proposition 12. *The category $\mathcal{S}_{\mathsf{GSeTT}}$ carries a structure of category with families, such that, for an object Γ of $\mathcal{S}_{\mathsf{GSeTT}}$, the set Ty^Γ consists in the types derivable in Γ and, for A such a type, the set Tm_A^Γ consists in terms of type A in Γ.*

Note 13. Here we have given a presentation with named variables, but one could also give a presentation of the same type theory using unnamed variables, such as de Bruijn indices. This would lead to a slightly different notion of the syntactic category, which is essentially the previously defined syntactic category quotiented under renaming (or α-conversion) of contexts. From now on, we suppose given such a presentation with unnamed variables, so that the renamings are not explicitly taken in account in the syntactic category. Since there is no variable binders, this operation of quotienting is straightforward.

Disks and spheres contexts. In the category $\mathcal{S}_{\mathsf{GSeTT}}$, there are two classes of contexts which play an important role, the *n-disk context* D^n and the *n-sphere context* S^n. Their precise role in our theory are made clear by the Lemma 16 and by the understanding of the syntactic category provided by the Theorem 22. These contexts are defined inductively by

$$\begin{array}{lcl} S^{-1} & = & \varnothing \\ S^n & = & (D^n, d_{2n+1} : U_n) \end{array} \qquad \begin{array}{lcl} D^0 & = & (d_0 : U_0) \\ D^{n+1} & = & (S^n, d_{2(n+1)} : U_{n+1}) \end{array}$$

where the types U_n are inductively defined by

$$\begin{array}{lcl} U_0 & = & \star \\ U_{n+1} & = & d_{2n-2} \underset{U_n}{\longrightarrow} d_{2n-1} \end{array}$$

and where the d_i are a family of distinct variables. We reserve the notation d_i for these specific variables throughout this paper. This is simply a convenient writing convention, since ultimately we consider everything up to renaming.

Proposition 14. *For any integer n, the contexts D^n and S^n are well-formed, i.e., the following rules are admissible.*

$$\frac{}{D^n \vdash} \qquad\qquad \frac{}{S^n \vdash}$$

Proof. We prove the validity of these contexts by induction. First notice that $S^{-1} = \varnothing$ is well defined by the rule (EC), and that by applying successively the rules (CE) and (OBJ), D^0 is also well defined. Then, suppose that S^{k-1} and D^k are valid contexts. The rule (AX) ensures that $D^k \vdash d_{2k} : U_k$, and by Lemma 6, this proves that $D^k \vdash U_k$, since moreover $d_{2k+1} \notin \mathrm{Var}(D^k)$, the rule (CE) applies and shows $S^k \vdash$. Moreover, the rule (AX) applies twice to show both $S^k \vdash d_{2k} : U_k$ and $S^k \vdash d_{2k+1} : U_k$, hence by the rule (HOM), this proves $S^k \vdash U_{k+1}$ and since $d_{2(k+1)} \notin S^k$, the rule (CE) applies and proves $D^{k+1} \vdash$. □

We can also define two substitutions $D^{n+1} \vdash s_n : D^n$ and $D^{n+1} \vdash t_n : D^n$, with the following formulas

$$\begin{aligned} s_n &= \langle d_0 \mapsto d_0, d_1 \mapsto d_1, \ldots, d_{2n-1} \mapsto d_{2n-1}, d_{2n} \mapsto d_{2n} \rangle \\ t_n &= \langle d_0 \mapsto d_0, d_1 \mapsto d_1, \ldots, d_{2n-1} \mapsto d_{2n-1}, d_{2n} \mapsto d_{2n+1} \rangle \end{aligned}$$

One can check that the morphisms define this way satisfy the globular relations (2), hence the disks objects are coglobular objects in the category $\mathcal{S}_{\mathsf{GSeTT}}$. We reformulate this fact by the following definition.

Definition 15. We define the functor $D^\bullet : \mathcal{G}^{\mathrm{op}} \to \mathcal{S}_{\mathsf{GSeTT}}$, sending every object n on the disk context D^n and the morphisms σ_n (resp. τ_n) on the substitution s_n (resp. t_n) in $\mathcal{S}_{\mathsf{GSeTT}}$.

Familial representability of types. The following lemma is central in our study of the type theory GSeTT. It allows to understand both types and terms as special cases of substitutions, and the action of substitution then becomes pre-composition.

Lemma 16. *For any natural number n, the map*

$$\begin{array}{rcl} \mathcal{S}_{\mathsf{GSeTT}}(\Gamma, S^{n-1}) & \to & \left\{ A \in \mathrm{Ty}^\Gamma \mid \dim(A) = n-1 \right\} \\ \gamma & \mapsto & U_n[\gamma] \end{array}$$

is an isomorphism natural in Γ. Given a type A of dimension $n-1$, we denote the associated substitution

$$\chi_A : \Gamma \to S^{n-1}$$

Moreover, the maps

$$\begin{array}{rcl} (\mathcal{S}_{\mathsf{GSeTT}}/S^{n-1})(\Gamma \xrightarrow{\chi_A} S^{n-1}, D^n \xrightarrow{\pi} S^{n-1}) & \to & \mathrm{Tm}^\Gamma_A \\ \gamma & \mapsto & d_{2n}[\gamma] \end{array}$$

are also isomorphisms, natural in Γ (the source is a hom-set in the slice category of $\mathcal{S}_{\mathsf{GSeTT}}$ over S^{n-1}). Given a term $t \in \mathrm{Tm}^\Gamma_A$ of type A, we denote the associated substitution over χ_A by $\chi_t : \Gamma \to D^n$, in such a way that the following diagram commutes

$$\begin{array}{ccc} \Gamma & \xrightarrow{\chi_t} & D^n \\ & \underset{\chi_A}{\searrow} & \downarrow{\scriptstyle \pi} \\ & & S^{n-1} \end{array}$$

Proof. We first prove that the first part of the statement implies the second one, for a given natural number n. This is a consequence of the fact that the context D^n is defined to be $(S^{n-1}, d_{2n} : U_n)$. Indeed, an object in $\mathcal{S}_{\mathsf{GSeTT}}/S^{n-1}$ is a context that comes equipped with a substitution $\gamma : \Gamma \to S^{n-1}$, and the universal property of the context comprehension operation states exactly that there is a natural isomorphism $\mathcal{S}_{\mathsf{GSeTT}}(\Gamma, D^n) \cong \mathrm{Tm}^{\Gamma}_{U_n[\gamma]}$. Using the previous natural isomorphism, one can write γ as χ_A and $U_n[\gamma]$ then simplifies to A, which proves the natural isomorphism $\mathcal{S}_{\mathsf{GSeTT}}(\Gamma, D^n) \cong \mathrm{Tm}^{\Gamma}_{A}$. We now prove by induction over the dimension n that the first part of the judgment holds.

- Case $n = 0$. The context $S^{-1} = \varnothing$ is terminal: there is always exactly one substitution $\Gamma \vdash \langle\rangle : \varnothing$. Similarly there is always exactly one type of dimension -1 derivable in Γ, which is the type $\star$, and which is the type U_0 by definition.
- Suppose that the result holds for the sphere S^{n-1}. Then, by the second part of the result that we have already proven, we get a natural isomorphism $(\mathcal{S}_{\mathsf{GSeTT}} / S^{n-1})(\Gamma, D^n) \cong \mathrm{Tm}^{\Gamma}_{A}$. Substitutions $\Gamma \vdash \gamma : S^n$ are exactly the ones of the form $\langle \gamma', u\rangle$ and are derived by the following application of the rule (SE)

  $$\frac{\Gamma \vdash \gamma' : D^n \qquad D^n \vdash U_n \qquad \Gamma \vdash u : U_n[\gamma']}{\Gamma \vdash \langle \gamma', u\rangle : S^n}$$

  The substitutions $\Gamma \vdash \gamma : S^n$ are thus naturally isomorphic to pairs γ', u, with $\Gamma \vdash \gamma' : D^n$ and $\Gamma \vdash u : U_n[\gamma']$. By induction, the substitutions $\Gamma \vdash \gamma' : D^n$ are of the form χ_t, for $\Gamma \vdash t : A$ a term in Γ. Then the type $U_n[\chi_t]$ rewrites as A by naturality of the previous transformation. So these substitutions are naturally isomorphic to pairs of terms of dimension n and of the same type in Γ, which are exactly the types in Γ of dimension n.

□

Note 17. This proof does not rely on how the terms are constructed, so no matter what the term constructors are, this result will always hold.

We can reformulate this result in several ways. First, we can collect together all the isomorphisms $\left\{A \in \mathrm{Ty}^{\Gamma} \mid \dim A = n-1\right\} \cong \mathcal{S}_{\mathsf{GSeTT}}(\Gamma, S^{n-1})$ into a single natural isomorphism

$$\mathrm{Ty}^{\Gamma} \cong \coprod_{n\in\mathbb{N}} \mathcal{S}_{\Gamma, S^{n-1}}$$

In other words, we have proven that the family $S^{\bullet}$ *familially represents* the functor Ty. We can also unravel a bit this proposition, showing that any type $\Gamma \vdash A$ corresponds uniquely to a substitution $\Gamma \vdash \chi_A : S^{n-1}$, and that any term $\Gamma \vdash t : A$ corresponds uniquely to a substitution $\Gamma \vdash \chi_t : D^n$, in such a way that the following diagram commutes

$$\begin{array}{ccc} \Gamma & \xrightarrow{\chi_t} & D^n \\ & \searrow^{\chi_A} & \downarrow \pi \\ & & S^{n-1} \end{array}$$

To simplify things further, we write $\mathrm{ty} : D^n \to S^{n-1}$ for the projection substitution, so that we have $\mathrm{ty} \circ \chi_t = \chi_A$. In other words, ty acts on terms by giving their associated types. The definition of the morphism ty along with Lemma 1 shows the following:

Lemma 18. *In the category $\mathcal{S}_{\mathsf{GSeTT}}$, a context of the form $(\Gamma, x : A)$ is obtained as the pullback*

$$\begin{array}{ccc} (\Gamma, x : A) & \longrightarrow & D^n \\ \pi \downarrow & \lrcorner & \downarrow \mathrm{ty} \\ \Gamma & \xrightarrow[\chi_A]{} & S^{n-1} \end{array}$$

It is straightforward using Lemma 18 to check that the sphere contexts can be obtained as iterated pullbacks of the disks contexts, dually to the way topological spheres can be obtained as pushout of topological disks.

Lemma 19. *The sphere context S^n is obtained as the following pullback*

$$\begin{array}{ccc} S^n & \longrightarrow & D^n \\ \downarrow & \lrcorner & \downarrow \pi \\ D^n & \xrightarrow{\ \pi\ } & S^{n-1} \end{array}$$

The syntactic category of GSeTT. We now characterize the syntactic category of GSeTT. This is an important step in order to study the models of the theory, since understanding precisely the syntactic category gives good insights on the functors mapping out of it. Interestingly, in all the cases we study here, it always turn out that the syntactic category is dual to the category of finitely presented objects that we are studying, in accordance with the Gabriel-Ülmer duality [18]. In order to prove this result, we introduce a functor that we denote V (the V stands for "variable"), that we describe as follows.

Definition 20. The functor $V : \mathcal{S}_{\mathsf{GSeTT}} \to \mathbf{FinGSet}^{\mathrm{op}}$ associates to any context $\Gamma = (x_i : A_i)$ the set

$$(V\Gamma)_n = \{x_i \mid \dim(A_i) = n\} = \{\text{derivable terms of dimension } n \text{ in } \Gamma\}$$

and to any substitution $\Delta \vdash \langle x_i : t_i \rangle : \Gamma$ associates the map

$$\begin{array}{rcrcl} V\gamma & : & V\Gamma & \to & V\Delta \\ & & x_i & \mapsto & t_i \end{array}$$

or equivalently, we require the equation $(V\gamma)x = V(x[\gamma])$.

Lemma 21. *The functor V is well-defined.*

Proof. For x of type A in Γ, with $\dim(A) = n+1$, by definition of the dimension, A is of the form $A = y \to z$, for two derivable terms y and z, with $\dim_\Gamma(y) = \dim_\Gamma(z) = n$. We therefore have $y, z \in (V\Gamma)_n$, and we define $s(x) = y$ and $t(x) = z$. The derivation rule for A implies that y and z have the same type, thus $s(y) = s(z)$ and $t(y) = t(z)$, which proves that the globular relations are satisfied, and that $V\Gamma$ is indeed a globular set.

Let $\Delta \vdash \gamma : \Gamma$ be a substitution, and write $\Gamma = (x_i : A_i)$, then the substitution γ is of the form $\gamma = \langle x_i : t_i \rangle$, where t_i is a derivable term in the context Δ, i.e., $t_i \in V\Delta$. Suppose that x is of type $y \to z$ in Γ, then $x[\gamma] = y[\gamma] \to z[\gamma]$ in Δ. This means that as an element of $F\Delta$, $x[\sigma]$ satisfies $s(x[\gamma]) = y[\gamma]$ and $t(x[\gamma]) = z[\gamma]$, or in other words, $s((F\gamma)x) = (F\gamma)(sx)$ and $t((F\gamma)x) = (F\gamma)(tx)$. Hence $F\gamma$ defines a morphism of globular sets. $\square$

Theorem 22. *The functor V is part of an equivalence of categories $\mathcal{S}_{\mathsf{GSeTT}} \simeq \mathbf{FinGSet}^{\mathrm{op}}$.*

Proof. We first show that V is full and faithful. Consider two substitutions γ and δ such that $V\gamma = V\delta$. This implies in particular that for all variables x in Γ, $(V\gamma)x = (V\delta)x$, thus $x[\gamma] = x[\delta]$. By Lemma 5, this proves that $\gamma = \delta$, hence V is faithful. Dually, consider two contexts Γ and Δ, where $\Delta = (x_i : A_i)_{0 \leq i\ \leq l}$, together with a morphism of globular sets $f : V\Gamma \to V\Delta$. Then one can define the substitution $\gamma_f = \langle x_i\ : f(x_i) \rangle_{1 \leq i\ \leq l}$. We check by induction on the length l of Δ that this produces a well-defined substitution γ_f such that $V(\gamma_f) = f$. If $l = 0$ then $\Delta = \varnothing$ and $\gamma_f = \langle\rangle$, then the rule (ES) gives a derivation of $\Gamma \vdash \langle\rangle : \varnothing$. If $\Delta = (\Delta', x_{l+1} : A_{l+1})$, then the natural inclusion $V\Delta' \hookrightarrow V\Delta$ induces by composition a map $f' : V\Delta' \to V\Gamma$. By induction hypothesis, we have $\Gamma \vdash \gamma_{f'} : \Delta'$, and since Δ is a context, we also have $\Delta \vdash A_{l+1}$. Moreover, if $A_{l+1} = \star$, then $\Gamma \vdash f(x_{l+1}) : \star$ since f preserves the dimension, and otherwise $A_{l+1} = y \to z$, and $\Gamma \vdash f(x_{l+1}) : f(y) \to f(z)$ since f is a morphism of globular sets. In both cases, this proves that $\Gamma \vdash f(x_{l+1}) : A_{l+1}[\gamma_{f'}]$. By application of the rule (SE), this proves that $\Gamma \vdash \langle \gamma_{f'}, x_{l+1} \mapsto f(n_{l+1}) \rangle : \Delta$. Since $\gamma_f = \langle \gamma_{f'}, x_{l+1} \mapsto f(x_{l+1}) \rangle$, this proves that γ_f is well defined, and by definition it satisfies $V\gamma = f$. Hence the functor V is full.

Moreover, V is essentially surjective. Indeed, considering a finite globular set X, we show by induction on the number of elements of X that we can construct a context Γ such that $V\Gamma = X$. If X is the empty globular set, then $\Gamma = \varnothing$ is well defined by the rule (EC), otherwise, if X is not empty, consider an element x of maximal dimension in X and consider the globular set Y obtained by removing this element from X. By induction, the context Δ constructed from Y is well-defined. Moreover, if x is of dimension 0, then define $A = \star$ and we have $\Delta \vdash A$, and otherwise, we have $\Delta \vdash sx : B$ and $\Delta \vdash tx : B$ since both sx and tx are parallel elements in Y, and define $A = s\, x \to tx$, this shows that $\Delta \vdash A$. In both cases, we have $\Delta \vdash A$, and the rule (CE) applies to prove that $\Delta, x : A \vdash$. Moreover $V(\Delta, x : A)$ is obtained from $V\Delta$ by adding one element x' of the same dimension as x, and such that $sx' = sx$ and $tx' = tx$ if this dimension is not 0. Since by induction $V\Delta = Y$, we deduce $V(\Delta, x : A) = X$. This construction is not canonical, and there are in general many contexts Γ such that $V\Gamma = X$, but the fact that we can construct one shows that V is essentially surjective. Since the functor V is fully faithful and essentially surjective, it is an equivalence of categories. □

We can give an alternative description of V in the light of Lemma 16. Indeed a term of dimension n in Γ is simply a substitution $\Gamma \to D^n$, hence $V(\Gamma)_n = \mathcal{S}_{\mathsf{GSeTT}}(\Gamma, D^n)$. Consider the generalized nerve functor $\mathcal{S}_{\mathsf{GSeTT}}(\_, D^\bullet) : \mathcal{S}_{\mathsf{GSeTT}}^{\mathrm{op}} \to \mathbf{GSet}$ associated to the inclusion $D^\bullet : \mathcal{G}^{\mathrm{op}} \to \mathcal{S}_{\mathsf{GSeTT}}$. This functor can be seen as a functor $\mathcal{S}_{\mathsf{GSeTT}} \to \mathbf{GSet}^{\mathrm{op}}$. By the previous remark, it coincides with V on objects, and hence it restricts to a functor $\mathcal{S}_{\mathsf{GSeTT}}(\Gamma, D^\bullet) \to \mathbf{FinGSet}$. Moreover, for any variable $\Gamma \vdash x : A$ and any substitution $\Delta \vdash \gamma : \Gamma$, we have the equalities

$$\chi_x \circ \gamma = \chi_{x[\gamma]}$$
$$V(\gamma)(x) = V(x[\gamma])$$

which show that the functors V and $\mathcal{S}_{\mathsf{GSeTT}}(\_, D^\bullet)$ coincide on morphisms. From now on, we thus identify V with the generalized nerve functor $\mathcal{S}_{\mathsf{GSeTT}}(\_, D^\bullet)$, and use this point of view in more involved situations of interest.

Remark 23. Under the equivalence of categories of Theorem 22, the globular set D^n corresponds exactly to the context D^n, and the globular set S^n corresponds to the context S^n. This justifies the choice of the same notations for the contexts and the globular sets.

2.4 Models of the type theory GSeTT.

We can use the characterization of the syntactic category of GSeTT obtained in previous section in order to study its models. This relies heavily on the fact that $\mathcal{S}_{\mathsf{GSeTT}} \simeq \mathbf{FinGSet}^{\mathrm{op}}$ is the free finite limit completion of the category $\mathcal{G}^{\mathrm{op}}$. It is helpful for to start with a small discussion on Kan extensions and their properties.

Properties of Kan extensions. We first recall a few important properties of the right Kan extensions. These are known results, on which our constructions rely. Given a functor $G : \mathcal{C} \to \mathcal{D}$ and an object $d \in \mathcal{D}$, there is a coma-category $d \downarrow G$ equipped with the forgetful functor

$\Pi_d : d \downarrow G \to \mathcal{C}$. Given a functor $F : \mathcal{C} \to \mathcal{E}$ such that for all d, the limit of the diagram $F\Pi_d$ exists, then it is a classical result [29, th.6.2.1 and 6.3.7] that the right Kan extension $\mathrm{Ran}_G F$ exists and it pointwise, i.e., it is given by the formula

$$(\mathrm{Ran}_G F)d = \lim \left(d \downarrow G \overset{\Pi_d}{\to} \mathcal{C} \overset{G}{\to} \mathcal{E} \right)$$

Define the nerve functors $N_G : \mathcal{D}^{\mathrm{op}} \to \widehat{\mathcal{C}^{\mathrm{op}}}$ by $N_G(d) = \mathcal{D}(d, G\_)$ and $N_F : \mathcal{E}^{\mathrm{op}} \to \widehat{\mathcal{C}^{\mathrm{op}}}$ by $N_F(e) = \mathcal{E}(e, F\_)$.

Lemma 24. *The pointwise right Kan extension is uniquely characterized by the existence of a natural isomorphism* $\mathcal{E}(e, (\mathrm{Ran}_G F)d) \cong \widehat{\mathcal{C}}(N_G d, N_F e)$.

Proof. The set of cones of apex e over the diagram $G\Pi_d$ is naturally isomorphic to $\widehat{\mathcal{C}}(N_G d, N_F e)$ (see [29, lemma 6.3.8]). Under this isomorphism, this is the universal property of the limit. □

Lemma 25. *If N_G sends finite limits to the corresponding finite colimits, and if the pointwise right Kan extension* $\mathrm{Ran}_G F$ *exists, then it preserves finite limits.*

Proof. Consider a finite diagram $A : I \to \mathcal{C}$, together with its limit $\lim A$. Then, for any object e in $\mathcal{E}$, we have the following isomorophisms, by Lemma 24, continuity of the Hom-functor and assumption on N_G

$$\begin{aligned}
\mathcal{E}(e, (\mathrm{Ran}_G F)(\lim A)) &\cong \widehat{\mathcal{C}}(N_G(\lim A), N_F e) \\
&\cong \widehat{\mathcal{C}}(\mathrm{colim}(N_G \circ A^{\mathrm{op}}), N_F e) \\
&\cong \lim(\widehat{\mathcal{C}}(N_G \circ A^{\mathrm{op}}, N_F e)) \\
&\cong \lim \mathcal{E}(e, \mathrm{Ran}_G(F \circ A)) \\
&\cong \mathcal{E}(e, \lim \mathrm{Ran}_G(F \circ A))
\end{aligned}$$

This shows that $(\mathrm{Ran}_G F)(\lim A)$ satisfies the characterization of $\lim \mathrm{Ran}_G(F \circ A)$ given in Lemma 24. □

Lemma 26. *If G is fully faithful and the right Kan extension* $\mathrm{Ran}_G F$ *exists, then the universal natural transformation* $\epsilon : (\mathrm{Ran}_G F)G \cong G$ *is a natural isomorphism.*

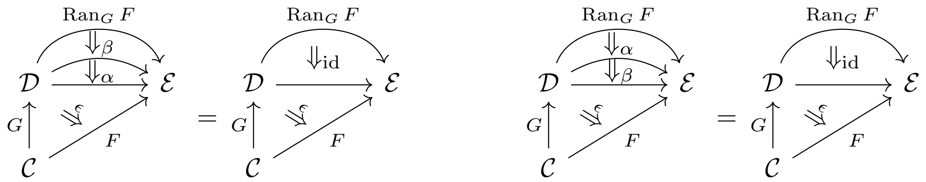
Proof. If G is fully faithful, then $N_G(Gc) = \mathcal{D}(Gc, G\_) = \mathcal{C}(c, \_)$ is a representable presheaf over $\mathcal{C}^{\mathrm{op}}$. By the characterization given by Lemma 24 and the Yoneda lemma,

$$\mathcal{E}(e, (\mathrm{Ran}_G F)Gc) \cong \widehat{\mathcal{C}^{\mathrm{op}}}(N_G(Gc), N_F e) \cong (N_F e)_c \cong \mathcal{E}(e, Fc)$$

Since this isomorphism is natural in e, it shows that $(\mathrm{Ran}_G F)Gc$ is the limit of the diagram with a single point Fc, hence $(\mathrm{Ran}_G F)Gc \cong Fc$. □

Lemma 27. *If G is fully faithful, and the right Kan extension* $\mathrm{Ran}_G F$ *exists, then for every natural transformation* $\alpha : \mathrm{Ran}_G F \Rightarrow \mathrm{Ran}_G F$ *such that the restriction* α_G *is a natural isomorphism,* α *is a natural isomorphism.*

Proof. Consider the universal natural transformation $\epsilon : (\mathrm{Ran}_G F)G \Rightarrow G$. By Lemma 26, it is an isomorphism. Then consider the composition $\epsilon(\alpha_G)^{-1} : (\mathrm{Ran}_G F)G \Rightarrow G$. By universality of the Kan extension, there exists $\beta : \mathrm{Ran}_G F \Rightarrow \mathrm{Ran}_G F$ such that $\epsilon\beta_G = \epsilon(\alpha_G)^{-1}$. We now show that β is the inverse of α. We have the following equalities of diagrams



By universal property of the Kan extension, this shows that $\beta\alpha = \mathrm{id}$ and $\alpha\beta = \mathrm{id}$. □

Application to the type theory. We now come back to $\mathcal{S}_{\mathsf{GSeTT}}$, and consider extensions along the disk functor $D^\bullet : \mathcal{G}^{\mathrm{op}} \to \mathcal{S}_{\mathsf{GSeTT}}$. Up to equivalence of Theorem 22, this functor is the Yoneda embedding, so it is fully faithful, and its nerve is the functor V defined in Section 2.3, which associates to each context the globular set of its variables. By Theorem 22, this functor is an equivalence of categories $\mathcal{S}_{\mathsf{GSeTT}}^{\mathrm{op}} \cong \mathbf{FinGSet}$, hence it sends limits onto colimits. Given a category $\mathcal{C}$ and a functor $F : \mathcal{G}^{\mathrm{op}} \to \mathcal{C}$ we denote its nerve T_F, and in the case where $\mathcal{C}$ is a category with families, this functors gives a class of terms.

Theorem 28. *Consider a finitely complete category $\mathcal{C}$ together with a functor $F : \mathcal{G}^{\mathrm{op}} \to \mathcal{C}$, then the following pair of functors defines an equivalence of categories*

$$\_ \circ D^\bullet \quad : \quad \mathbf{Cat}(\mathcal{S}_{\mathsf{GSeTT}}, \mathcal{C})_{\mathrm{flim}} \quad \simeq \quad \mathbf{Cat}(\mathcal{G}^{\mathrm{op}}, \mathcal{C}) \quad : \quad \mathrm{Ran}_{D^\bullet}$$

Proof. Since $\mathcal{C}$ is finitely complete, the right Kan extension exists. Lemma 26 shows that for any functor $F : \mathcal{G}^{\mathrm{op}} \to \mathcal{C}$, we have a natural transformation $\mathrm{Ran}_{D^\bullet}(F \circ D^\bullet) \circ D^\bullet \cong F \circ D^\bullet$. Conversely, we show that there is a natural isomorphism $\mathrm{Ran}_{D^\bullet}(F \circ D^\bullet) \cong F$, i.e., that every functor preserving finite limits is isomorphic to the Kan extension of its restriction. Lemma 26 shows that F and $\mathrm{Ran}_{D^\bullet}(F \circ D^\bullet)$ coincide on all the disk objects. Moreover, any object Γ is a finite limit of disk objects, by Theorem 22 alongside with the fact that every presheaf is a colimit of representables. Since both F and $\mathrm{Ran}_{D^\bullet}(F \circ D^\bullet)$ coincide on disk objects and preserve finite limits, they coincide on Γ, hence they are naturally isomorphic. □

Considering the Kan extension of the disk $D^\bullet : \mathcal{G}^{\mathrm{op}} \to \mathcal{S}_{\mathsf{GSeTT}}$ along itself, this theorem implies that $\mathrm{Ran}_{D^\bullet} D^\bullet$ is the identity functor. Lemma 24 then states that $\mathcal{S}_{\mathsf{GSeTT}}(\Gamma, \Delta) \cong \widehat{\mathcal{G}}(V\Delta, V\Gamma)$, that is, substitutions from Γ to Δ are given by the data of a variable of Γ for every variable of Δ in a way that is compatible with the source and targets. In Section 4 we present a type theory with term constructors, for which a substitution associates a term to every variable. We discuss a of a way to generalize this result to that theory in Section 6.5. Note that Theorem 22 characterizes the syntactic category $\mathcal{S}_{\mathsf{GSeTT}}$ as the opposite of finite globular sets. Since all the representable disks D^n are themselves finite, the finite globular sets are exactly the universal cocompletion of the category $\mathcal{G}$ by finite colimits and Theorem 28 is the universal property of this characterization. This is a standard construction (see for instance [23, Theorem 5.37]), but the tools that we have introduced serve as preparatory work for the study of the models of CaTT.

Models of the theory GSeTT. We can now characterize the models of the type theory GSeTT. This characterization of the models relies the following lemma

Lemma 29. *Given a functor $F : \mathcal{S}_{\mathsf{GSeTT}} \to \mathcal{C}$ which preserves the terminal object and sends pullbacks along the display maps $\left\{\pi : D^n \to S^{n-1}\right\}$ to pullbacks, then F is naturally isomorphic to the pointwise right Kan extension of its restriction $F \cong \mathrm{Ran}_{D^\bullet} F D^\bullet$.*

Proof. We prove this property by induction on the context. Specifically, we show that for every object Γ of $\mathcal{C}$ and every context Δ, we have the isomorphism $\mathcal{C}(\Gamma, F\Delta) \cong \widehat{\mathcal{G}}(V\Delta, T_F\Gamma)$, which characterizes the right Kan extension. We first prove this property for the disk objects. This is given by the Yoneda lemma: since VD^n is the representable presheaf in $\widehat{\mathcal{G}}$, we have $\widehat{\mathcal{G}}(VD^n, T_F\Gamma) \cong (T_F\Gamma)_n \cong \mathcal{C}(\Gamma, FD^n)$. We now prove this property on all spheres by induction on the dimension:

- The sphere S^{-1} is the empty context $\varnothing$, which is terminal in $\mathcal{S}_{\mathsf{GSeTT}}$, hence FS^{-1} is terminal in $\mathcal{C}$. Moreover, VS^{-1} is the empty presheaf since there are no variable in the empty context, and thus it is initial. This proves that $\mathcal{C}(\Gamma, FS^{-1}) = \widehat{\mathcal{G}}(VS^{-1}, T_F\Gamma) = \{\bullet\}$.
- Assume the property for the sphere S^{n-1}. The sphere S^{n+1} is obtained as a pushout as follows

$$\begin{array}{ccc} S^n & \longrightarrow & D^n \\ \big\downarrow & \lrcorner & \big\downarrow \\ D^n & \longrightarrow & S^{n-1} \end{array}$$

The equivalence is then shown by the following computation, using the inductive hypothesis as well as the preservation of this pushout square by F, and the continuity properties of the involved hom-functors, and the fact that V is an equivalence of categories.

$$\begin{aligned}
\mathcal{C}(\Gamma, FS^n) &\cong \mathcal{C}(\Gamma, F(\lim(D^n \to S^{n-1} \leftarrow D^n))) \\
&\cong \lim(\mathcal{C}(\Gamma, FD^n) \to \mathcal{C}(\Gamma, FS^{n-1}) \leftarrow \mathcal{C}(\Gamma, FD^n)) \\
&\cong \lim(\widehat{\mathcal{G}}(VD^n, T_F\Gamma) \to \widehat{\mathcal{G}}(VS^{n-1}, T_F\Gamma) \leftarrow \widehat{\mathcal{G}}(VD^n, T_F\Gamma)) \\
&\cong \widehat{\mathcal{G}}(\operatorname{colim}(VD^n \leftarrow VS^{n-1} \to VD^n), T_F\Gamma) \\
&\cong \widehat{\mathcal{G}}(V(\lim(D^n \to S^{n-1} \leftarrow D^n)), T_F\Gamma) \\
&\cong \widehat{\mathcal{G}}(VS^n, T_F\Gamma)
\end{aligned}$$

We now prove the desired property by induction on the contexts.

- A context of length 0 is the empty context, which is also the sphere S^{-1}, for which we have already proven the property.
- The context (Δ, A) is obtained as the following pullback

$$\begin{array}{ccc} (\Delta, A) & \longrightarrow & D^n \\ \big\downarrow & \lrcorner & \big\downarrow \\ \Delta & \xrightarrow{\chi_A} & S^{n-1} \end{array}$$

Then, by a computation similar to the case of the sphere, using induction, the preservation of this pullback by F, and continuity property of the hom-functors, we have

$$\begin{aligned}
\mathcal{C}(\Gamma, F(\Delta, A)) &\cong \lim(\widehat{\mathcal{G}}(V\Delta, T_F\Gamma) \to \widehat{\mathcal{G}}(VS^{n-1}, T_F\Gamma) \leftarrow \widehat{\mathcal{G}}(VD^n, T_F\Gamma)) \\
&\cong \widehat{\mathcal{G}}(V(\Delta, A), T_F\Gamma)
\end{aligned}$$

□

Theorem 30. *There is an equivalence of categories* $\mathbf{Mod}(\mathcal{S}_{\mathsf{GSeTT}}) \simeq \mathbf{GSet}$.

Proof. Theorem 28 shows that $\mathbf{GSet} \simeq \mathbf{Cat}(\mathcal{S}_{\mathsf{GSeTT}}, \mathbf{Set})_{\mathrm{flim}}$, so it suffices to prove that the models are equivalent to that category. Define the pair of functors

$$\mathbf{Mod}(\mathcal{S}_{\mathsf{GSeTT}}) \rightleftarrows \mathbf{Cat}(\mathcal{S}_{\mathsf{GSeTT}}, \mathbf{Set})_{\mathrm{flim}}$$

associating to every morphism of categories with families $F : \mathcal{S}_{\mathsf{GSeTT}} \to \mathbf{Set}$ the right Kan extension, and to every finite limit preserving functor $G : \mathcal{S}_{\mathsf{GSeTT}} \to \mathbf{Set}$, the morphism of category with families that it defines by Lemma 3. Then Lemmas 26 and 29 show that this defines an equivalence of categories. □

Note that this proof consists in two parts:

1. restate the models as being equivalent to the functors preserving finite limits,
2. use standard categorical machinery to show that these are equivalent to globular sets.

The first step abstracts away the category with families structure, to restate the problem as a plain category theory one. This approach is contingent to recognizing the syntactic category to be $\mathbf{FinGSet}^{\mathrm{op}}$ (Theorem 22) and does not generalize. For this reason, we give a reformulation of the proof, using an initiality theorem for the category $\mathcal{S}_{\mathsf{GSeTT}}$, which provides a better account of the category with families structure.

2.5 Globular categories with families.

We now introduce the notion of *globular categories with families*, which are particular categories with families that share a lot of structural properties with the category $\mathcal{S}_{\mathsf{GSeTT}}$.

Definition 31. A globular category with families is a category with families $\mathcal{C}$ equipped with two families of objects $(S^{n-1})_{n\in\mathbb{N}}$ and $(D^n)_{n\in\mathbb{N}}$ and a family of types $U_n \in \mathrm{Ty}_{S^n}$ such that the following equations are satisfied

$$S^{-1} = \varnothing$$
$$S^n = (D^n, U_{n-1}[\partial_n]) \qquad D^n = (S^{n-1}, U_{n-1})$$

where ∂_n denotes the projection map $\partial_n = \pi_{S^{n-1},U_{n-1}} : D^n \to S^{n-1}$.

We suppose given a globular category with families $\mathcal{C}$, we denote $\frown_n : S^n \to D^n$ the display map given by the category with families. Moreover, we also define another map $\smile_n : S^n \to D^n$ by $\smile_n = \langle \partial_n \frown_n, p_n \rangle$, where p_n denotes the universal term of S^n of type $U_{n-1}[\partial_n \frown_n]$ obtained by context comprehension. With the help of these maps, we define a pair of maps

$$\begin{array}{lcl} s_n & : & D^{n+1} \to D^n \\ s_n & = & \frown_n \partial_{n+1} \end{array} \qquad \begin{array}{lcl} t_n & : & D^{n+1} \to D^n \\ t_n & = & \smile_n \partial_{n+1} \end{array}$$

Lemma 32. *The maps s_n and t_n satisfy the globular relations* (2).

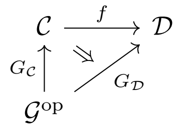
Proof. Using the properties of substitution extension in categories with families, we have the following computations.

$$\begin{aligned} s_i t_{i+1} &= \frown_i \partial_{i+1} \smile_{i+1} \partial_{i+2} \\ &= \frown_i \partial_{i+1} \langle \partial_{i+1} \frown_{i+1}, p_{i+1} \rangle \partial_{i+2} \\ &= \frown_i \partial_{i+1} \frown_{i+1} \partial_{i+2} \\ &= s_i s_{i+1} \end{aligned} \qquad \begin{aligned} t_i t_{i+1} &= \smile_i \partial_{i+1} \smile_{i+1} \partial_{i+2} \\ &= \smile_i \partial_{i+1} \langle \partial_{i+1} \frown_{i+1}, p_{i+1} \rangle \partial_{i+2} \\ &= \smile_i \partial_{i+1} \frown_{i+1} \partial_{i+2} \\ &= t_i s_{i+1} \end{aligned}$$

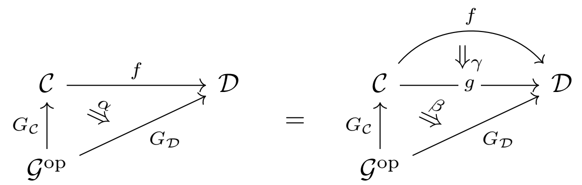
□

This shows that given a category with families $\mathcal{C}$, we can construct a functor $G_{\mathcal{C}} : \mathcal{G}^{\mathrm{op}} \to \mathcal{C}$, by setting $G_{\mathcal{C}}(n) = D^n$, $G_{\mathcal{C}}(\sigma_n) = s_n$ and $G_{\mathcal{C}}(\tau_n) = t_n$. We call it the *induced globular structure* of the globular category with families $\mathcal{C}$.

Definition 33. A morphism of globular category with families between $\mathcal{C}$ and $\mathcal{D}$ is a morphism f of categories with families between them, along with a natural transformation between their induced globular structures as follows



We denote **gCwF** the category of globular categories with families defined this way. We also define a 2-morphism between two morphisms of globular categories with families (f, α) and (g, β) to be a natural transformation $\gamma : f \Rightarrow g$ which commutes with the natural transformations of the induced globular structures, that is such that we have the following equality



We denote $\underline{\textbf{gCwF}}$ the resulting 2-category.

Initiality of $\mathcal{S}_{\mathsf{GSeTT}}$. We now reformulate Lemma 26 about Kan extensions in terms of the notion of globular category with families. This provides a local form of 2-categorical initiality of the category $\mathcal{S}_{\mathsf{GSeTT}}$ in the 2-category of globular categories with families.

Lemma 34. *For every globular category with families $\mathcal{C}$, $\mathrm{Ran}_{D^\bullet} G_{\mathcal{C}} : \mathcal{S}_{\mathsf{GSeTT}} \to \mathcal{C}$ exists and is a pointwise.*

Proof. We construct by induction a functor $r : \mathcal{S}_{\mathsf{GSeTT}} \to \mathcal{C}$ such that $r(D^n) = G_{\mathcal{C}}(n)$, and r preserves the pullbacks along the projection maps $\pi : D^n \to S^{n-1}$ for all $n \in \mathbb{N}$, and preserves the terminal object. We define by induction on Γ, an object $r(\Gamma)$ of $\mathcal{C}$ such that $r(D^n) = G_{\mathcal{C}}(n)$ and $r(S^{n-1}) = S^{n-1}$, along with, for every morphism $m : \Gamma \to S^n$ (resp. $m : \Gamma \to D^n$), a corresponding morphism $m' : r(\Gamma) \to S^n$ (resp. $m' : r(\Gamma) \to D^n$), such that $\partial'_n = \partial_n$, $\smile'_n = \smile_n$, $\frown'_n = \frown_n$ and $\mathrm{id}'_{S^n} = \mathrm{id}_{S^n}$.

- For the empty context, $\varnothing$, we define $r(\varnothing)$ to be the terminal object $\varnothing$ in the category $\mathcal{C}$. Since $\varnothing = S^{-1}$ both in $\mathcal{S}_{\mathsf{GSeTT}}$ and in $\mathcal{C}$, this gives the commutation property that we want. Moreover, the only map from the empty context to a disk or a sphere context is the identity map $\mathrm{id}_\varnothing = \varnothing \to \varnothing$, and we define $\mathrm{id}'_\varnothing = \mathrm{id}_\varnothing$.
- For a context of the form (Γ, A), where $\dim A = k$, we define $r(\Gamma, A) = (r(\Gamma), U_k[\chi'_A])$. Moreover, given a morphism $\chi_x = (\Gamma, A) \to D^n$, either the variable x is a variable in Γ, in which case χ_x factors as $\chi_y \pi_{\Gamma,A}$ (where y denotes the same variable x but seen as variable in Γ), and we chose $\chi'_x = \chi'_y \pi_{r(\Gamma), U_k[\chi'_A]}$, or x is the last variable in Γ, in which case we

have the pullback square on the left, and we chose χ'_x as displayed on the pullback square on the right.

$$\begin{array}{ccc} (\Gamma, A) & \xrightarrow{\chi_x} & D^{k+1} \\ \big\downarrow & \lrcorner & \big\downarrow \partial_{k+1} \\ \Gamma & \xrightarrow{\chi_A} & S^k \end{array} \qquad\qquad \begin{array}{ccc} r(\Gamma, A) & \xrightarrow{\chi'_x} & D^{k+1} \\ \big\downarrow & \lrcorner & \big\downarrow \partial_{k+1} \\ r(\Gamma) & \xrightarrow{\chi'_A} & S^k \end{array}$$

Given a map $m : (\Gamma, A) \to S^n$, if $n = -1$, then, m is the unique map to the terminal object, and we chose m' to be the unique map to the terminal object in $\mathcal{C}$, otherwise, $S^n = (D^n, U_{n-1}[\partial_n])$, and we have $m = \langle \frown_k m, t\rangle$. We then chose

$$m' = \langle (\frown_k m)', p_{S^{n-1}, U_{n-1}}[\chi'_t]\rangle.$$

In the case where $(\Gamma, A) = D^{k+1}$, then $\Gamma = S^k$ and $A = U_k$. Then $\chi_{U_k} = \mathrm{id}_{S^k}$, so by induction $\chi'_{U_k} = \mathrm{id}_{S^k}$, and thus, $r(D^{k+1}) = (S^k, U_k) = D^{k+1}$. Moreover, given the map $\partial_{k+1} : D^{k+1} \to S^k$, we have $(\frown_k \partial_{k+1})' = \frown'_k \partial_{k+1}$, and $\chi'_{p_k[\partial_{k+1}]} = \chi'_{p_k}\partial_{k+1}$. Moreover, $\chi_{p_k} = \smile_k$. By induction, this shows that we have

$$\begin{aligned} \partial'_{k+1} &= \langle (\frown_k \partial_{k+1})', p_{S^k, U_k}[\chi'_{p_k[\partial_{k+1}]}]\rangle \\ &= \langle \frown_k \partial_{k+1}, p_{S^k, U_k}[\smile_k \partial_{k+1}]\rangle \\ &= \langle \frown_k \partial_{k+1}, p_k[\partial_{k+1}]\rangle \\ &= \partial_{k+1} \end{aligned}$$

In the case where $(\Gamma, A) = S^{k+1}$, then $\Gamma = D^{k+1}$ and $A = U_k \partial_{k+1}$, so $\chi_A = \partial_{k+1}$, and by induction $\chi'_A = \partial_{k+1}$, and $r(\Gamma) = D^{k+1}$. This shows that

$$r(S^{k+1}) = (D^{k+1}, U_k[\partial_{k+1}]) = S^{k+1}.$$

Moreover, the map $\frown_{k+1}$ is the display map, characterising the previous to last variable in S^{n+1}, hence $\frown'_{k+1}$ is defined to be $\mathrm{id}_{D^{k+1}} \frown_{k+1} = \frown_{k+1}$. We have the following square defines a pullback both in $\mathcal{S}_{\mathsf{GSeTT}}$ and in $\mathcal{C}$, showing the equality for $\smile'_{k+1} = \smile_k$

$$\begin{array}{ccc} S^{k+1} & \xrightarrow{\frown_{k+1}} & D^{k+1} \\ \big\downarrow & \lrcorner & \big\downarrow \partial_{k+1} \\ D^{k+1} & \xrightarrow{\partial_{k+1}} & S^k \end{array}$$

Finally, note that we have $\chi_{p_{k+1}} = \smile_{k+1}$, and so by definition and by induction hypothesis, we have

$$\begin{aligned} \mathrm{id}'_{S^{k+1}} &= \langle \frown_{k+1}, p_{k+1}\rangle' \\ &= \langle \frown'_{k+1}, p_{S^k, U_k}[\smile'_{k+1}]\rangle \\ &= \langle \frown_{k+1}, p_{S^k, U_k}[\smile_{k+1}]\rangle \\ &= \langle \frown_{k+1}, p_{k+1}\rangle \\ &= \mathrm{id}_{S^{k+1}} \end{aligned}$$

We now define for a substitution $\gamma : \Delta \to \Gamma$, the map $r(\gamma) : r(\Delta) \to r(\Gamma)$.

– For $\gamma = \langle\rangle$ the empty substitution, $\Gamma = \varnothing$, so $r(\Gamma) = \varnothing$ is terminal, and we define $r(\gamma)$ to be the unique map $r(\Delta) \to r(\Gamma)$.

– For the map $\langle \gamma, t\rangle : \Delta \to (\Gamma, A)$, denote $n = \dim A$. Define $r(\langle\gamma, t\rangle) = \langle r(\gamma), p_{S^n, U_n}[\chi'_t]\rangle$

This assignment is functorial and for every map $m : \Gamma \to S^n$ or $m : \Gamma \to D^n$, we have $r(m) = m'$. Moreover, by definition, r preserves the terminal object and the pullbacks along the maps $\{\partial_n\}$ to pullbacks, hence by Lemma 29, we then have $r \cong \mathrm{Ran}_{D^\bullet} G_{\mathcal{C}}$. □

Lemma 35. *For a globular category with families $\mathcal{C}$, there is a morphism of category with families*

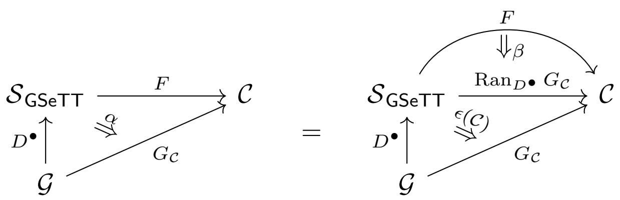
$$(\mathrm{Ran}_{D^\bullet} G_{\mathcal{C}}, \epsilon(\mathcal{C})) : \mathcal{S}_{\mathsf{GSeTT}} \to \mathcal{C}$$

where the transformation $\epsilon(\mathcal{C}) : \mathrm{Ran}_{D^\bullet} G_{\mathcal{C}} \circ D^\bullet \to G_{\mathcal{C}}$ is a natural isomorphism.

Proof. the Kan extension $\mathrm{Ran}_{D^\bullet} G_{\mathcal{C}}$ exists by Lemma 34. Lemma 25 shows that $\mathrm{Ran}_{D^\bullet} G_{\mathcal{C}}$ preserves the limits, and hence by Lemma 3, it defines a unique morphism of categories with families. Let $\epsilon(\mathcal{C}) : \mathrm{Ran}_{D^\bullet} G_{\mathcal{C}} \circ D^\bullet \Rightarrow G_{\mathcal{C}}$ be the universal natural transformation obtained as the Kan extension. Lemma 26 then shows that $\epsilon(\mathcal{C})$ is a natural isomorphism. □

Theorem 36 (local initiality of the syntactic category)**.** *The morphism of globular categories with families* $(\mathrm{Ran}_{D^\bullet} G_{\mathcal{C}}, \epsilon(\mathcal{C})) : \mathcal{S}_{\mathsf{GSeTT}} \to \mathcal{C}$ *is a terminal object in the category* $\underline{\mathbf{gCwF}}(\mathcal{S}_{\mathsf{GSeTT}}, \mathcal{C})$.

Proof. This is exactly the universal property of the right Kan extension. Indeed, consider a morphism of globular category with families $(F, \alpha) : \mathcal{S}_{\mathsf{GSeTT}} \to \mathcal{C}$. The universal property of the right Kan extension lets us construct a natural transformation $\beta : F \Rightarrow \mathrm{Ran}_{D^\bullet} G_{\mathcal{C}}$ such that we have the following equality:



Thus, β is a natural transformation satisfying $\beta : (F, \alpha) \to (\mathrm{Ran}_{D^\bullet} G_{\mathcal{C}}, \epsilon(\mathcal{C}))$. □

Models of the theory GSeTT. Using the machinery of globular categories with families, we can give a new proof that the models of GSeTT are the globular sets, in a way that is more generalisable to more complicated cases. For this, we consider the forgetful functor $\mathcal{U} : \mathbf{gCwF} \to \mathbf{CwF}$, and we consider, for a given category with families $\mathcal{C}$, the category $\mathcal{U}^{-1}_{\mathrm{id}}(\mathcal{C})$ whose objects are the globular categories with families $\mathcal{D}$ such that $\mathcal{U}(\mathcal{D}) = \mathcal{C}$, morphisms are the morphisms of globular categories with families which project onto $\mathrm{id}_{\mathcal{C}}$ by $\mathcal{U}$. Note that since $\mathcal{U}$ is an isofibration, this construction is invariant by equivalence of categories. Intuitively, $\mathcal{U}^{-1}_{\mathrm{id}}(\mathcal{C})$ is the category of choices of disk and sphere objects in $\mathcal{C}$ in a way that is compatible with the structure of category with families.

Proposition 37. *Consider a category with families $\mathcal{C}$, there is an equivalence of categories* $\underline{\mathbf{CwF}}(\mathcal{S}_{\mathsf{GSeTT}}, \mathcal{C}) \simeq \mathcal{U}^{-1}_{\mathrm{id}}(\mathcal{C})$.

Proof. We build a pair of functors

$$\underline{\mathbf{CwF}}(\mathcal{S}_{\mathsf{GSeTT}}, \mathcal{C}) \underset{\mathrm{Ran}_{D^\bullet}}{\overset{D^*}{\rightleftarrows}} \mathcal{U}^{-1}_{\mathrm{id}}(\mathcal{C})$$

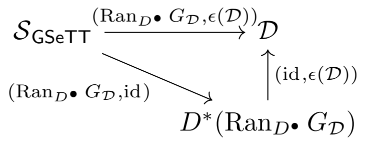
and show that they define an equivalence of categories.

Definition of the functor D^*. Consider a functor of category with families $F : \mathcal{S}_{\mathsf{GSeTT}} \to \mathcal{C}$, we define a globular category with families structure on $\mathcal{C}$ that we call $D^*(F)$ by choosing the disk and sphere objects to be $F(D^n)$ and $F(S^{n-1})$, so (F, id) defines a morphism of globular categories with families $\mathcal{S}_{\mathsf{GSeTT}} \to D^*(F)$. Given two morphisms of categories with families $F, G : \mathcal{S}_{\mathsf{GSeTT}} \to \mathcal{C}$ and a 2-cell $\alpha : F \Rightarrow G$, we define an induced morphism of globular category with families $D^*(\alpha) = (\mathrm{id}_{\mathcal{C}}, \alpha_{D^\bullet}) : D^*(F) \to D^*(G)$.

Definition of the functor $\mathrm{Ran}_{D^\bullet}$. As the notation suggests, this functor is constructed as a right Kan extension. Given an object $\mathcal{D}$ in $\mathcal{U}_{\mathrm{id}}^{-1}(\mathcal{C})$, $\mathrm{Ran}_{D^\bullet} G_{\mathcal{D}} : \mathcal{S}_{\mathsf{GSeTT}} \to \mathcal{C}$ exists and is a morphism of categories with families by Lemma 35. Consider a morphism $(\mathrm{id}_{\mathcal{C}}, \alpha) : \mathcal{D} \to \mathcal{D}'$ in $\mathcal{U}_{\mathrm{id}}^{-1}(\mathcal{C})$, we have $\alpha\epsilon(\mathcal{D}) \in \underline{\mathbf{Cat}}((\mathrm{Ran}_{D^\bullet} G_{\mathcal{D}})D^\bullet, G_{\mathcal{D}'})$. We then define $\mathrm{Ran}_{D^\bullet}(\alpha)$ by universal property of the Kan extension $\mathrm{Ran}_{D^\bullet} G_{\mathcal{D}'}$, to be the unique map such that

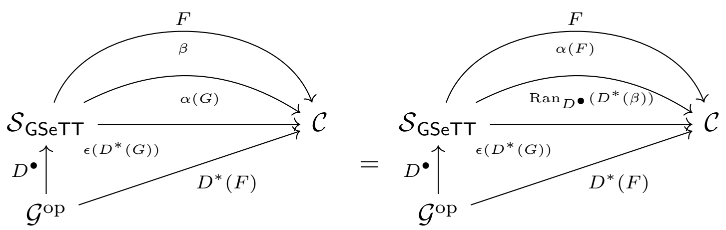
$$\epsilon(\mathcal{D}') \,\mathrm{Ran}_{D^\bullet}(\alpha) = \alpha\epsilon(\mathcal{D}).$$

Equivalence $D^* \circ \mathrm{Ran}_{D^\bullet} \simeq \mathrm{id}$. Consider an object $\mathcal{D}$ in $\mathcal{U}_{\mathrm{id}}^{-1}(\mathcal{C})$, then Lemma 35 shows that we have a morphism of globular category with families $(\mathrm{Ran}_{\mathcal{D}} G_{\mathcal{D}}, \epsilon(\mathcal{D})) : \mathcal{S}_{\mathsf{GSeTT}} \to \mathcal{D}$, with $\epsilon(\mathcal{D}) : (\mathrm{Ran}_{\mathcal{D}} G_{\mathcal{D}}) \circ D^\bullet \Rightarrow G_{\mathcal{D}}$ a natural isomorphism. We then have the following commutative diagram of globular categories with families



Since $\epsilon(\mathcal{D})$ is a natural isomorphism, $(\mathrm{id}, \epsilon(\mathcal{D}))$ is an isomorphism in the category $\mathcal{U}_{\mathrm{id}}^{-1}(\mathcal{C})$, whose inverse is $(\mathrm{id}, \epsilon(\mathcal{D})^{-1})$. This therefore shows that we have a family of isomorphisms $(\mathrm{id}, \epsilon(\mathcal{D})) : D^*(\mathrm{Ran}_{D^\bullet} G_{\mathcal{D}}) \simeq \mathcal{D}$. This family is natural in $\mathcal{D}$, as witnessed by the equality $\epsilon(\mathcal{D}') \,\mathrm{Ran}_{D^\bullet}(\alpha)_{D^\bullet} = \alpha_{D^\bullet}\epsilon(\mathcal{D})$ characterizing $\mathrm{Ran}_{D^\bullet}(\alpha)$.

Equivalence $\mathrm{Ran}_{D^\bullet} \circ D^* \simeq \mathrm{id}$. A morphism of categories with families $F : \mathcal{S}_{\mathsf{GSeTT}} \to \mathcal{C}$ defines a morphism of globular categories with families $(F, \mathrm{id}) : \mathcal{S}_{\mathsf{GSeTT}} \to D^*(F)$. Then by Theorem 36, we have a natural transformation $\alpha(F) : F \Rightarrow \mathrm{Ran}_{D^\bullet} G_{D^*(F)}$ obtained by universality of the Kan extension. Since $\epsilon(D^*(F))$ is an isomorphism, so is $\alpha(F)_{D^\bullet}$. Consider the isomorphism $\gamma : F \cong \mathrm{Ran}_{D^\bullet}(G_{D^*(F)})$ obtained by Lemma 29. Then $(\alpha(F)\gamma^{-1})_{D^\bullet}$ is a natural isomorphism, Lemma 27 then shows that $\alpha(F)\gamma^{-1}$ is a natural isomorphism, hence so is $\alpha(F)$. We now show that the family $\alpha(F)$ is natural in F: Given two morphisms of categories with families $F, G : \mathcal{S}_{\mathsf{GSeTT}} \to \mathcal{C}$, we consider the two following diagram, whose compositions are both equal to $D^*(\beta)$, using the equations that characterize ϵ and $\mathrm{Ran}_{D^\bullet}(D^*(\beta))$.



By universality of the Kan extension, this shows the equation $\alpha(G)\beta = \mathrm{Ran}_{D^\bullet}(D^*(\beta))\alpha(F)$, which is exactly the naturality of α. □

Set-theoretic models of GSeTT. Applying the previous result lets us give a second proof of the characterization of the models of the theory GSeTT.

Proposition 38. *There is an equivalence of categories* $\mathcal{U}_{\mathrm{id}}^{-1}(\mathbf{Set}) \simeq \mathbf{GSet}$.

Proof. We define a functor $\mathcal{M} : \mathbf{GSet} \to \mathcal{U}_{\mathrm{id}}^{-1}(\mathbf{Set})$, as follows: for a globular set $G : \mathcal{G}^{\mathrm{op}} \to \mathbf{Set}$, we consider $\mathcal{M}(G)$ to be the globular category with families on $\mathbf{Set}$ obtained by defining

$$D^n = \mathcal{G}(n) = \mathbf{GSet}(\mathrm{Y}(n), G) \qquad S^n = \mathbf{GSet}(\partial\,\mathrm{Y}(n), G) \qquad U_n = (i_n^{1}\langle x\rangle)_{x\in\mathrm{Y}(n)}$$

where $\partial\,\mathrm{Y}(n)$ is the globular set obtained by removing its top dimensional cell to $\mathrm{Y}(n)$ and $i_n : \partial\,\mathrm{Y}(n) \to \mathrm{Y}(n)$ is the inclusion morphism and $i_n^{-1}\langle x\rangle$ is the preimage of i_n at x. By definition, the globular structure induced by $\mathcal{M}(G)$ on $\mathbf{Set}$ is exactly G. Thus, $\mathcal{M}$ sends a morphism of globular sets $\alpha : G \to G'$ to α seen as a morphism in $\mathcal{U}_{\mathrm{id}}^{-1}(\mathbf{Set})$. We show that $\mathcal{M}$ defines an equivalence of categories, by showing that it is fully faithful and essentially surjective.

Consider an object M in $\mathcal{U}_{\mathrm{id}}^{-1}(\mathbf{Set})$, it is a globular category with families, so it induces a globular structure G_M on $\mathbf{Set}$, such that $\mathcal{M}(G_M) = M$. Hence $\mathcal{M}$ is essentially surjective. Moreover, considering two globular sets G, G', the moprhisms $\mathcal{M}(G) \to \mathcal{M}(G')$ in $\mathcal{U}_{\mathrm{id}}^{-1}(\mathbf{Set})$ are by definition exactly the same as the morphisms of globular sets $G \to G'$. Hence $\mathcal{M}$ is fully faithful. $\mathcal{M}$ is thus an equivalence of categories. □

Theorem 30. *There is an equivalence of categories* $\mathbf{Mod}(\mathcal{S}_{\mathsf{GSeTT}}) \simeq \mathbf{GSet}$.

Proof. By Proposition 37 and Proposition 38 we have the following equivalences of categories

$$\mathbf{Mod}(\mathcal{S}_{\mathsf{GSeTT}}) = \underline{\mathbf{CwF}}(\mathcal{S}_{\mathsf{GSeTT}}, \mathbf{Set}) \simeq \mathcal{U}_{\mathrm{id}}^{-1}(\mathbf{Set}) \simeq \mathbf{GSet}$$ □

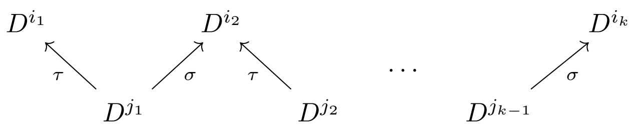
3 The Grothendieck-Maltsiniotis definition of ω-categories

This entire section is a quick presentation of the definition of weak ω-categories given by Maltsionitis [28], based on the definition of a weak ω-groupoid introduced by Grothendieck [19]. We introduce here the notions on which the type theory CaTT relies, as well as the notations. For a more in-depth study of this definition, we refer the reader to the original article by Maltsionitis [28] or the PhD thesis of Ara [2].

3.1 Pasting schemes.

We define a subcategory of globular sets, called the *pasting schemes*. These are meant to represent composable situations in a globular set, and thus serve as the arities of the operations in ω-categories.

Globular sums. Consider a category $\mathcal{C}$. A *globular structure* on a category $\mathcal{C}$ consists in a functor $F : \mathcal{G} \to \mathcal{C}$. When given such a structure, we often denote respectively by D^n, σ_i and τ_i the images under F of n, σ_i and τ_i. When there is no ambiguity, we may write σ and τ, leaving the index implicit, moreover, we also write σ (resp. τ) to denote a composite of maps of the form σ (resp. τ). In the category $\mathcal{C}$, a *globular sum* is a colimit of a diagram of the form



In this diagram, we always assume that the iterated sources σ and the iterated targets τ are not identity, so that we always have the inequality $i_k > j_k < i_{k+1}$. Given a non-canonical diagram (i.e., a diagram which can contain identities), one can contract away all the identity morphisms without changing the colimit of the diagram. It is be useful to encode such a colimit by its *table of dimensions*

$$\begin{pmatrix} i_1 & & i_2 & & \cdots & & i_k \\ & j_1 & & j_2 & \cdots & j_{k-1} & \end{pmatrix}$$

Dually, if a category $\mathcal{C}$ is endowed with a contravariant functor $F : \mathcal{G}^{\mathrm{op}} \to \mathcal{C}$, called a *contravariant globular structure*, we denote respectively by D^n, s and t the images by F of n, σ and τ. We call a *globular product* a limit of the diagram of the following form, which we can also encode with a table of dimensions.

$$\begin{array}{ccccccccc} D^{i_1} & & & & D^{i_2} & & & & D^{i_k} \\ & \overset{t}{\searrow} & & \overset{s}{\swarrow} & & \overset{t}{\searrow} & \cdots & \overset{s}{\swarrow} & \\ & & D^{j_1} & & & & D^{j_2} & D^{j_{k-1}} & \end{array}$$

If $\mathcal{C}$ has a globular structure and $\mathcal{D}$ has a contravariant globular structure, we say that a globular sum in $\mathcal{C}$ and a globular product in $\mathcal{D}$ are *dual* to each other if they share the same table of dimensions.

The category of pasting schemes. The category **GSet** is equipped with a globular structure given by the Yoneda embedding $\mathrm{Y} : \mathcal{G} \to \mathbf{GSet}$, defined by $\mathrm{Y}(n) = \mathbf{GSet}(\_, n)$. In this situation we call all the globular sets that are obtained as globular sums the *pasting schemes*, and we denote by $\mathbf{\Theta}_0$ the full subcategory of **GSet** whose objects are the pasting schemes. Note that since the globular sum diagrams are finite, the pasting schemes are finite globular sets. A few examples and counter-examples of pasting schemes are depicted in Figure 2, using the diagrammatic notation for finite globular sets.

A relation characterizing the pasting schemes. Apart from tables of dimensions, there are several ways of parametrizing pasting schemes using combinatorial structures, such as Batanin trees [3]. In fact these trees also assemble into a category, which can be proved to be equivalent to the category $\mathbf{\Theta}_0$ [2, 8, 21]. Other combinatorial descriptions of pasting schemes are also possible, such as Dyck words, or non-decreasing parking functions, as well as inductive definitions. We refer the reader to [4] for a brief presentation of these views. We focus here on a characterization due to Finster and Mimram [16], using a binary relation.

Definition 39. Consider a globular set G, we introduce the relation $\triangleleft$ on its set of cells to be the transitive closure of the relation generated, for every cell x of G, by

$$s(x) \triangleleft x \triangleleft t(x)$$

This relation can be used to characterize the pasting schemes among all the finite globular sets:

Theorem 40 (Finster, Mimram [16]). *The pasting schemes are exactly the non-empty finite globular sets such that $\triangleleft$ is total and antisymmetric, that is, when we have*

$$x \neq y \iff (x \triangleleft y \text{ or } y \triangleleft x)$$

We also say in this case that the globular set is $\triangleleft$-linear.

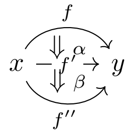
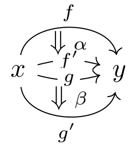
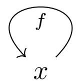
globular set	relation $\lhd$	pasting scheme?
$x \xrightarrow{f} y \xrightarrow{g} z$	$x \lhd f \lhd y \lhd g \lhd z$	yes
$x \xrightarrow{f} y \xleftarrow{g} z$	$x \lhd f$, $f \lhd y$, $g \lhd y$, $z \lhd g$	no
	$x \lhd f \lhd \alpha \lhd f' \lhd \beta \lhd f'' \lhd y \lhd g \lhd z$	yes
	$f \lhd \alpha \lhd f'$, $x \lhd f$, $f' \lhd y$, $x \lhd g$, $g \lhd \beta \lhd g'$, $g' \lhd y$	no
	$x \lhd f$, $f \lhd x$	no

Figure 2: Globular sets and the relation $\lhd$

We refer the reader to the original article [16] for a proof of this theorem, and illustrate the relation $\lhd$ on a few examples, pasting schemes and non pasting schemes, in Figure 2.

The proof of Theorem 40 given in [16] relies on constructing a globular sum diagram associated to a $\lhd$-linear globular set. Since this association reaches all the globular sums and is injective, this also proves the following result.

Lemma 41. *Every pasting scheme can be written as a globular sum in a unique way.*

Source and target of pasting schemes. A pasting scheme X canonically comes equipped with a source and a target, that are two distinguished sub globular sets of X which are also pasting schemes. Since the source and target are isomorphic globular sets, we will define a unique object ∂X along with the two inclusions which identify ∂X as a subobject of X in two different ways.

$$\sigma_X, \tau_X : \partial X \to X$$

When X is given by a table of dimensions as above, write $i = \max(i_1, \ldots, i_n)$ the dimension of X. We then define its boundary ∂X to be given by the table

$$\begin{pmatrix} \overline{i_1} & & \overline{i_2} & & \cdots & & \overline{i_k} \\ & j_1 & & j_2 & \cdots & j_{k-1} & \end{pmatrix} \qquad \text{where } \overline{i_k} = \begin{cases} i_k & \text{if } i_m < i \\ i-1 & \text{if } i_m = i \end{cases}$$

Fig. 3 shows an example of a pasting scheme along with its boundary, represented as pasting schemes as well as as diagrams. Note that the definition of the boundary of a pasting scheme may produce tables that do not strictly comply with the definition of globular sums, as presented before, since it is possible to have the equality

$$\overline{i_m} = j_m = \overline{i_{m+1}} = i - 1$$

	dimension table	diagram reprepsentation
X	$\begin{pmatrix} 2 & & 1 \\ & 0 & \end{pmatrix}$	$\bullet \overset{\longrightarrow}{\underset{\longrightarrow}{\Downarrow}} \bullet \longrightarrow \bullet$
∂X	$\begin{pmatrix} 1 & & 1 \\ & 0 & \end{pmatrix}$	$\bullet \longrightarrow \bullet \longrightarrow \bullet$

Figure 3: A pasting scheme and its boundary

However, when it is the case we will chose the corresponding iterated sources and target to be the identity maps (i.e., the map iterated 0 times). We can then normalize with the following rewriting rule, that does not change the colimit and thus exhibits ∂X as a pasting scheme

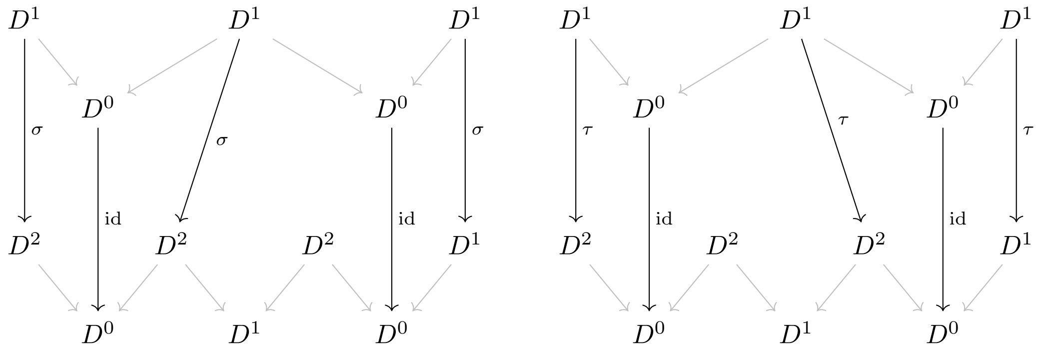
$$\begin{pmatrix} \cdots & i-1 & & i-1 & \cdots \\ \cdots & & i-1 & & \cdots \end{pmatrix} \rightsquigarrow \begin{pmatrix} \cdots & i-1 & \cdots \\ \cdots & & \cdots \end{pmatrix}$$

Now, we can define the two inclusion maps σ_X and τ_X to induced by the families

$$\overline{\sigma_{i_m}} : D_{\overline{i_m}} \longrightarrow D_{i_m} \qquad\qquad \overline{\tau_{i_m}} : D_{\overline{i_m}} \longrightarrow D_{i_m}$$

$$\overline{\sigma_{i_m}} = \begin{cases} \text{id} & \text{if } i_m < i \\ \sigma_{i-1} & \text{if } i_m = i \end{cases} \qquad\qquad \overline{\tau_{i_m}} = \begin{cases} \text{id} & \text{if } i_m < i \\ \tau_{i-1} & \text{if } i_m = i \end{cases}$$

Note that there is a subtlety whenever there are two or more successive cells of maximal dimension n composed in dimension $n-1$. In this case we have to renormalize the dimension of table of ∂X in order to remove multiple successive instances of $n-1$. Defining σ_X and τ_X in this case requires to handle carefully this renormalization, as illustrated in the following example:



exhibiting the diagram on the left respectively as source and target of the one on the right.

By convention, in the case of the pasting scheme D_0, we chose ∂D_0 to be the empty globular set, which is not a pasting scheme.

Characterization of the source and target via the relation $\triangleleft$. The notions of source and target are defined for all the pasting schemes, and are closely related to the relation $\triangleleft$ defined above. Given a pasting scheme X and two parallel cells x and y in X, we denote by $X(x, y)$

the set of cells with source x and target y. Then the relation $\triangleleft$ on the entire pasting scheme X is a preorder, and therefore also induces a preorder on the set $X(x, y)$. We thus define two sub-globular sets of the pasting scheme X of dimension n, denoted $\partial^- X$ and $\partial^+ X$ as follows

$$\begin{array}{rccc} \text{For } k < n-1 & (\partial^- X)_k = X_k & & (\partial^+ X)_k = X_k \\ \text{For all } x, y \in X_{n-2} & \partial^- X(x,y) = \min X(x,y) & & \partial^+ X(x,y) = \max X(x,y) \\ \text{For } k \geq n & (\partial^- X)_k = \emptyset & & (\partial^+ X)_k = \emptyset \end{array}$$

Where the max and the min are respectively the maximal and minimal elements for the preorder $\triangleleft$.

Proposition 42. *The globular set $\partial^- X$ (resp. $\partial^+ X$) is the image of the source morphism $\sigma_X : \partial X \to X$ (resp. target morphism $\tau_X : \partial X \to X$).*

Proof. One can check these images, by definition morphisms σ_X and τ_X, since it removes the variables of maximal dimension n, and keeps the variables of dimension $n-2$. Proving the equality in dimension $n-1$ requires a careful handling of the subtlety that appears in the case of several successive cells dimension n composed in dimension $n-1$. ☐

Maps in the category $\mathbf{\Theta}_0$. The characterization of pasting schemes using the relation $\triangleleft$ allows us show that the maps in the category $\mathbf{\Theta}_0$ are very restricted.

Lemma 43. *Any map $f : X \to Y$ in the category $\mathbf{\Theta}_0$ is injective.*

Proof. A map of globular sets has to preserve the relation $\triangleleft$, since it preserves the source and target. Consider two distinct elements x and y in the pasting scheme X, then Theorem 40 proves that either $x \triangleleft y$ or $y \triangleleft x$, hence we have either $f(x) \triangleleft f(y)$ or $f(y) \triangleleft f(x)$, which by applying Theorem 40 again shows that $f(x) \neq f(y)$. ☐

Lemma 44. *A pasting scheme has no non-trivial automorphism.*

Proof. Consider a pasting scheme X together with an automorphism $f : X \to X$. Suppose that there exists an element $x \in X$ such that $x \neq f(x)$. Then by Theorem 40, we have either $x \triangleleft f(x)$ or $f(x) \triangleleft x$: we suppose that we are in the first case, the second one being similar. Since f preserves the relation $\triangleleft$, this provides us with an infinite chain

$$x \triangleleft f(x) \triangleleft f(f(x)) \triangleleft f(f(f(x))) \triangleleft \cdots$$

which is impossible since X has only finitely many elements. The automorphism f is thus necessarily the identity. ☐

3.2 Globular extensions and globular theories.

In order to define weak ω-categories, we rely on the notion of a *coherator* which is a category whose objects are the arities of the operations expected in ω-categories and the morphisms encode the algebraic operations that they should have. It can be thought of as an analogue of Lawvere theories in the dependently sorted case. Recall that in Lawvere theories, one requires the set of objects to be freely generated by the finite products of a single object, the coherator satisfies an analogous condition for the dependently sorted case, which is captured by the notion of *globular theory*

Globular extensions. A category $\mathcal{C}$ with a globular structure F is called a *globular extension* when all the globular sums exist in $\mathcal{C}$. Given two globular extensions $F : \mathcal{G} \to \mathcal{C}$ and $G : \mathcal{G} \to \mathcal{D}$, a morphism of globular extensions is a functor $H : \mathcal{C} \to \mathcal{D}$ such that $H \circ F = G$, which preserves globular sums. Dually, a category equipped with a contravariant globular structure and which has all globular products is called a *contravariant globular extension*, the notion of morphism contravariant globular extensions is dual to that of globular extensions.

The universal globular completion. There is a canonical functor $\mathcal{G} \to \mathbf{\Theta}_0$, sending an object n to the disk D^n, which is an object of $\mathbf{\Theta}_0$ as being obtained as the globular sum corresponding to the table of dimensions (n). This exhibits $\mathbf{\Theta}_0$ as the completion of $\mathcal{G}$ under globular sums: we sometimes say that it is the universal *globular completion* of $\mathcal{G}$.

Proposition 45. *The category $\mathbf{\Theta}_0$ is the universal globular extension: for any globular extension $F : \mathcal{G} \to \mathcal{C}$, there is an essentially unique morphism of globular extensions $\mathbf{\Theta}_0 \to \mathcal{C}$.*

Proof. Consider a globular extension $F : \mathcal{G} \to \mathcal{C}$ together with a morphism of globular extensions $f : \mathbf{\Theta}_0 \to \mathcal{C}$. Then an object X in $\mathbf{\Theta}_0$ decomposes as a globular sum induced by a table of dimensions

$$\begin{pmatrix} i_1 & & i_2 & & \cdots & & i_n \\ & j_1 & & j_2 & \cdots & j_{n-1} & \end{pmatrix}$$

By definition, $f(X)$ is the globular sum of the same diagram in $\mathcal{C}$, hence f is determined up to natural isomorphism. Conversely, we can define $f(D^n) = Fn$ and extends this definition to all the pasting schemes while preserving the globular sums since by Lemma 41 every pasting scheme is written as a globular sum in a unique way. □

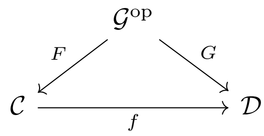
Dually $\mathbf{\Theta}_0^{\mathrm{op}}$ is a *globular cocompletion*: for every contravariant globular extension $\mathcal{G}^{\mathrm{op}} \to \mathcal{C}$, there is an essentially unique morphism $\mathbf{\Theta}_0^{\mathrm{op}} \to \mathcal{C}$.

The category of globular extensions. Globular extensions are characterized by the fact that they have globular sums, and the globular sums factorize through the category $\mathbf{\Theta}_0$. We can thus use the universality of the category $\mathbf{\Theta}_0$ in order to characterize the category of globular extensions as follows.

Lemma 46. *The universal property of the category $\mathbf{\Theta}_0$ induces an equivalence of categories between the category of globular extensions and the full subcategory of the coslice category $\mathbf{\Theta}_0 \backslash \mathbf{Cat}$ whose objects are the functors preserving the globular sums.*

Dually, there is an equivalence of categories between the category of contravariant globular extensions and the full subcategory of the coslice category $\mathbf{\Theta}_0^{\mathrm{op}} \backslash \mathbf{Cat}$, whose objects are the functors preserving the globular products.

Proof. By the universal property of $\mathbf{\Theta}_0$, a globular extension $F : \mathcal{G} \to \mathcal{C}$ induces a morphism of globular extensions $\mathbf{\Theta}_0 \to \mathcal{C}$, which is an object of the coslice $\mathbf{\Theta}_0 \backslash \mathbf{Cat}$ preserving globular sums, and this assignment is functorial. Conversely, consider a functor $F : \mathbf{\Theta}_0 \to \mathcal{C}$ preserving globular sums. Then, by precomposition by the canonical functor $\mathcal{G} \to \mathbf{\Theta}_0$, it induces a globular structure on $\mathcal{C}$, and F is a morphism from of globular structures from $\mathbf{\Theta}_0$ to $\mathcal{C}$. Any globular sum diagram for this structure in $\mathcal{C}$ factorizes through F. Since $\mathbf{\Theta}_0$ has all the globular sums, this diagram has a globular sum in $\mathbf{\Theta}_0$, and since F preserves those, this diagram has a globular sum in $\mathcal{C}$, hence $\mathcal{C}$ has all the globular sums and is a globular extension. Moreover, consider a commutative triangle of the form



with F and G preserving the globular sums. Then, by the previous statement, $\mathcal{C}$ has all the globular sums, which all factor through F. Since $f \circ F = G$ preserves the globular sums, it follows that necessarily f preserves the globular sums and thus defines a morphism of globular extensions. This proves the equivalence of categories. □

Globular theories. Given a globular extension $\mathcal{G} \to \mathcal{C}$, by universality of the globular completion, there exists a unique morphism of globular extensions $F : \mathbf{\Theta}_0 \to \mathcal{C}$. The functor $\mathcal{G} \to \mathcal{C}$ is called a *globular theory* if the induced functor F is faithful and bijective on the isomorphism classes of objects. Whenever it is the case, we can up to equivalence identify $\mathbf{\Theta}_0$ as a subcategory of $\mathcal{C}$. A *morphism of globular theories* is just a morphism of the underlying globular extensions. A morphism f of a globular theory $\mathcal{C}$ is said to be *globular* if it is in $\mathbf{\Theta}_0$. Dually, a contravariant globular extension $\mathcal{G}^{\mathrm{op}} \to \mathcal{C}$ is called a *contravariant globular theory* if $\mathcal{C}^{\mathrm{op}}$ is a globular theory.

3.3 Weak ω-categories.

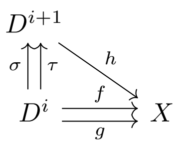
We have introduced the notion of globular theory, which plays the role of Lawvere theories, in the case where we have dependent sorts with the dependency described by the category of globes and the arities are given by pasting schemes. There are various such theories, and we now introduce the one we will be interested in for weak ω-categories. As it is often the case for higher structures, there is not a single theory of weak ω-categories, but several of them, called *coherators*. We introduce here one such coherator.

Admissible pair of arrows. Let $\mathcal{G} \to \mathcal{C}$ be a globular extension, two arrows $f, g : D^i \to X$ in $\mathcal{C}$ are said to be *parallel* when

$$f \circ \sigma = g \circ \sigma \qquad\qquad f \circ \tau = g \circ \tau$$

If $\mathcal{C}$ is a globular theory, then an arrow f of $\mathcal{C}$ is said to be *algebraic*, when for every decomposition $f = g \circ f'$, with g globular, then g is an identity. A pair of parallel arrows $f, g : D^i \to X$ is called an *admissible pair* if either both f and g are algebraic, or there exists a decomposition $f = \sigma_X \circ f'$ and $g = \tau_X \circ g'$, with f' and g' algebraic.

Definition 47. Given an admissible pair of maps $f, g : D^i \to X$, a *lift* is a map $h : D^{i+1} \to X$ such that $h \circ \sigma = f$ and $h \circ \tau = g$

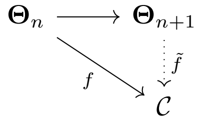


We also say that an arrow is algebraic or that a pair is admissible in a contravariant globular theory $\mathcal{C}$, to mean that it is the same in $\mathcal{C}^{\mathrm{op}}$, and a lift for an admissible is a lift in the opposite category in $\mathcal{C}^{\mathrm{op}}$.

Cat-coherator. We introduce here the Batanin-Leinster cat-coherator, which is the one we will be using for our type theory. For a more general definition of cat-coherators, as well as other examples, see [28]. For the rest of this paper, we will simply say cat-coherator to refer to the Batanin-Leinster cat-coherator. The cat-coherator $\mathbf{\Theta}_\infty$ is defined to be the colimit

$$\mathbf{\Theta}_\infty = \mathrm{colim}(\mathbf{\Theta}_0 \to \mathbf{\Theta}_1 \to \mathbf{\Theta}_2 \to \cdots \to \mathbf{\Theta}_n \to \cdots)$$

where the categories $\mathbf{\Theta}_n$ are defined by induction on n. Given $n \in \mathbb{N}$, define E_n to be the set of all pairs of admissible arrows of $\mathbf{\Theta}_n$ that are not in $E_{n'}$ for any $n' < n$. Then we can define $\mathbf{\Theta}_{n+1}$ to be the universal globular extension of $\mathbf{\Theta}_n$ obtained by formally adding a lift for each pair in E_n. In other words $\mathbf{\Theta}_{n+1}$ is the category such that, for each globular extension $f : \mathbf{\Theta}_n \to \mathcal{C}$ equipped with the choice of a lift in $\mathcal{C}$ for all the images by f of the pairs of arrows in E_n, $\tilde{f}$ which makes the following triangle commute



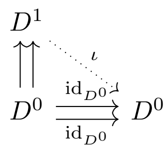
and for every pair of arrows in E_n, sending the chosen lift of the pair in $\mathbf{\Theta}_{n+1}$ to the chosen lift of its image in $\mathcal{C}$.

Weak ω-categories. We define a *weak ω-category* to be functor $F : \mathbf{\Theta}_\infty^{\mathrm{op}} \to \mathbf{Set}$ which sends globular sums in $\mathbf{\Theta}_\infty^{\mathrm{op}}$ to the globular product on the opposite diagram, for the globular structure on $\mathbf{Set}$ induced by F. Given $n \in \mathbb{N}$, the elements of the set FD^n are called the *n-cells* of the weak ω-category. The category ω-**Cat** of weak ω-categories is the full subcategory of $\widehat{\mathbf{\Theta}_\infty}$ whose objects are the presheaves that are weak ω-categories.

3.4 Identity and composition.

In order to illustrate the above definition, we show that a weak ω-category $F : \mathbf{\Theta}_\infty^{\mathrm{op}} \to \mathbf{Set}$ has identities on 0-cells and composites of composable 1-cells. We refer the reader to [28, 2] for more examples of the same nature. More advanced examples of operations are presented in Section 4.3, where they are described in a type theoretic style.

Identities on 0-cells. The pair of maps $(\mathrm{id}_{D^0}, \mathrm{id}_{D^0})$ is admissible. Hence, there exists a lift

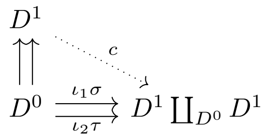


Given a 0-cell $x \in FD^0$, its *identity* 1-*cell* $i(x) \in FD^1$ is $i(x) = F\iota(x)$. Moreover, by definition, we have $s(i(x)) = t(i(x)) = x$ as expected for the identity 1-cell on x.

Composition of 1-cells. Consider the globular sum given as $D^1 \coprod_{D^0} D^1$. There are two canonical maps $\iota_1, \iota_2 : D^1 \to D^1 \coprod_{D^0} D^1$, and we consider the admissible pair

$$(\iota_1\sigma, \iota_2\tau) : D^0 \to D^1 \coprod_{D^0} D^1$$

which provides the lift



A pair of composable 1-cell is the same as an element $(f, g) \in F(D^1 \coprod_{D^0} D^1)$, and the element $F(c)(f, g)$ in FD^1 defines the composition $f \cdot g$. By definition, $s(f \cdot g) = s(f)$ and $t(f \cdot g) = t(g)$, as expected for the composition.

4 Type theory for weak ω-categories

Our aim is now to extend the type theory GSeTT presented in Section 2, by adding term constructors corresponding to the algebraic structure that one need to add to globular sets in order to obtain weak ω-categories. We call the resulting theory CaTT and motivate its introduction by following the ideas of the Grothendieck-Maltsiniotis definition of weak ω-categories recalled in Section 3.

4.1 Ps-contexts.

We have proved in Theorem 22 that the syntactic category of the theory GSeTT is equivalent to the opposite of the category of finite globular sets. The Grothendieck-Maltsiniotis definition of weak ω-categories strongly relies on a particular class of such finite globular sets, namely the pasting schemes, obtained as globular sums. In order to translate this definition in a type theory, it is useful to transfer this notion of pasting scheme in a type theoretic framework.

Recognition algorithm. We introduce a new kind of judgment to the theory, that we denote

$$\Gamma \vdash_{\mathsf{ps}}$$

A context Γ such that the judgment $\Gamma \vdash_{\mathsf{ps}}$ is derivable is called a *ps-context*. It intuitively corresponds to a situation where the context Γ is a pasting scheme, as formally shown in Theorem 53. In order to define this judgment by induction, we also introduce an auxiliary judgment

$$\Gamma \vdash_{\mathsf{ps}} x : A$$

where the variable x is called the *dangling variable*. We require these judgments to be subject to the following inference rules:

$$\frac{}{(x : \star) \vdash_{\mathsf{ps}} x : \star}\,(\text{PSS}) \qquad \frac{\Gamma \vdash_{\mathsf{ps}} f : x \underset{A}{\longrightarrow} y}{\Gamma \vdash_{\mathsf{ps}} y : A}\,(\text{PSD})$$

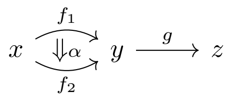
$$\frac{\Gamma \vdash_{\mathsf{ps}} x : A}{\Gamma, y : A, f : x \underset{A}{\longrightarrow} y \vdash_{\mathsf{ps}} f : x \underset{A}{\longrightarrow} y}\,(\text{PSE}) \quad \text{when } y, f \notin \mathrm{Var}(\Gamma) \qquad \frac{\Gamma \vdash_{\mathsf{ps}} x : \star}{\Gamma \vdash_{\mathsf{ps}}}\,(\text{PS})$$

Note that every derivation of the judgment $\Gamma \vdash_{\mathsf{ps}}$ starts with the rule (PSS) and ends with the rule (PS), with an equal number of applications of the rules (PSE) and (PSD) in between.

An example of a derivation. In order to understand how a derivation of this judgment works, we have illustrated in Figure 4 the derivation of $\Gamma \vdash_{\mathsf{ps}}$ where Γ is the context

$$\Gamma = (x : \star, y : \star, f_1 : x \to y, f_2 : x \to y, \alpha : f_1 \to f_2, z : \star, g : y \to z)$$

which corresponds to the globular set



We follow the step-by-step derivation of the judgment $\Gamma \vdash_{\mathsf{ps}}$, and give a graphical representation of the globular corresponding globular set being constructed, where we encircle the dangling variable on the judgment.

The rules that we have given for recognizing ps-contexts do in particular recognize usual contexts in the theory GSeTT, in other words, the following holds [16, 4]:

Proposition 48. *The following rules are admissible*

$$\frac{\Gamma \vdash_{\mathsf{ps}} x : A}{\Gamma \vdash} \qquad \frac{\Gamma \vdash_{\mathsf{ps}} x : A}{\Gamma \vdash A} \qquad \frac{\Gamma \vdash_{\mathsf{ps}} x : A}{\Gamma \vdash x : A} \qquad \frac{\Gamma \vdash_{\mathsf{ps}}}{\Gamma \vdash}$$

Proof. The admissibility of the first three of these rules can be shown by mutual induction, and the admissibility of the last one is then a consequence of the former, see [4] for a detailed proof. □

The category of ps-contexts. Note that the notion of ps-context is not invariant under equivalence: a context can be isomorphic to a ps-context without being a ps-context itself. For the sake of simplicity, we consider the subcategory $\mathcal{S}_{\mathsf{ps}}$ of $\mathcal{S}_{\mathsf{GSeTT}}$, whose objects are exactly the ps-contexts, not including the contexts that are invariant under equivalence.

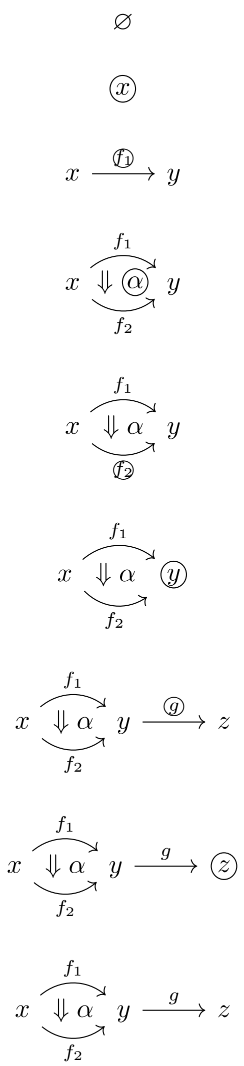
The correspondence between ps-contexts and pasting schemes. We now show that ps-contexts correspond to pasting schemes. In order to do so, we use the following useful lemma, which involves the functor $V : \mathcal{S}_{\mathsf{GSeTT}} \to \mathbf{FinGSet}^{\mathrm{op}}$, introduced in Section 2 and defined by $V\Gamma = \mathcal{S}_{\mathsf{GSeTT}}(\Gamma, D^\bullet)$, and the relation $\triangleleft$, recalled in Section 3.1.

Proposition 49. *For every ps-context $\Gamma \vdash_{\mathsf{ps}}$, the globular set $V\Gamma$ is $\triangleleft$-linear.*

Proof. The proof requires the introduction of subtle invariants beforehand. One can check by induction on the derivation tree, that whenever a judgment of the form $\Gamma \vdash_{\mathsf{ps}} x : A$ is derivable, there is no variable f whose type in Γ is $y \to z$ where y is an iterated target of x (i.e., there exists a sequence of terms $x = x_0, x_1, x_2, \ldots, x_n = y$ with $\Gamma \vdash x_i : a \to x_{i+1}$ for a variable a). As a consequence, in this situation, every relation of the form $x \triangleleft y$ with $x, y \in V\Gamma$ is such that y is an iterated target of x. Using this fact, it can be shown by induction that the set $V\Gamma$ is $\triangleleft$-linear.

Now, suppose fixed a ps-context $\Gamma \vdash_{\mathsf{ps}}$. We first show that if we have a, b in Γ such that $a \neq b$ then necessarily $a \triangleleft b$ or $b \triangleleft a$, by induction on the form of the pasting scheme.

- For the pasting scheme $(x : \star)$, the statement is vacuously true since there are no two disjoint variables.
- For a ps-context of the form $\Gamma = (\Gamma', y : A, f : x \to y)$, we distinguish different cases.



$$\dfrac{}{x : \star \vdash_{\mathsf{ps}} x : \star}\ \text{(PSS)}$$

$$\dfrac{x : \star \vdash_{\mathsf{ps}} x : \star}{(x : \star, y : \star, f_1 : x \to y) \vdash_{\mathsf{ps}} f_1 : x \to y}\ \text{(PSE)}$$

$$\dfrac{(x : \star, y : \star, f_1 : x \to y) \vdash_{\mathsf{ps}} f_1 : x \to y}{(x : \star, y : \star, f_1 : x \to y, f_2 : x \to y, \alpha : f_1 \to f_2) \vdash_{\mathsf{ps}} \alpha : f_1 \to f_2}\ \text{(PSE)}$$

$$\dfrac{(x : \star, y : \star, f_1 : x \to y, f_2 : x \to y, \alpha : f_1 \to f_2) \vdash_{\mathsf{ps}} \alpha : f_1 \to f_2}{(x : \star, y : \star, f_1 : x \to y, f_2 : x \to y, \alpha : f_1 \to f_2) \vdash_{\mathsf{ps}} f_2 : x \to y}\ \text{(PSD)}$$

$$\dfrac{(x : \star, y : \star, f_1 : x \to y, f_2 : x \to y, \alpha : f_1 \to f_2) \vdash_{\mathsf{ps}} f_2 : x \to y}{(x : \star, y : \star, f_1 : x \to y, f_2 : x \to y, \alpha : f_1 \to f_2) \vdash_{\mathsf{ps}} y : \star}\ \text{(PSD)}$$

$$\dfrac{(x : \star, y : \star, f_1 : x \to y, f_2 : x \to y, \alpha : f_1 \to f_2) \vdash_{\mathsf{ps}} y : \star}{(x : \star, y : \star, f_1 : x \to y, f_2 : x \to y, \alpha : f_1 \to f_2\ z : \star, g : y \to z) \vdash_{\mathsf{ps}} g : y \to z}\ \text{(PSE)}$$

$$\dfrac{(x : \star, y : \star, f_1 : x \to y, f_2 : x \to y, \alpha : f_1 \to f_2\ z : \star, g : y \to z) \vdash_{\mathsf{ps}} g : y \to z}{(x : \star, y : \star, f_1 : x \to y, f_2 : x \to y, \alpha : f_1 \to f_2\ z : \star, g : y \to z) \vdash_{\mathsf{ps}} z : \star}\ \text{(PSD)}$$

$$\dfrac{(x : \star, y : \star, f_1 : x \to y, f_2 : x \to y, \alpha : f_1 \to f_2\ z : \star, g : y \to z) \vdash_{\mathsf{ps}} z : \star}{\Gamma \vdash_{\mathsf{ps}}}\ \text{(PS)}$$

Figure 4: Derivation of the judgment $\Gamma \vdash_{\mathsf{ps}}$

– If both a and b are in Γ' then by induction, either $a \lhd b$ or $b \lhd a$ in Γ', and thus the same holds in Γ.

– If a is in Γ', but not b, either $b = y$ or $b = f$, then by induction, either $a = x$, or $x \lhd x$ or $x \lhd a$. In the first two cases, since we have $x \lhd b$, the transitivity shows that $a \lhd b$. We can thus assume that $x \lhd a$. In this case, by the fact that we have proved, a is an iterated target of x. Since y is parallel to x and y is a target of f, in either case, a is also an iterated target of b, which shows that $b \lhd a$.

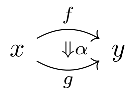
– If b is in Γ' but not a, the situation is symmetric to the previous case.

– If neither a nor b is in Γ', then necessarily they are f and y, and we have $f \lhd y$.

Conversely, we show that for every ps-context Γ, we never have $x \lhd x$. In order to prove this, we first note that whenever we have a relation of the form $a \lhd b$ in the ps-context $(\Gamma, y : A, f : x \to y)$ with a and b variables of Γ, we also have the same relation in Γ. Indeed, considering the chain of generating relations $a \lhd a_1 \lhd \cdots \lhd b$, it suffices to prove that there is a chain completely included in Γ. If it is not the case, that means that there are occurrences of the form $s \lhd y \lhd t$ or $x \lhd f \lhd y \lhd t$ with s the source of y and t its target (these are the only possibilities because of the fact that y can never be a source). In the first case, one can replace the occurrence with $s \lhd x \lhd t$ and in the second case, one can replace it with $s \lhd t$, in order to obtain a chain proving $a \lhd b$ in Γ. Proving that there is no variable x such that $x \lhd x$ in ps-context Γ is then a straightforward induction over the derivation tree of the judgment $\Gamma \vdash_{\mathsf{ps}}$. □

We now prove the converse, that any $\lhd$-linear globular set corresponds to a ps-context. In order to do this, we introduce the notion of *locally maximal element* of a pasting scheme as an element x such that there is no variable y such that $s(x) \lhd y \lhd x$ or $x \lhd y \lhd t(x)$. Alternatively, the locally maximal elements are the elements corresponding to the peaks in the decomposition as a globular sum.

Example 50. In the globular set



we have that α is maximal but f is not maximal because $f \lhd \alpha \lhd y = t(f)$.

In order to prove the result, we use the following lemma:

Lemma 51. *Consider a globular set G with two elements x, y such that $x \lhd y$ and $\dim x > \dim y$. Then $t(x) \lhd y$ or $t(x) = y$.*

Proof. Suppose that $x \lhd y$. By definition of the relation $\lhd$ there exists a sequence of elements $x = x_0 \lhd x_1 \lhd \ldots \lhd x_n = y$ such that, for every index i, $x_i = s(x_{i+1})$ or $x_{i+1} = t(x_i)$. We reason by induction on the length n of this sequence.

– If $n = 1$, then necessarily, either $y = t(x)$ or $x = s(y)$, and the condition on the dimensions implies that we have $y = t(x)$.

– Suppose that the result holds for all chains of length at most $n - 1$. Note that either $x_1 = t(x)$, or $x = s(x_1)$. The first case gives the result immediately. In the second case, the induction shows that we have a relation $t(x_1) \lhd y$, given by a chain of length less than $n - 1$, so applying again the induction hypothesis proves that $t(t(x_1)) \lhd y$. And we conclude by using the fact that $t(t(x_1)) = t(s(x_1)) = t(x)$. □

Proposition 52. *For any $\triangleleft$-linear non-empty finite globular set G, there exists a unique (up-to α-equivalence) ps-context Γ such that $V\Gamma = G$.*

Proof. We construct the context Γ inductively and then prove that it satisfies $\Gamma \vdash_{\mathsf{ps}} x : A$, where $V(x)$ is the greatest (for the relation $\triangleleft$) locally maximal element of G.

- If the globular set G has a unique element, then this element is necessarily of dimension 0 and we then associate the context $\Gamma = (x : \star)$, where the derivation of $\Gamma \vdash_{\mathsf{ps}} x : \star$ is given by the rule (PSS).
- If G has more than one element, write a for the greatest locally maximal element of G. We can consider the globular set G' obtained by removing a and $t(a)$ from G: indeed, by definition of locally maximal element, there is no element whose source or target is a, hence a can safely be removed. Moreover, any element x distinct from a and whose target is $t(a)$ satisfies $x \triangleleft t(a)$, thus either it is a or it compares to a by linearity. Since a is locally maximal, we then cannot have $a \triangleleft x$, so we necessarily have $x \triangleleft a$ and thus $x \triangleleft s(a)$. Lemma 51 then applies to show that $t(x) \triangleleft s(a)$. Since we have $t(x) = t(a)$ and also $s(a) \triangleleft t(a)$, this implies in particular that $t(a) \triangleleft t(a)$, which contradicts the linearity of G. So any element whose target is $t(a)$ is necessarily a, and since a is the greatest locally maximal element, there cannot be any element whose source is $t(a)$. Hence, after removing a, one can also remove $t(a)$ safely. In fact, this analysis shows that the resulting globular set G' is still a non-empty finite $\triangleleft$-linear set, so by induction, one can construct a context Γ' such that $V(\Gamma') = G'$ and $\Gamma' \vdash_{\mathsf{ps}} x : A$, where x is the greatest locally maximal element in G'.
  - Either the greatest locally maximal element of G' is $s(a)$. In this case, we define the context $\Gamma = (\Gamma', t(a) : A, a : s(a) \to t(a))$, we then have $V(\Gamma) = G$ by definition, and the rule (PSE) gives a derivation of $\Gamma \vdash_{\mathsf{ps}} a : s(a) \to t(a)$.
  - Or the greatest locally maximal element x of G' is such that $x \triangleleft s(a)$. In this case, since x is locally maximal, $s(a)$ is necessarily an iterated target of x, and we write n for the number of iterations. Then applying the rule (PSD) n times gives a derivation of $\Gamma \vdash s(a) : B$. We define $\Gamma = (\Gamma', t(a) : B, a : s(a) \to t(a))$, in such a way that we have $V(\Gamma) = G$ and $\Gamma \vdash_{\mathsf{ps}} a : s(a) \to t(a)$ obtained from the previous derivation by applying the rule (PSE).

  Since a is the greatest locally maximal variable, these are the two only cases, and in both cases, we constructed a suitable preimage. ☐

The two previous propositions, together with Theorem 22 finally allows us to conclude:

Theorem 53. *There is an isomorphism of categories*

$$\mathcal{S}_{\mathsf{ps}} \cong \mathbf{\Theta}_0^{\mathrm{op}}$$

Proof. By Proposition 49, the functor V induces a functor $\mathcal{S}_{\mathsf{ps}} \to \mathbf{\Theta}_0^{\mathrm{op}}$. Moreover, we have shown by Theorem 22 that V is fully faithful, so the restriction is also fully faithful, and Proposition 52 shows that this restriction defines a bijection on objects (note that the objects of $\mathcal{S}_{\mathsf{ps}}$ are assumed to be quotiented by α-equivalences), hence it is an isomorphism categories. ☐

We illustrate in Figure 5 the correspondence between the ps-contexts and the $\triangleleft$-linear contexts with our previous example of derivation, showing how we construct $\triangleleft$ to be a preorder.

Note that the notion of ps-context is not invariant under isomorphism in the category $\mathcal{S}_{\mathsf{GSeTT}}$. As an example, one can consider the two following Γ and Γ' which are isomorphic, as they only differ from the order of the variables, but the context Γ is a ps-context whereas the context Γ' is not:

$$\Gamma = (x : \star, y : \star, f : x \to y, z : \star, g : y \to z)$$
$$\Gamma' = (x : \star, y : \star, z : \star, f : x \to y, g : y \to z)$$

Thus one can understand the notion of ps-context as a recognition algorithm for a particular representative of a context in each equivalence classes of contexts corresponding to a pasting scheme.

Uniqueness of derivation. The following results rely on a more detailed analysis of the allowed derivation trees and show that these rules enjoy good computational properties.

Proposition 54. *Given a context Γ, the derivability of the judgment $\Gamma \vdash_{\mathsf{ps}}$ is decidable, and when this judgment is derivable, it has a unique derivation.*

Proof. The proof is more subtle than it may appear at first glance, as one cannot just use a straightforward induction to prove this result. Indeed, any derivation of the judgment $\Gamma \vdash$ is obtained from a derivation of the judgment $\Gamma \vdash_{\mathsf{ps}} x : \star$, but there is no guarantee a priori that this the variable $x : \star$ is the same for all possible derivations. However, in the proof of Proposition 49 we have characterized the variable x in a judgment of the form $\Gamma \vdash_{\mathsf{ps}} x : A$ as an iterated target of the greatest locally maximal variable. This proves that whenever we have two derivations of the form $\Gamma \vdash_{\mathsf{ps}} x : A$ and $\Gamma \vdash_{\mathsf{ps}} y : B$ with $\dim A = \dim B$, then necessarily $x = y$. Moreover, Proposition 49 also characterizes the judgments $\Gamma \vdash_{\mathsf{ps}} x : A$ obtained from the rule (PSE) as those when x is locally maximal in Γ, all the other ones are obtained from the rule (PSD). These two facts together combine allow for a straightforward proof by induction on the structure of the derivation trees, that each judgment of this form has a single derivation. □

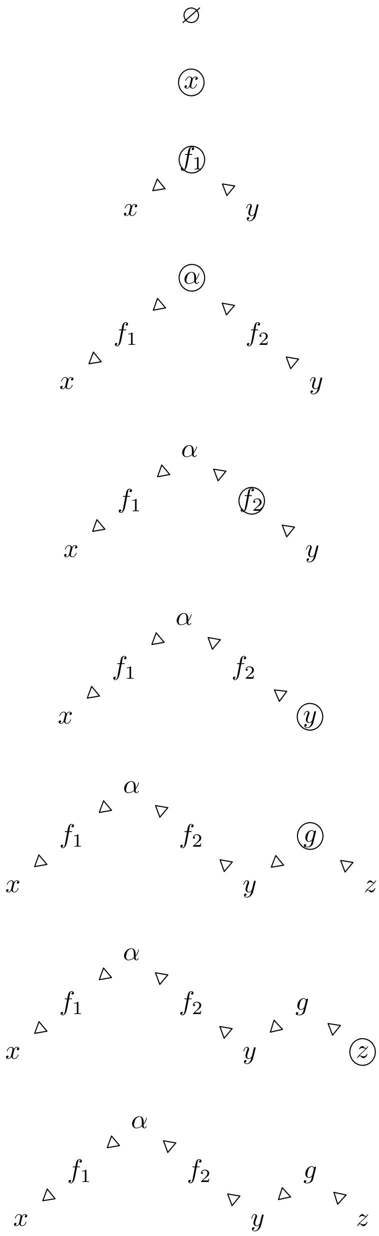
Source and target of a ps-context. The ps-contexts come equipped with a notion of source and target, which mirror the corresponding operations on pasting scheme, already presented in Section 3. Following the proofs that we have given, one could already figure out how to define these: indeed, it suffices to use the correspondence of ps-contexts and pasting schemes in order to compute the source, or the target of a pasting scheme, and then use the correspondence in the other direction to get back a ps-context. We give here a direct computation by induction on the syntax of a ps-context of this process. We define for all $i \in \mathbb{N}$ the *i-source* of a ps-context Γ induction on the length of Γ, by setting $\partial_i^-(x : \star) = (x : \star)$ and

$$\partial_i^-(\Gamma, y : A, f : x \to y) = \begin{cases} \partial_i^- \Gamma & \text{if } \dim A \ \geq i - 1 \\ \partial_i^- \Gamma, y : A, f : x \to y & \text{otherwise} \end{cases}$$

and similarly the *i-target* of Γ is defined by $\partial_i^+(x : \star) = (x : \star)$, and

$$\partial_i^+(\Gamma, y : A, f : x \to y) = \begin{cases} \partial_i^+ \Gamma & \text{if } \dim A \ \geq i \\ \mathrm{drop}(\partial_i^+ \Gamma), y : A & \text{if } \dim A \ = i - 1 \\ \partial_i^+ \Gamma, y : A, f : x \to y & \text{otherwise} \end{cases}$$

where $\mathrm{drop}(\Gamma)$ is the context Γ with its last variable removed. One can check by induction on the derivation of the judgment $\Gamma \vdash_{\mathsf{ps}}$ that whenever Γ is a ps-context of non-zero dimension, both $\partial_i^- \Gamma$ and $\partial_i^+ \Gamma$ are also ps-contexts. It is straightforward in the case of the i-source, and for the i-target, it relies on the fact that whenever the drop operator is used, immediately afterwards a variable of the same type that the one that was removed is added. We denote $\partial^-(\Gamma) = \partial^-_{\dim \Gamma - 1} \Gamma$ and $\partial^+(\Gamma) = \partial^+_{\dim \Gamma - 1} \Gamma$ and call these the *source* and *target* of Γ.



$$\frac{}{x:\star \vdash_{\mathsf{ps}} x:\star}\ \text{(PSS)}$$

$$\frac{x:\star \vdash_{\mathsf{ps}} x:\star}{(x:\star, y:\star, f_1:x\to y) \vdash_{\mathsf{ps}} f_1 : x \to y}\ \text{(PSE)}$$

$$\frac{(x:\star, y:\star, f_1:x\to y) \vdash_{\mathsf{ps}} f_1 : x \to y}{(x:\star, y:\star, f_1:x\to y, f_2 : x \to y, \alpha : f_1 \to f_2) \vdash_{\mathsf{ps}} \alpha : f_1 \to f_2}\ \text{(PSE)}$$

$$\frac{}{(x:\star, y:\star, f_1:x\to y, f_2 : x \to y, \alpha : f_1 \to f_2) \vdash_{\mathsf{ps}} f_2 : x \to y}\ \text{(PSD)}$$

$$\frac{}{(x:\star, y:\star, f_1:x\to y, f_2 : x \to y, \alpha : f_1 \to f_2) \vdash_{\mathsf{ps}} y : \star}\ \text{(PSD)}$$

$$\frac{}{(x:\star, y:\star, f_1:x\to y, f_2 : x \to y, \alpha : f_1 \to f_2, z:\star, g : y \to z) \vdash_{\mathsf{ps}} g : y \to z}\ \text{(PSE)}$$

$$\frac{}{(x:\star, y:\star, f_1:x\to y, f_2 : x \to y, \alpha : f_1 \to f_2, z:\star, g : y \to z) \vdash_{\mathsf{ps}} z : \star}\ \text{(PSD)}$$

$$\frac{}{\Gamma \vdash_{\mathsf{ps}}}\ \text{(PS)}$$

Figure 5: $\triangleleft$-linearity of a ps-context

Lemma 55. *For every ps-context Γ, the globular set $V(\partial^-(\Gamma))$ is exactly the sub-globular set $\partial^-(V\Gamma)$ of $V\Gamma$, and similarly $V(\partial^+(\Gamma))$ is the sub-globular set $\partial^+(V\Gamma)$.*

Proof. By definition, $V(\partial^-(\Gamma))$ contains the same elements as $V(\Gamma)$ in dimension up to $\dim\Gamma-2$, and is empty in dimensions $\dim\Gamma$ and higher. So by Proposition 42, it suffices to check that in dimension $\dim\Gamma - 1$, the globular set $V(\partial^-(\Gamma))$ contains exactly the minimal elements for the preorder $\triangleleft$ in $V(\Gamma)$, with source and target fixed. This is true by a straightforward induction. In the case of the target, it, it is similar, except one has to check that we only keep the maximal element. For the induction to work, we thus have to also show that in a derivation of the form $(\Gamma, y : A, f : x \to y) \vdash_{\mathsf{ps}}$ with $\dim A = \dim\Gamma - 2$ the last variable in the context $\partial^+\Gamma$ is the maximal element of type A in Γ. □

4.2 Operations and coherences.

In order to translate the Grothendieck-Maltsiniotis definition of weak ω-categories in type theory, we extend the type theory GSeTT with term constructors which correspond the operations present in those categories.

Signature of the theory. We extend the signature of the theory GSeTT with two term constructors op and coh, corresponding to the liftings that are formally added in the Grothendieck–Maltsiniotis definition of weak ω-categories. Both of these constructors take as arguments a context, a type and a substitution, in such a way that terms in the theory are now either variables, or of the form $\mathsf{op}_{\Gamma,A}[\gamma]$ or $\mathsf{coh}_{\Gamma,A}[\gamma]$, with Γ a context, A a type and γ a substitution. We define the set of variables of a term constructed this way as

$$\mathrm{Var}(\mathsf{op}_{\Gamma,A}[\gamma]) = \mathrm{Var}(\gamma) \qquad\qquad \mathrm{Var}(\mathsf{coh}_{\Gamma,A}[\gamma]) = \mathrm{Var}(\gamma)$$

Importantly, the variables that appear in the ps-context Γ are not accounted for in the variables of the term constructed this way. One can understand this by thinking of the term constructors op and coh as binders for these variables. We also need to extend the action of substitutions on terms to these new terms. This has to be defined together with the composition of substitution, as they are mutually inductive notions:

$$t[\langle\rangle] = t \qquad\qquad y[\langle\gamma, x \mapsto u\rangle] = \begin{cases} u & \text{if } y = x \\ y[\gamma] & \text{otherwise} \end{cases}$$

$$\mathsf{op}_{\Gamma,A}[\gamma][\delta] = \mathsf{op}[\gamma\circ\delta] \qquad\qquad \mathsf{coh}_{\Gamma,A}[\gamma][\delta] = \mathsf{coh}_{\Gamma,A}[\gamma\circ\delta]$$

$$\star[\gamma] = \star \qquad\qquad (t \underset{A}{\to} u)[\gamma] = (t[\gamma]) \xrightarrow[(A[\gamma])]{} (u[\gamma])$$

$$\langle\rangle\circ\gamma = \langle\rangle \qquad\qquad \langle\delta, x\mapsto t\rangle\circ\gamma = \langle\delta\circ\gamma, x\mapsto t[\gamma]\rangle$$

Rules for coherences. The introduction rules for these two term constructors are subject to two side conditions, expressing the fact that some terms use all of the variables of a context. In order to express these conditions in a more compact way, we write $\mathrm{Var}(t : A) = \mathrm{Var}(t)\cup\mathrm{Var}(A)$ for the union of the set of variables of the term t and the set of variables of the type A. In this notation, it is always implicit that the term t is of type A in the context we are considering. The introduction rules for the term constructors op and coh are then given as follows.

For contexts:

$$\frac{}{\varnothing \vdash}(\text{EC}) \qquad \frac{\Gamma \vdash A}{\Gamma, x : A \vdash}(\text{CE}) \quad \text{when } x \notin \mathrm{Var}(\Gamma)$$

For types:

$$\frac{\Gamma \vdash}{\Gamma \vdash \star}(\star\text{-INTRO}) \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash t : A \qquad \Gamma \vdash u : A}{\Gamma \vdash t \underset{A}{\longrightarrow} u}(\rightarrow\text{-INTRO})$$

For terms:

$$\frac{\Gamma \vdash \qquad (x : A) \in \Gamma}{\Gamma \vdash x : A}(\text{VAR})$$

$$\frac{\Gamma \vdash_{\mathsf{ps}} \qquad \partial^-(\Gamma) \vdash t : A \qquad \partial^+(\Gamma) \vdash u : A \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \mathsf{op}_{\Gamma, t \underset{A}{\rightarrow} u}[\gamma] : t[\gamma] \underset{A[\gamma]}{\longrightarrow} u[\gamma]}(\text{OP}) \quad \text{when } (\mathrm{C}_{\mathsf{op}})$$

$$\frac{\Gamma \vdash_{\mathsf{ps}} \qquad \Gamma \vdash t : A \qquad \Gamma \vdash u : A \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \mathsf{coh}_{\Gamma, t \underset{A}{\rightarrow} u}[\gamma] : t[\gamma] \underset{A[\gamma]}{\longrightarrow} u[\gamma]}(\text{COH}) \quad \text{when } (\mathrm{C}_{\mathsf{coh}})$$

For substitutions:

$$\frac{\Delta \vdash}{\Delta \vdash \langle\rangle : \varnothing}(\text{ES}) \qquad \frac{\Delta \vdash \gamma : \Gamma \qquad \Gamma, x : A \vdash \qquad \Delta \vdash t : A[\gamma]}{\Delta \vdash \langle \gamma, x \mapsto t \rangle : (\Gamma, x : A)}(\text{SE})$$

Figure 6: Derivation rules of the theory CaTT.

– For the constructor op, the rule is

$$\frac{\Gamma \vdash_{\mathsf{ps}} \qquad \partial^-(\Gamma) \vdash t : A \qquad \partial^+(\Gamma) \vdash u : A \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \mathsf{op}_{\Gamma, t \underset{A}{\rightarrow} u}[\gamma] : t[\gamma] \underset{A[\gamma]}{\longrightarrow} u[\gamma]}(\text{OP})$$

subject to the side conditions

$$\mathrm{Var}(t : A) = \mathrm{Var}(\partial^-(\Gamma)) \qquad \text{and} \qquad \mathrm{Var}(u : A) = \mathrm{Var}(\partial^+(\Gamma)) \tag{C_{op}}$$

– For the constructor coh, the rule is

$$\frac{\Gamma \vdash_{\mathsf{ps}} \qquad \Gamma \vdash t : A \qquad \Gamma \vdash u : A \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \mathsf{coh}_{\Gamma, t \underset{A}{\rightarrow} u}[\gamma] : t[\gamma] \underset{A[\gamma]}{\longrightarrow} u[\gamma]}(\text{COH})$$

subject to the side conditions

$$\mathrm{Var}(t : A) = \mathrm{Var}(\Gamma) \qquad \text{and} \qquad \mathrm{Var}(u : A) = \mathrm{Var}(\Gamma) \tag{$\mathrm{C}_{\mathsf{coh}}$}$$

Note that the rule (COH) presented here is slightly different from the one introduced in [16]: it is equivalent but makes the presentation closer to the conditions of Maltsiniotis' definition of weak ω-categories [28]. A detailed account of the equivalence between the two presentations is given in [4, Section 3.5.1]. We give in Figure 6 a full summary of all the rules of the theory CaTT.

Interpretation. These rules are to be understood as follows. A derivable judgment $\Gamma \vdash t : A$ can be thought of as a given composite of various cells that are supposed to be known in the context Γ, and in the case of a ps-context, adding the side condition $\mathrm{Var}(t : A) = \mathrm{Var}(\Gamma)$ enforces that the composite uses all the cells of Γ. This intuition is made more formal in Section 5, where we show that the contexts of this theory are finite polygraphs for weak ω-categories. In the light of this identification, a term $\Gamma \vdash t : A$ is a cell in the free category generated by the polygraph Γ. The use of the two rules can be detailed as follows.

- Rule (OP). Given a pasting scheme Γ and a way to compose entirely its source and its target encoded as the terms $\partial^-(\Gamma) \vdash t : A$ and $\partial^+(\Gamma) \vdash u : A$ satisfying the condition $(\mathrm{C}_{\mathsf{op}})$, this rule provides a way to compose entirely Γ. The result of this composition goes from the specified composition of the source to the specified composition of the target, and is encoded as the term $\Gamma \vdash \mathsf{op}_{\Gamma,t\to u}[\mathrm{id}_\Gamma] : t \to u$.
- Rule (COH). Given two ways of composing entirely the pasting scheme Γ, encoded as a pair of terms $\Gamma \vdash t : A$ and $\Gamma \vdash u : A$ satisfying the condition $(\mathrm{C}_{\mathsf{coh}})$, the rule provides a cell between these two compositions, encoded as the term $\Gamma \vdash \mathsf{coh}_{\Gamma,t\to u}[\mathrm{id}_\Gamma] : t \to u$. It turns out that this rule only produces invertible cells, and thus it can be reformulated as: "any two ways of composing entirely a pasting scheme are weakly equivalent", or by adopting a more topological view it expresses that the space of ways to compose a pasting scheme is contractible.

4.3 Some examples of derivations.

We provide some examples of derivations that one may compute in CaTT, using the actual syntax implemented in the tool [7]. This software relies on the fact that the derivability of the judgments in the theory CaTT is decidable and provides an algorithm to decide it. The inputs from the user are interpreted by the system as typing judgments, and the software accepts an input whenever it is able to find a derivation for the corresponding judgement. This implementation follows the convention introduced by Finster and Mimram [16] and does not distinguish between the term constructors `op` and `coh`, assuming a single term constructor with two rules that are mutually exclusive. As a result, all the new constructions are introduced with the keyword `coh`, followed by a name to identify it. Then comes a list of arguments which is the description of a ps-context followed by a column and a type. For instance the following line

```
coh id (x : *) : x -> x
```

defines a coherence called `id`, which corresponds to the construction $\mathsf{coh}_{(x:\star):x\to x}$. Note that this expression is not a complete term, as it lacks a substitution. Implicitly, we may assume that we have in fact defined the term

$$(x : \star) \vdash \mathsf{coh}_{(x:\star),x \underset{\star}{\to} x}[\mathrm{id}_{(x:\star)}] : x \underset{\star}{\to} x$$

The derivation of this judgments is then guaranteed by the software (in this example, it follows from an application of the rule (COH)). We can then use the admissibility of the action of substitutions (given by Lemma 56) to define the term $\mathsf{coh}_{(x:\star),x\to x}[\gamma]$ for any substitution. Thus further references to this coherence just have to specify the substitution γ towards the context $(x : \star)$. We encode such a substitution as a list of arguments, for instance one may write `id y` to refer to the term identity, in a context containing a variable `y` of type `*`. In general, we only specify some of the argument for instance, considering the following declaration defining composition

```
coh comp (x : *) (y : *) (f : x -> y) (z : *) (g : y -> z) : x -> z
```

one needs to write only `comp f g` instead of `comp x y f z g` when referring to it, as the terms `x`, `y` and `z` can be inferred from the data of `f` and `g`. Lemma 64 proves that it suffices to provide the terms corresponding to the locally maximal variables of the target ps-context, and the software implements an elaboration mechanism that builds a full substitution out of only these arguments. Thus it can detect automatically which argument should be left implicit and allows the user to write shorter terms. Other examples of declarations one may define in CaTT include

- left unitality and its inverse

  ```
  coh unitl (x : *) (y : *) (f : x -> y) : comp (id x) f -> f

  coh unitl- (x : *) (y : *) (f : x -> y) : f -> comp (id x) f
  ```

- right unitality and its inverse

  ```
  coh unitr (x : *) (y : *) (f : x -> y) : comp f (id y) -> f

  coh unitr- (x : *) (y : *) (f : x -> y) : f -> comp f (id y)
  ```

- associativity and its inverse

  ```
  coh assoc (x : *) (y : *) (f : x -> y) (z : *)
            (g : y -> z) (w : *) (h : z -> w)
            : comp f (comp g h) -> comp (comp f g) h

  coh assoc- (x : *) (y : *) (f : x -> y) (z : *)
             (g : y -> z) (w : *) (h : z -> w)
             : comp (comp f g) h -> comp f (comp g h)
  ```

- vertical composition of 2-cells

  ```
  coh vcomp (x : *) (y : *) (f : x -> y) (g : x -> y)
            (a : f -> g) (h : x -> y) (b : g -> h) : f -> h
  ```

- horizontal composition of 2-cells

  ```
  coh hcomp (x : *) (y : *) (f : x -> y) (f' : x -> y) (a : f -> f')
            (z : *) (g : y -> z) (g' : y -> z) (b : g -> g')
            : comp f g -> comp f' g'
  ```

- left whiskering

  ```
  coh whiskl (x : *) (y : *) (f : x -> y) (z : *) (g : y -> z)
             (g' : y -> z) (b : g -> g') : comp f g -> comp f g'
  ```

- right whiskering

  ```
  coh whiskr (x : *) (y : *) (f : x -> y) (f' : x -> y)
             (a : f -> f') (z : *) (g : y -> z) : comp f g -> comp f' g
  ```

We also provide a syntax to define arbitrary compositions of the above declarations in an arbitrary context. The corresponding keyword is `let` followed with an identifier and a context, the symbol `=`, and a full definition of the term using previously defined terms and declarations. For instance, the following term defines the squaring of an endomorphism

```
let sq (x : *) (f : x -> x) = comp f f
```

Note that the context associated to the keyword `coh` is necessarily a ps-context, whereas any context can be associated to the keyword `let`.

4.4 Properties of the theory CaTT.

In order to reason and prove results about CaTT, we mostly reason by induction on its terms. We thus first need to study some of the properties of the syntax which, even though quite simple, will prove quite useful in the following.

Preservation of the basic properties. The first thing that one can check about this theory is that the term constructor are nice enough, so that the basic properties established for GSeTT still hold for this new type theory.

Lemma 56. *All the properties of Lemma 6 still hold in* CaTT*, and every derivable judgment in* CaTT *has exactly one derivation.*

Proof. These results are proved as in the case of GSeTT, by mutual induction on the derivation trees of the various judgments. The added term constructors make things a slightly more involved than in the case with only variables, and some of the properties that could be proved on a syntactical level in the theory GSeTT only hold for derivable judgments in the theory CaTT. Apart for these technical subtleties, the generalization is straightforward. □

The syntactic category. As for the theory GSeTT, the identity substitution id_Γ associated to a context Γ is always derivable, as well as the composition of derivable substitutions, using the action of substitution on raw terms. Moreover, all the results that we have stated for the theory GSeTT still hold for the theory CaTT: it is in particular the case for Proposition 9, Proposition 10 and Proposition 12. This shows that the derivable contexts of the theory CaTT assemble into a category, whose morphisms are the derivable substitution. We write $\mathcal{S}_{\mathsf{CaTT}}$ for this category and call it the *syntactic category* of the theory CaTT. The aforementioned results imply that $\mathcal{S}_{\mathsf{CaTT}}$ is equipped with a canonical structure of category with families, where Ty^Γ is the set of derivable types in the context Γ, and Tm^Γ_A is the set of terms of type A in the context Γ.

Inclusion of $\mathcal{S}_{\mathsf{GSeTT}}$. The theory CaTT contains the theory GSeTT as a sub-theory: anything that can be derived using variables (in GSeTT) only can still be derived using variables and term constructors (in CaTT). In particular, any valid context Γ in the theory GSeTT is also a valid context in the category CaTT: this is in particular the case for the disk contexts D^n and the sphere contexts S^{n-1}. Moreover, this induces a functor between the syntactic categories $\mathcal{S}_{\mathsf{GSeTT}} \to \mathcal{S}_{\mathsf{CaTT}}$. It is immediate that this functor is a morphism of categories with families, by taking any type (resp. any term) in the theory GSeTT to the same type (resp. to the same term) in the theory CaTT.

Familial representability of types. Central in the study of GSeTT was the Lemma 16, which establishes that the family $S^\bullet$ familially represents the functor Ty. We have noted that its proof does not really depend on the extra terms which are present in the theory, and thus immediately extends to the case of CaTT:

Lemma 57. *For any natural number n, the map*

$$\begin{array}{rcl} \mathcal{S}_{\mathsf{CaTT}}(\Gamma, S^{n-1}) & \to & \left\{A \in \mathrm{Ty}^\Gamma \mid \dim(A) = n-1\right\} \\ \gamma & \mapsto & U_n[\gamma] \end{array}$$

is an isomorphism natural in Γ. Given a type A of dimension $n-1$, we denote the associated substitution

$$\chi_A : \Gamma \to S^{n-1}$$

Moreover, the maps

$$\begin{array}{rcl} (\mathcal{S}_{\mathsf{CaTT}}/S^{n-1})(\Gamma \xrightarrow{\chi_A} S^{n-1}, D^n \xrightarrow{\pi} S^{n-1}) & \to & \mathrm{Tm}_A^\Gamma \\ \gamma & \mapsto & d_{2n}[\gamma] \end{array}$$

are isomorphisms, natural in Γ (the source is a hom-set in the slice category of $\mathcal{S}_{\mathsf{CaTT}}$ over S^{n-1}). Given a term $t \in \mathrm{Tm}_A^\Gamma$ of type A, we denote the associated substitution over χ_A by $\chi_t : \Gamma \to D^n$, in such a way that the following diagram commutes

$$\begin{array}{ccc} \Gamma & \xrightarrow{\chi_t} & D^n \\ & \searrow_{\chi_A} & \downarrow \pi \\ & & S^{n-1} \end{array}$$

Depth of a term. In order to study the theory CaTT, we often reason by structural induction on terms. In order to justify that these inductions are well-founded, we introduce the notion of *depth* of a term and of a substitution. It is the natural number $\mathrm{depth}(t)$ (resp. $\mathrm{depth}(\gamma)$) defined by induction on the term t (resp. substitution γ) by

$$\begin{array}{ll} \mathrm{depth}(x) = 0 & \mathrm{depth}(\mathsf{coh}_{\Gamma,A}[\gamma]) = 1 + \mathrm{depth}(\gamma) \\ \mathrm{depth}(\langle\rangle) = 0 & \mathrm{depth}(\langle \gamma, t \mapsto u\rangle) = \max(\mathrm{depth}(\gamma), \mathrm{depth}(u)) \end{array}$$

Informally, the depth of a term expresses how many nested term constructors are needed to write it, and similarly for substitution. It should not be confused with the notion of "coherence depth", introduced in Section 5.

Terms in the empty context. An important property that we can prove on the theory CaTT, using induction on the depth of terms, is that the there is no way to build a term in the empty context.

Lemma 58. *In the theory CaTT there is no term derivable in the empty context.*

Proof. We prove this result by induction on the depth of the term. First note that no variable is derivable in the empty context. A term of depth $d+1$ in the empty context has to be constructed using a substitution $\varnothing \to \Delta$ of depth d, where Δ is a ps-context. Since Δ is non-empty, such a substitution has to be built out of terms that are derivable in the empty context. Since the substitution is of depth at most d, these terms are of depth at most d also, and by induction there is no such term, hence there is no such substitution. This proves that there is no term of depth $d+1$ in the context $\varnothing$. □

Variables of the characteristic substitution. Using Lemma 57, we can slightly reformulate the side conditions of the rules (OP) and (COH), which involve expressions of the form $\mathrm{Var}(t : A)$.

Lemma 59. *Consider a context $\Gamma \vdash$, together with a term $\Gamma \vdash t : A$, then the following sets of variables are equal:*

$$\mathrm{Var}(A) = \mathrm{Var}(\chi_A) \qquad\qquad \mathrm{Var}(t : A) = \mathrm{Var}(\chi_t)$$

Proof. We prove these two results by mutual induction, on the dimension of A.

- If $\dim(A) = 0$, then necessarily $A = \star$. We have $\chi_\star = \langle\rangle$ and $\mathrm{Var}(\star) = \mathrm{Var}(\langle\rangle) = \emptyset$.
- If $\dim(A) > 0$, we can write $A = t \underset{B}{\to} u$, and we have $\chi_A = \langle \chi_t, d_{2n+1} \mapsto u\rangle$ with $\dim(B) + 1 = \dim(A)$. Moreover, we have by definition

$$\begin{aligned}\mathrm{Var}(A) &= \mathrm{Var}(B) \cup \mathrm{Var}(t) \cup \mathrm{Var}(u)\\ &= \mathrm{Var}(t : B) \cup \mathrm{Var}(u)\end{aligned}$$

  and on the other hand, we have

$$\mathrm{Var}(\chi_A) = \mathrm{Var}(\chi_t) \cup \mathrm{Var}(u)$$

  The induction case for terms then shows $\mathrm{Var}(A) = \mathrm{Var}(\chi_A)$.
- For A of arbitrary dimension, we have $\chi_t = \langle \chi_A, d_{2n} \mapsto t\rangle$, and thus

$$\begin{aligned}\mathrm{Var}(t : A) &= \mathrm{Var}(A) \cup \mathrm{Var}(t)\\ \mathrm{Var}(\chi_t) &= \mathrm{Var}(\chi_A) \cup \mathrm{Var}(t)\end{aligned}$$

  The induction case for types then shows $\mathrm{Var}(t : A) = \mathrm{Var}(\chi_t)$. □

Globular set of variables of a term. The contexts in the theory CaTT coming from the theory GSeTT play a particular role in the theory, and we call them *globular contexts*. They are recognizable by the fact that they are built out only from variables, and as we have shown in the definition of the functor V, their variables form into a globular set. For instance, of the two following contexts, the first one is a globular context, whereas the second one is not.

$$(x : \star, y : \star, z : \star, f : x \to y, g : z \to y) \qquad\qquad (x : \star, \alpha : \mathtt{id}\ x \to \mathtt{id}\ x)$$

Lemma 60. *Consider a globular context Γ in the theory CaTT together with a derivable term $\Gamma \vdash t : A$. For every variable x in the set $\mathrm{Var}(t : A)$, its source and target also belong to this set. This equips the set $\mathrm{Var}(t : A)$ with a structure of a globular set which is a globular subset of $V\Gamma$.*

Proof. We prove this result by induction on the depth of the term t.

- Since Γ is a context in $\mathcal{S}_{\mathsf{GSeTT}}$, if the term t is of depth 0, then it is a variable $t = x$ and the map $\chi_x : \Gamma \to D^n$ defines a map in $\mathcal{S}_{\mathsf{GSeTT}}$. Then $\mathrm{Var}(\chi_x)$ is the set of elements of the image of the map $V(\chi_x) : V(D^n) \to V(\Gamma)$, so it is stable under source and target and is naturally a globular subset of $V(\Gamma)$. The result is then given by Lemma 59.
- If the term t is of depth $d + 1$, it is of the form $t = \mathsf{op}_{\Delta,B}[\gamma]$ or $t = \mathsf{coh}_{\Delta,B}[\gamma]$, with γ a substitution of depth at most d. Consider a variable $x \in \mathrm{Var}(t : A)$ and denote respectively by y and z its source and target in Γ. Necessarily we have $x \in \mathrm{Var}(\gamma)$, and thus there exists a variable x' in Δ such that $x \in \mathrm{Var}(x'[\gamma])$. Then consider the variables y' and z' that are respectively the source and target of x' in Δ, in such a way that we have $\Delta \vdash x' : y' \to z'$. Then we have $\Gamma \vdash x'[\gamma] : y'[\gamma] \to z'[\gamma]$ with $x \in \mathrm{Var}(x'[\gamma] : y'[\gamma] \to z'[\gamma])$ and $x'[\gamma]$ of depth at most d. By induction this proves that $y, z \in \mathrm{Var}(x'[\gamma] : y'[\gamma] \to z'[\gamma]) \subseteq \mathrm{Var}(t)$. Hence the source and target of x belong to $\mathrm{Var}(t : A)$. □

5 The syntactic categories associated to CaTT

This section is dedicated to the study of the syntactic category $\mathcal{S}_{\mathsf{CaTT}}$. We have shown in Theorem 53 that the subcategory $\mathcal{S}_{\mathsf{ps}}$ of the syntactic category $\mathcal{S}_{\mathsf{GSeTT}}$ is equivalent to the category $\mathbf{\Theta}_0$. We now show that adding the term constructors op and coh allow us to recover exactly the missing pieces of information to obtain weak ω-categories: we exhibit a subcategory $\mathcal{S}_{\mathsf{ps},\infty}$ of the category $\mathcal{S}_{\mathsf{CaTT}}$ which is equivalent to the cat-coherator $\mathbf{\Theta}_\infty^{\mathrm{op}}$.

5.1 A filtration in $\mathcal{S}_{\mathsf{CaTT}}$.

We consider the full subcategory $\mathcal{S}_{\mathsf{ps},\infty}$ of $\mathcal{S}_{\mathsf{CaTT}}$, whose objects are ps-contexts. Our aim is to exhibit this category as a colimit of the form

$$\mathcal{S}_{\mathsf{ps},\infty} = \operatorname{colim}\left(\mathcal{S}_{\mathsf{ps},0} \to \mathcal{S}_{\mathsf{ps},1} \to \mathcal{S}_{\mathsf{ps},2} \to \ldots\right)$$

that mimics the iterative construction of $\mathbf{\Theta}_\infty$ as a colimit of the $\mathbf{\Theta}_n$ in the Grothendieck-Maltsiniotis definition of weak ω-categories.

Coherence depth. We introduce the notion of *coherence depth* of a term, type or substitution in order to construct the categories $\mathcal{S}_{\mathsf{ps},n}$. It is defined inductively by

$$\begin{array}{rr}
\mathrm{cd}(v : A) = \mathrm{cd}(A) & \mathrm{cd}(\mathsf{op}_{\Gamma,A}[\gamma]) = \max(\mathrm{cd}(A)+1, \mathrm{cd}(\gamma)) \\
 & \mathrm{cd}(\mathsf{coh}_{\Gamma,A}[\gamma]) = \max(\mathrm{cd}(A)+1, \mathrm{cd}(\gamma)) \\
\mathrm{cd}(\star) = 0 & \mathrm{cd}(t \underset{A}{\to} u) = \max(\mathrm{cd}(A), \mathrm{cd}(t), \mathrm{cd}(u)) \\
\mathrm{cd}(\langle\rangle) = 0 & \mathrm{cd}(\langle \gamma, x \mapsto t\rangle) = \max(\mathrm{cd}(\gamma), \mathrm{cd}(t))
\end{array}$$

Note that this definition is distinct from the one of depth that we have introduced in Section 4 for reasoning on syntax.

The filtration. We define the category $\mathcal{S}_{\mathsf{ps},n}$ to be the graph that has the same objects as the category $\mathcal{S}_{\mathsf{ps},\infty}$ and whose morphisms are substitutions of coherence depth at most n. Lemma 61 shows that this is actually a subcategory of $\mathcal{S}_{\mathsf{ps},\infty}$. Note that in the case $n = 0$, the substitutions of coherence depth 0 are the substitutions containing only variables, and thus they are exactly the substitutions of $\mathcal{S}_{\mathsf{GSeTT}}$, i.e.,

$$\mathcal{S}_{\mathsf{ps},0} = \mathcal{S}_{\mathsf{ps}}$$

We can sum up the situation with the following diagram of inclusions

$$\begin{array}{ccccccccc}
\mathcal{S}_{\mathsf{GSeTT}} & \longrightarrow & & & & & & & \mathcal{S}_{\mathsf{CaTT},\infty} \\
\uparrow & & & & & & & & \uparrow \\
\mathcal{S}_{\mathsf{ps}} = \mathcal{S}_{\mathsf{ps},0} & \longrightarrow & \mathcal{S}_{\mathsf{ps},1} & \longrightarrow & \mathcal{S}_{\mathsf{ps},2} & \longrightarrow & \cdots & \longrightarrow & \mathcal{S}_{\mathsf{ps},\infty} \\
\uparrow & & & & & & & & \\
\mathcal{G}^{\mathrm{op}} & & & & & & & &
\end{array}$$

It is straightforward from the definition that $\mathcal{S}_{\mathsf{ps},\infty}$ is the colimit of this sequence of morphisms of categories

$$\mathcal{S}_{\mathsf{ps},\infty} = \operatorname{colim}\left(\mathcal{S}_{\mathsf{ps}} \to \mathcal{S}_{ps,1} \to \mathcal{S}_{ps,2} \to \cdots\right)$$

Indeed, since all these functors are the identity on the objects, it amounts to taking the colimit of the hom-sets, which define a filtration of sets:

$$\{\Delta \vdash \gamma : \Gamma\} = \bigcup_{n\in\mathbb{N}} \{\Delta \vdash \gamma : \Gamma \mid \mathrm{cd}(\gamma) \leq n\}$$

Properties of the coherence depth. The notion of coherence depth sometimes behaves awkwardly with respect to the structure of the type theory. To illustrate this, consider the context $(x : \star, \alpha : \texttt{id}\ x \to \texttt{id}\ x)$: although the term α is of coherence depth 0, its type is of coherence depth 1. This may be an issue when reasoning inductively on the coherence depth, as one cannot consider all the terms and its types (an example of this issue appears in Lemma 66). However, we show below that such issues do not arise in globular contexts, which are the only ones for which we are going to consider coherence depths.

First note that the application of a substitution cannot increase the coherence depth arbitrarily:

Lemma 61. *Given a substitution γ we have*

- *for any type A,* $\mathrm{cd}(A[\gamma]) \leq \max(\mathrm{cd}(A), \mathrm{cd}(\gamma))$,
- *for any term t,* $\mathrm{cd}(t[\gamma]) \leq \max(\mathrm{cd}(t), \mathrm{cd}(\gamma))$,
- *for any substitution δ,* $\mathrm{cd}(\delta \circ \gamma) \leq \max(\mathrm{cd}(\delta), \mathrm{cd}(\gamma))$.

Proof. We prove this result by mutual induction on the type, term and substitution.

- For the type $\star$, we have $\star[\gamma] = \star$, and hence $\mathrm{cd}(\star[\gamma]) = 0 \leq \max(0, \mathrm{cd}(\gamma))$.
- For the type $t \underset{A}{\to} u$, we have

$$\begin{aligned}
\mathrm{cd}((t \underset{A}{\to} u)[\gamma]) &= \mathrm{cd}(t[\gamma] \xrightarrow[A[\gamma]]{} u[\gamma]) \\
&= \max(\mathrm{cd}(A[\gamma]), \mathrm{cd}(t[\gamma]), \mathrm{cd}(u[\gamma])) \\
&\leq \max(\mathrm{cd}(A), \mathrm{cd}(t), \mathrm{cd}(u), \mathrm{cd}(\gamma)) && \text{by induction} \\
&\leq \max(\mathrm{cd}(t \underset{A}{\to} u), \mathrm{cd}(\gamma))
\end{aligned}$$

- For a variable x, we have $\mathrm{cd}(x[\gamma]) \leq \mathrm{cd}(\gamma)$ by definition of the coherence depth of a substitution.
- For the term $t = \mathsf{op}_{\Delta,A}[\delta]$, or for the term $t = \mathsf{coh}_{\Delta,A}[\delta]$, we have

$$\begin{aligned}
\mathrm{cd}(t[\gamma]) &= \max(\mathrm{cd}(A) + 1, \mathrm{cd}(\delta \circ \gamma)) \\
&\leq \max(\mathrm{cd}(A) + 1, \mathrm{cd}(\delta), \mathrm{cd}\,\gamma) && \text{by induction} \\
&\leq \max(\mathrm{cd}(t), \mathrm{cd}(\gamma))
\end{aligned}$$

- For the substitution $\langle\rangle$, we have $\mathrm{cd}(\langle\rangle \circ \gamma) = 0 \leq \mathrm{cd}\,\gamma$.
- For the substitution $\langle \delta, x \mapsto t\rangle$, we have

$$\begin{aligned}
\mathrm{cd}(\langle \delta, x \mapsto t\rangle \circ \gamma) &= \mathrm{cd}(\langle \delta \circ \gamma, x \mapsto t[\gamma]\rangle) \\
&= \max(\mathrm{cd}(\delta \circ \gamma), \mathrm{cd}(t[\gamma])) \\
&\leq \max(\mathrm{cd}(\delta), \mathrm{cd}(t), \mathrm{cd}(\gamma)) && \text{by induction} \\
&\leq \max(\mathrm{cd}(\langle \delta, x \mapsto t\rangle), \mathrm{cd}(\gamma))
\end{aligned}$$

From which we conclude. □

Lemma 62. *In a globular context* Γ*, for every derivable term* $\Gamma \vdash t : A$*, we have* $\mathrm{cd}(A) \leq \mathrm{cd}(t)$.

Proof. We distinguish between the case where t is a variable and the case where t is obtained by application of a term constructors.

- If $t = x$ is a variable, and since it is derivable in a globular context, its type is derivable in the theory GSeTT and hence is of depth 0.
- A term t is not a variable, it is either of the form $t = \mathsf{op}_{\Delta,B}[\delta]$ or $t = \mathsf{coh}_{\Delta,B}[\delta]$, and in both cases we have $\mathrm{cd}(t) = \max(\mathrm{cd}(B) + 1, \mathrm{cd}(\delta))$, and the type A is obtained as $A = B[\delta]$. Lemma 61 then shows that $\mathrm{cd}(B) \leq \max(\mathrm{cd}(B), \mathrm{cd}(\delta)) \leq \mathrm{cd}(t)$. □

Corollary 63. *In a globular context* Γ*, for every type* $\Gamma \vdash A$*, we have* $\mathrm{cd}(A) = \mathrm{cd}(\chi_A)$ *and for every term* $\Gamma \vdash t : A$*, we have* $\mathrm{cd}(t) = \mathrm{cd}(\chi_t)$.

Proof. We prove these two results by mutual induction on the dimension,

- For the type $\Gamma \vdash \star$, we have $\chi_\star = \langle\rangle$, and by definition, $\mathrm{cd}(\star) = \mathrm{cd}(\langle\rangle) = 0$.
- For the type $\Gamma \vdash A$ of dimension $n \geq 0$, we can write $A = t \underset{B}{\to} u$, and we have $\chi_A = \langle \chi_t, d_{2n+1} \mapsto u \rangle$. Applying Lemma 62 shows that $\mathrm{cd}(A) = \max(\mathrm{cd}(t), \mathrm{cd}(u))$. Moreover, by definition $\mathrm{cd}(\chi_A) = \max(\mathrm{cd}(\chi_t), \mathrm{cd}(u))$. The induction case for term then shows that $\mathrm{cd}(A) = \mathrm{cd}(\chi_A)$.
- For a term $\Gamma \vdash t : A$ of dimension n, we have $\chi_t = \langle \chi_A, d_{2n} \mapsto t \rangle$, and we have by definition $\mathrm{cd}(\chi_t) = \max(\mathrm{cd}(\chi_A), \mathrm{cd}(t))$. The induction case for types together with Lemma 62 show that $\mathrm{cd}(\chi_A) = \mathrm{cd}(A) \leq \mathrm{cd}(t)$ and hence $\mathrm{cd}(\chi_t) = \mathrm{cd}(t)$. □

5.2 Globular products in the category $\mathcal{S}_{\mathsf{CaTT}}$.

In order to show that $\mathcal{S}_{\mathsf{ps},\infty}$ dualizes the construction of the category $\mathbf{\Theta}_\infty$, we characterize the globular products in this category.

$\mathcal{S}_{\mathsf{CaTT}}$ as a globular category with families. The inclusion functor $I : \mathcal{S}_{\mathsf{GSeTT}} \to \mathcal{S}_{\mathsf{CaTT}}$ induces a structure of category with families on the category $\mathcal{S}_{\mathsf{CaTT}}$, which coincides exactly with the one given by Lemma 57. Hence for this structure, I is a morphism of globular categories with families.

Lemma 64. *The inclusion functor* $I : \mathcal{S}_{\mathsf{GSeTT}} \to \mathcal{S}_{\mathsf{CaTT}}$ *preserves globular products.*

Proof. Since I is a morphism of globular categories, it preserves the pullbacks along the display maps $D^n \to S^{n-1}$. By Lemma 29, we have $I \cong \mathrm{Ran}_{D^\bullet} ID^\bullet$, and thus it preserves globular products because $\mathrm{Ran}_{D^\bullet} ID^\bullet$ does, by Lemma 25. □

Lemma 65. *The inclusion functor* $I : \mathcal{S}_{\mathsf{ps}} \to \mathcal{S}_{\mathsf{ps},\infty}$ *preserves globular products.*

Proof. Note that the inclusion of the full subcategory $\mathcal{S}_{\mathsf{ps},\infty} \hookrightarrow \mathcal{S}_{\mathsf{CaTT}}$ reflects all limits. Moreover, by Lemma 64, the composite

$$\mathcal{S}_{\mathsf{ps}} \xrightarrow{I} \mathcal{S}_{\mathsf{ps},\infty} \hookrightarrow \mathcal{S}_{\mathsf{CaTT}}$$

preserves the globular products. Hence the functor I also preserves the globular products. □

Reflexivity of the depth-bounded inclusion. There is a canonical functor $\mathcal{S}_{\mathsf{ps},n} \to \mathcal{S}_{\mathsf{ps},\infty}$, which consists in forgetting that a substitution is of bounded coherence depth. In order to understand the globular product in the categories $\mathcal{S}_{\mathsf{ps},n}$, it is useful to study the behavior of this functor with respect to globular products.

Lemma 66. *The functor $\mathcal{S}_{\mathsf{ps},n} \to \mathcal{S}_{\mathsf{ps},\infty}$ reflects globular products.*

Proof. Consider an object Γ which is a globular product in the category $\mathcal{S}_{\mathsf{ps},\infty}$, it suffices to show that is also a globular product in the category $\mathcal{S}_{\mathsf{ps},n}$. Any cone of apex Δ over the diagram of Γ in $\mathcal{S}_{\mathsf{ps},n}$ induces a cone over the diagram of Γ in $\mathcal{S}_{\mathsf{ps},\infty}$, which by definition of a limit defines a unique substitution $\gamma : \Delta \to \Gamma$, and it suffices to show that this substitution is in fact in $\mathcal{S}_{\mathsf{ps},n}$. By definition, all the maps $\chi_{x[\gamma]}$ where x is a maximal variable appear in the legs of the cone of apex Δ. Since these legs are chosen in the category $\mathcal{S}_{\mathsf{ps},n}$, this shows that for every locally maximal variable $\chi_{x[\gamma]}$ is of depth at most n, and hence by Corollary 63, $x[\gamma]$ is of depth at most n. Applying Lemma 62 ensures that all the iterated sources and targets of all the $x[\gamma]$ are of depth at most n, and since every variable of Γ is obtained as an iterated source or target of variables of dimension locally maximal in Γ, all the $x[\gamma]$ for every variable x in Γ is of depth at most n. By definition, this means that γ is of depth at most n, and hence γ is a substitution in $\mathcal{S}_{\mathsf{ps},n}$. □

Globular products in the category $\mathcal{S}_{\mathsf{ps},n}$. All the categories $\mathcal{S}_{\mathsf{ps},n}$, for $n \in \mathbb{N} \cup \{\infty\}$, have the same objects, and there are more and more morphisms when n increases. None of these categories are equipped with a structure of category with families, since they lack the possibility of extending the context by any type. However for $n = 0$ and $n = \infty$ we can exhibit them as full subcategories of categories with families. Using these structure of category with families, we could prove that the functor $\mathcal{S}_{\mathsf{ps}} \to \mathcal{S}_{\mathsf{ps},\infty}$ preserves globular products (Lemma 65), and we now use this result to study globular products in all the categories $\mathcal{S}_{\mathsf{ps},n}$.

Lemma 67. *The functors $\mathcal{S}_{\mathsf{ps},0} \to \mathcal{S}_{\mathsf{ps},n}$ preserve globular products.*

Proof. We have the commutative triangle

$$\begin{array}{ccc} \mathcal{S}_{\mathsf{ps},n} & \longrightarrow & \mathcal{S}_{\mathsf{ps},\infty} \\ \uparrow & \nearrow & \\ \mathcal{S}_{\mathsf{ps},0} & & \end{array}$$

By Lemma 64, the functor $\mathcal{S}_{\mathsf{ps},0} \to \mathcal{S}_{\mathsf{ps},\infty}$ preserves the globular products and, by Lemma 66, the functor $\mathcal{S}_{\mathsf{ps},n} \to \mathcal{S}_{\mathsf{ps},\infty}$ reflects the globular products. This implies the the functor $\mathcal{S}_{\mathsf{ps},0} \to \mathcal{S}_{\mathsf{ps},n}$ preserves globular products. □

Lemma 68. *The categories $\mathcal{S}_{\mathsf{ps},n}$ are contravariant globular extensions and the inclusion functors $\mathcal{S}_{\mathsf{ps},n} \to \mathcal{S}_{\mathsf{ps},n+1}$ are morphisms of contravariant globular extensions.*

Proof. Lemma 67 in conjunction with Lemma 46 and Theorem 53 shows that the functor $\mathcal{S}_{\mathsf{ps},0} \to \mathcal{S}_{\mathsf{ps},n}$ endows $\mathcal{S}_{\mathsf{ps},n}$ with a structure of contravariant globular extension. Moreover, Lemma 46 lifts the commutative triangle

$$\begin{array}{ccc} \mathcal{S}_{\mathsf{ps},n} & \longrightarrow & \mathcal{S}_{\mathsf{ps},n+1} \\ \uparrow & \nearrow & \\ \mathcal{S}_{\mathsf{ps},0} & & \end{array}$$

into a morphism of contravariant globular extension $\mathcal{S}_{\mathsf{ps},n} \to \mathcal{S}_{\mathsf{ps},n+1}$. □

$\mathcal{S}_{\mathsf{ps},n}$ as a contravariant globular theory. Assembling altogether the results we have proved about the categories $\mathcal{S}_{\mathsf{ps},n}$, with $n \in \mathbb{N} \cup \{\infty\}$, we have the following:

Proposition 69. *For $n \in \mathbb{N} \cup \{\infty\}$, the category $\mathcal{S}_{\mathsf{ps},n}$ is equipped with a structure of a contravariant globular theory, and the functors $\mathcal{S}_{\mathsf{ps},n} \to \mathcal{S}_{\mathsf{ps},n+1}$ are morphisms of contravariant globular theories.*

Proof. Lemma 68 already shows that $\mathcal{S}_{\mathsf{ps},n}$ is a contravariant globular extension, moreover note that $\mathcal{S}_{\mathsf{ps}}$ and $\mathcal{S}_{\mathsf{ps},n}$ have the same objects, but $\mathcal{S}_{\mathsf{ps},n}$ has strictly more morphisms, and the functor $\mathcal{S}_{\mathsf{ps}} \to \mathcal{S}_{\mathsf{ps},n}$ sends every object to itself and defines the inclusion of the morphisms. Hence it defines a contravariant globular theory. The same reasoning starting from Lemma 64 shows that $\mathcal{S}_{\mathsf{ps},\infty}$ is also a contravariant globular theory. By Lemma 68 the functor $\mathcal{S}_{\mathsf{ps},n} \to \mathcal{S}_{\mathsf{ps},n+1}$ is a morphism of contravariant globular extensions, and hence it is also a morphism of contravariant globular theories. □

5.3 Admissible pairs of substitutions.

In the contravariant globular theory $\mathcal{S}_{\mathsf{ps},m}$, for $m \in \mathbb{N}\cup\{\infty\}$, we consider a morphism $\xi : \Delta \to D^n$. By Lemma 57, such a morphism can be written as $\xi = \chi_t$ for some term t of type A in Δ. Using the notations introduced in Section 2 for defining the disk and sphere contexts, the term t in Δ can be recovered as $t = d_{2n}[\xi]$ and the type of t in Δ is $U_n[\xi]$. Note that U_n contains all the variables of D^n, except for the variable d_{2n}, and hence $\mathrm{Var}(d_{2n}) \cup \mathrm{Var}(U_n) = \mathrm{Var}(D^n)$. This equality shows

$$\begin{aligned}\mathrm{Var}(\xi) &= \mathrm{Var}(d_{2n}[\xi]) \cup \mathrm{Var}(U_n[\xi]) \\ &= \mathrm{Var}(t) \cup \mathrm{Var}(A)\end{aligned}$$

Lemma 70. *Given a term $\Delta \vdash t : A$, the morphism $\chi_t : \Delta \to D^n$ is algebraic in $\mathcal{S}_{\mathsf{ps},m}$ if and only if* $\mathrm{Var}(t : A) = \mathrm{Var}(\Delta)$

Proof. First suppose that $\mathrm{Var}(t : A) = \mathrm{Var}(\Delta)$, and consider a factorization of the form

$$\Delta \xrightarrow[\gamma]{} \Gamma \xrightarrow[\chi_u]{} D^n \qquad (\chi_t : \Delta \to D^n)$$

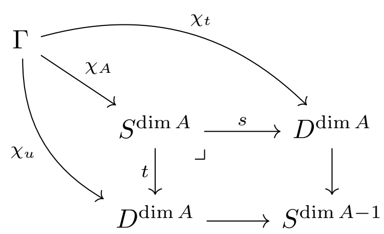
with γ a globular substitution, i.e., a substitution in $\mathcal{S}_{\mathsf{ps},0}$. Then we have a term $\Gamma \vdash u : B$ such that $B[\gamma] = A$ and $u[\gamma] = t$. The condition $\mathrm{Var}(t : A) = \mathrm{Var}(\Delta)$ then implies in particular that $\mathrm{Var}(\Delta) \subset \mathrm{Var}(\gamma)$. Note that under the correspondence of Theorem 22, $\mathrm{Var}(\gamma)$ is the set of elements in the image of the map $V(\gamma) : V\Gamma \to V\Delta$, and the equation $\mathrm{Var}(\Delta) \subset \mathrm{Var}(\gamma)$ then shows that the map $V(\gamma)$ is a surjective map of globular sets. By Lemma 43, any map between two pasting schemes is injective so in particular that $V(\gamma)$ is an isomorphism, and Lemma 44 shows that then it is an identity. By Theorem 53, V is an isomorphism of categories between $\mathcal{S}_{\mathsf{ps},0}$ and $\mathbf{\Theta}_0$. Hence γ is an identity. This proves that χ_t is an algebraic morphism. Conversely suppose that the morphism χ_t is algebraic. Lemma 60 shows that the set $\mathrm{Var}(t : A)$ can be viewed as a sub globular set of $V(\Delta)$. By the equivalence of Theorem 22 the inclusion $\mathrm{Var}(t : A) \to V(\Delta)$ provides a globular substitution $\Delta \vdash \gamma : \Gamma$. Moreover, by definition, we have $\Gamma \vdash t : A$ and $A[\gamma] = A$, $t[\gamma] = t$. Hence by algebraicity of χ_t, this shows that γ is an identity. This proves the inclusion $\mathrm{Var}(()t : A) \to V(\Delta)$ is the identity and thus $\mathrm{Var}(t : A) = V(\Delta)$, which by forgetting the globular set structure implies $\mathrm{Var}(t : A) = \mathrm{Var}(\Delta)$. □

Lemma 71. *The pairs of admissible morphisms in* Γ *are classified by the types* $\Gamma \vdash A$ *satisfying either* (C_{op}) *or* ($\mathrm{C}_{\mathsf{coh}}$)*. For such a type* $\Gamma \vdash A$*, the terms* $\Gamma \vdash t : A$ *classify exactly the lifts of the corresponding admissible pair. More precisely, there is a natural isomorphisms between pairs of admissible morphisms in* Γ *and types satisfying* (C_{op}) *or* ($\mathrm{C}_{\mathsf{coh}}$)*, as well as a natural isomorphism between terms of of such types and lifts of the corresponding isomorphisms.*

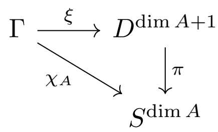
Proof. The types $\Gamma \vdash A$ of non-zero dimension, with A of the form $t \underset{B}{\to} u$, classify the pairs of terms (t, u) of same type B, which are exactly the pairs of parallel maps (χ_t, χ_u). Moreover, such a pair is admissible whenever we have one of the following.

- Both χ_t and χ_u are algebraic, which by Lemma 70 translates to the two conditions $\mathrm{Var}(t : B) = \mathrm{Var}(\Gamma)$ and $\mathrm{Var}(u : B) = \mathrm{Var}(\Gamma)$: this is exactly the condition ($\mathrm{C}_{\mathsf{coh}}$).
- χ_t factors through the source inclusion of Γ as a algebraic morphism and χ_u factors through the target as a algebraic morphism. Again, by Lemma 70, these conditions translate to $\partial^-(\Gamma) \vdash t : B$ with $\mathrm{Var}(t : B) = \mathrm{Var}(\partial^-(\Gamma))$ and $\partial^+(\Gamma) \vdash u : B$ with $\mathrm{Var}(u : B) = \mathrm{Var}(\partial^+(\Gamma))$: this is the condition (C_{op}).

A lift for such a admissible pair is a map $\xi : \Gamma \to D^{\dim A+1}$, such that we have both $s(\xi) = \chi_t$ and $t(\xi) = \chi_u$. In the category $\mathcal{S}_{\mathsf{CaTT}}$ we can encode this data as a substitution $\chi_A : \Gamma \to S^{\dim A}$:



A lift thus amounts to a morphism $\xi : \Gamma \to D^{\dim A+1}$ in $\mathcal{S}_{\mathsf{CaTT}}$ which makes the following triangle commute:



By Lemma 57, these are classified by the terms $\Gamma \vdash t : A$ in the theory CaTT. $\square$

5.4 Equivalence between $\mathcal{S}_{\mathsf{ps},\infty}$ and $\boldsymbol{\Theta}_\infty^{\mathrm{op}}$.

We now prove the main theorem, that the category $\mathcal{S}_{\mathsf{ps},\infty}$ is equivalent to the opposite of the cat-coherator $\boldsymbol{\Theta}_\infty$. This result thus identifies the cat-coherator $\boldsymbol{\Theta}_\infty$ as a full subcategory of the category of the category with families $\mathcal{S}_{\mathsf{CaTT}}$. We define the set F_n to be the set of all types $\Gamma \vdash t \underset{A}{\to} u$ of coherence depth exactly n in a ps-context Γ, satisfying (C_{op}) or ($\mathrm{C}_{\mathsf{coh}}$). By Lemma 71, the family F_n can be defined inductively as the set of all pair of admissible maps in $\mathcal{S}_{\mathsf{ps},n}$ that do not belong to any $F_{n'}$ for $n < n'$.

Lemma 72. *The inclusion* $\mathcal{S}_{\mathsf{ps},n} \to \mathcal{S}_{\mathsf{ps},n+1}$ *exhibits* $\mathcal{S}_{\mathsf{ps},n+1}$ *as the universal coglobular extension of* $\mathcal{S}_{\mathsf{ps},n}$ *equipped with a lift for all pair of morphisms in* F_n*.*

Proof. By Lemma 68, this functor is a morphism of coglobular theories. Moreover consider a admissible pair $(f, g) : \Gamma \to D^n$ in F_n corresponding to a type $\Gamma \vdash A$ in the ps-context Γ, which satisfies $(\mathrm{C}_{\mathsf{op}})$ or $(\mathrm{C}_{\mathsf{coh}})$ and which is of depth n. We can derive a term t by $\Gamma \vdash \mathsf{op}_{\Gamma,A}[\mathrm{id}_\Gamma] : A$ if A satisfies $(\mathrm{C}_{\mathsf{op}})$, or $\Gamma \vdash \mathsf{coh}_{\Gamma,A}[\mathrm{id}_\Gamma] : A$ if A satisfies $(\mathrm{C}_{\mathsf{coh}})$, the term t is then of coherence depth $n+1$. Hence t defines a map χ_t in the category $\mathcal{S}_{\mathsf{ps},n+1}$, which by Lemma 71 is a lift for the admissible pair (f, g). We have thus proved that $\mathcal{S}_{\mathsf{ps},n+1}$ is a contravariant globular extension which contains a lift for all pairs in F_n. We now show that this extension is universal: consider another extension $F : \mathcal{S}_{\mathsf{ps},n} \to C$ that defines a lift for all the pairs in F_n, we show that there exists a unique $\tilde{F}$ that preserves the chosen lifts and makes the following diagram commute

$$\begin{array}{ccc} \mathcal{S}_{\mathsf{ps},n} & \xrightarrow{F} & C \\ \big\downarrow & \nearrow_{\tilde{F}} & \\ \mathcal{S}_{\mathsf{ps},n+1} & & \end{array}$$

Indeed, the map $\tilde{F}$ is already defined on all objects of $\mathcal{S}_{\mathsf{ps},n+1}$, and all maps of coherence depth less than n, so that it coincides with F, so it suffices to show that there is a unique extension to the maps of coherence depth $n+1$. Since all the objects in $\mathcal{S}_{\mathsf{ps},n+1}$ are globular products, it suffices to show this for the maps of the form $\Gamma \to D^n$. We can thus reformulate the condition by saying that it suffices to show that there is a unique map $\tilde{F}$ on terms, satisfying the condition $\tilde{F}(t[\gamma]) = \tilde{F}t \circ \tilde{F}\gamma$. We proceed by induction on the depth, noticing that a term of coherence depth $n+1$ cannot be a variable, hence we have already defined a unique value for $\tilde{F}$ on terms of depth 0, by our previous condition, and thus the induction is already initialized.

- For a term $\Delta \vdash \mathsf{op}_{\Gamma,A}[\gamma] : A[\gamma]$ of depth $d+1$, the value of F is uniquely determined by $\tilde{F}(\mathsf{op}_{\Gamma,A}[\gamma]) = \tilde{F}(\mathsf{op}_{\Gamma,A}[\mathrm{id}_\Gamma])\tilde{F}\gamma$, and since γ is of depth d, by induction $\tilde{F}(\gamma)$ is defined, and $\tilde{F}(\mathsf{op}_{\Gamma,A}[\mathrm{id}_\Gamma])$ is uniquely defined by the condition of preserving the lifts for the pairs in F_n.
- Similarly, for a term $\Delta \vdash \mathsf{coh}_{\Gamma,A}[\gamma] : A[\gamma]$ of depth $d+1$, the value of F is uniquely determined by $\tilde{F}(\mathsf{op}_{\Gamma,A}[\gamma]) = \tilde{F}(\mathsf{op}_{\Gamma,A}[\mathrm{id}_\Gamma])\tilde{F}\gamma$, and since γ is of depth d, by induction $\tilde{F}(\gamma)$ is defined, and $\tilde{F}(\mathsf{coh}_{\Gamma,A}[\mathrm{id}_\Gamma])$ is uniquely defined by the condition of preserving the lifts for the pairs in F_n.

This proves that there exists a unique $\tilde{F}$ satisfying the condition, and hence $\mathcal{S}_{\mathsf{ps},n+1}$ is the universal coglobular extension obtained by adding a lift for all arrows in F_n to $\mathcal{S}_{\mathsf{ps},n}$ □

This establishes a close correspondence between the categories $\mathcal{S}_{\mathsf{ps},n}$ and $\mathbf{\Theta}_n$, and enables us to prove the following theorem.

Theorem 73. *We have an equivalence of categories*

$$\mathcal{S}_{\mathsf{ps},\infty} \simeq \mathbf{\Theta}_\infty^{\mathrm{op}}$$

Proof. By construction $\mathcal{S}_{\mathsf{ps},\infty}$ is obtained as the colimit of the inclusions of categories

$$\mathcal{G}^{\mathrm{op}} \to \mathcal{S}_{\mathsf{ps},0} \to \mathcal{S}_{\mathsf{ps},1} \to \cdots \to \mathcal{S}_{\mathsf{ps},n} \to \cdots \to \mathcal{S}_{\mathsf{ps},\infty} = \mathrm{colim}_n\, \mathcal{S}_{\mathsf{ps},n}$$

It is therefore enough to prove that $\mathcal{S}_{\mathsf{ps},n}$ is equivalent to $\mathbf{\Theta}_n^{\mathrm{op}}$, which we do by induction.

- We have already proved that $\mathcal{S}_{\mathsf{ps},0}$ is equivalent to $\mathbf{\Theta}_0^{\mathrm{op}}$ in Theorem 53.

– Suppose that $\mathcal{S}_{\mathsf{ps},k}$ is equivalent to $\mathbf{\Theta}_k^{\mathrm{op}}$ for every $k \leq n$. Lemma 72 shows that $\mathcal{S}_{\mathsf{ps},n+1}$ is the universal contravariant globular extension that adds a lift for each pair in the set F_n. Moreover, the set F_n coincides with the set E_n defined in Section 3 and, by definition, $\mathbf{\Theta}_{n+1}$ is the universal globular extension. Hence $\mathcal{S}_{\mathsf{ps},n+1}$ and $\mathbf{\Theta}_{n+1}^{\mathrm{op}}$ satisfy the same universal property and are therefore equivalent. □

6 Models of CaTT

This section is dedicated to the study of the models of the type theory CaTT using tools that generalize the ones developed in Section 2.5. In particular, we prove an initiality result analogous to Theorem 36 for the category $\mathcal{S}_{\mathsf{CaTT}}$. We then apply this result to characterize the **Set**-models of the theory and prove that they are equivalent to the weak ω-categories in the sense of Grothendieck-Maltsiniotis, presented in Section 3. We also give a detailed syntactic interpretation of the construction that we develop here, showing that although it uses abstract categorical machinery, it translates closely the intuition coming from type theory.

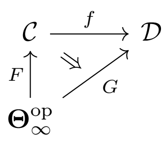
6.1 CaTT-categories with families.

In the case of the category GSeTT, we have introduced the notion of globular category with families, and proved that $\mathcal{S}_{\mathsf{GSeTT}}$ is initial among them (Theorem 36), which implies that we can compute the semantics of this theory in any category with families. We further prove this result by defining the structure of a CaTT-*category with families*, which plays an analogue role for the theory $\mathcal{S}_{\mathsf{CaTT}}$. We denote D_P the functor $\mathcal{G}^{\mathrm{op}} \to \mathbf{\Theta}_\infty^{\mathrm{op}}$, which defines the disk objects in the category $\mathbf{\Theta}_\infty^{\mathrm{op}} \simeq \mathcal{S}_{\mathsf{ps},\infty}$. It is the corestriction of the functor $D^\bullet$ to ps-contexts.

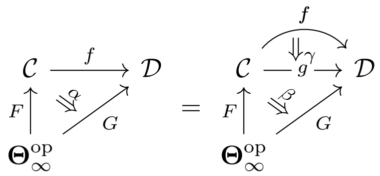
Definition 74. A CaTT-category with families is a globular category with families $\mathcal{C}$ together with a functor $F : \mathbf{\Theta}_\infty^{\mathrm{op}} \to \mathcal{C}$ sending globular sums to globular products, such that $G_{\mathcal{C}} = FD_P$.

Our main example of a CaTT-category with families is the syntactic category $\mathcal{S}_{\mathsf{CaTT}}$. The associated functor, that we write $P_\infty : \mathbf{\Theta}_\infty^{\mathrm{op}} \to \mathcal{S}_{\mathsf{CaTT}}$, is given by the inclusion $\mathcal{S}_{\mathsf{ps},\infty} \to \mathcal{S}_{\mathsf{CaTT}}$, together with the identification given by Theorem 73. In fact a CaTT-category with families can be thought of as a category with families which supports a type $\star$ along with all its iterated types $\to$, and for which the term constructor op and coh exist, like in the theory CaTT. From now on, we use Theorem 73 implicitly to identify the categories $\mathcal{S}_{\mathsf{ps},\infty}$ and the categories $\mathbf{\Theta}_\infty^{\mathrm{op}}$ and we think of an object of $\mathbf{\Theta}_\infty$ as a ps-context, and of a map $\gamma \in \mathbf{\Theta}_\infty(\Gamma, \Delta)$ as a substitution $\Delta \vdash \gamma : \Gamma$ in CaTT. Lemma 57, lets us think of maps $f : \mathbf{\Theta}_\infty(D^n, \Gamma)$ as terms in the ps-context Γ.

Morphisms of CaTT-categories with families. A morphism between two CaTT-categories with families $F : \mathbf{\Theta}_\infty^{\mathrm{op}} \to \mathcal{C}$ and $G : \mathbf{\Theta}_\infty^{\mathrm{op}} \to \mathcal{D}$ is a morphism of categories with families $f : \mathcal{C} \to \mathcal{D}$ together with a natural transformation



We denote **cCwF** the category of CaTT-categories with families defined this way. We also define a 2-cell of CaTT-category with families between two morphisms $(f, \alpha), (g, \beta) : \mathcal{C} \to \mathcal{D}$ to be a natural transformation $\gamma : f \Rightarrow g$ such that



We denote **<u>cCwF</u>** the 2-category obtained this way.

The two nerve functors. Given a CaTT-category with families $F : \Theta_\infty^{\mathrm{op}} \to \mathcal{C}$. We define its associated nerve functor N_F.

$$\begin{aligned} N_F : \mathcal{C} &\to \widehat{\Theta_\infty} \\ \Gamma &\mapsto \mathcal{C}(\Gamma, F\_) \end{aligned}$$

Recall that the nerve functor associated to FD_P plays the role of classifying terms, as per the theory of globular categories with families. We denote it T_F.

$$\begin{aligned} T_F : \mathcal{C} &\to \widehat{\mathcal{G}} \\ \Gamma &\mapsto \mathcal{C}(\Gamma, FD_P\_) \end{aligned}$$

In the case of the CaTT-category with families $\mathcal{S}_{\mathsf{CaTT}}$, we simply denote N and T these two functors. Our aim is to reproduce the arguments given in Section 2.5 to show the equivalence between CaTT-category with families structures and morphism from the syntactic category $\mathcal{S}_{\mathsf{CaTT}}$. In Section 2.5, the construction relies on the fact that the functor $D^\bullet$ is fully faithful and its associated nerve functor sends limits onto colimits. In our case, the functor P_∞ is also fully faithful, but there is no reason for its nerve functor N to send limits onto colimits. We introduce the concept of algebraic natural transformation and show in Lemma 81 that they satisfy a condition of preservation of pullbacks along display maps as a workaround for N not preserving limits.

6.2 Algebraic natural transformations.

Consider two CaTT-categories with families given by $F : \Theta_\infty^{\mathrm{op}} \to \mathcal{C}$ and $G : \Theta_\infty^{\mathrm{op}} \to \mathcal{D}$, along with an object Γ in $\mathcal{C}$ and an object Δ in $\mathcal{D}$. We define a notion of *algebraic natural transformation* between $T_G\Delta$ and $T_F\Gamma$ in the category $\widehat{\mathcal{G}}$. This can be seen as a compatibility condition, and might seem ad-hoc at first, but the reason why we are interested in such transformations will be apparent in Proposition 79, and we provide in Section 6.5 a discussion showing that from the point of view of type theory, they are actually a very natural notion to consider.

Induced nerve transformation. Consider two CaTT-category with families $F : \Theta_\infty^{\mathrm{op}} \to \mathcal{C}$ and $G : \Theta_\infty^{\mathrm{op}} \to \mathcal{D}$, together with a natural transformation $\eta \in \widehat{\mathcal{G}}(T_G\Delta, T_F\Gamma)$. For every globular set X, η induces by composition, the following transformation natural in X:

$$\begin{aligned} \eta^\star : \widehat{\mathcal{G}}(X, T_G\Delta) &\to \widehat{\mathcal{G}}(X, T_F\Gamma) \\ \xi &\mapsto \eta\xi \end{aligned}$$

In the case where X is of the form $V\Theta$ for a ps-context $\Theta \in \Theta_0$, Proposition 37, we have $G \simeq \mathrm{Ran}_{D^\bullet}(GD^\bullet)$, and the characterization of Kan extensions given by Lemma 24, shows that we have the following two natural isomorphisms:

$$\widehat{\mathcal{G}}(V\Theta, T_G\Delta) \cong \mathcal{D}(\Delta, G\Theta) = (N_G\Delta)_\Theta$$
$$\widehat{\mathcal{G}}(V\Theta, T_F\Delta) \cong \mathcal{C}(\Gamma, F\Theta) = (N_F\Gamma)_\Theta$$

This construction is natural in $\Theta \in \mathbf{\Theta}_0$, hence $\eta^\star \in \widehat{\mathbf{\Theta}_0}(N_G\Delta, N_F\Gamma)$. We thus have constructed the following transformation, which is natural in both Δ and Γ

$$\_^\star : \widehat{\mathcal{G}}(T_G\Delta, T_F\Gamma) \to \widehat{\mathbf{\Theta}_0}(N_G\Delta, N_F\Gamma)$$

More explicitly, for $\gamma \in \mathcal{D}(\Delta, GX)$, the transformation $\eta^\star(\gamma)$ is characterized by

$$\text{For every } \chi \in \mathbf{\Theta}_0(D^n, X),\ (F\chi)\eta^\star(\gamma) = \eta((G\chi)\gamma)$$

Definition 75. A natural transformation $\eta \in \widehat{\mathcal{G}}(T_G\Delta, T_F\Gamma)$ is *algebraic* if

$$\text{For every } \theta \in \mathcal{D}(\Delta, G\Theta) \text{ and } \chi \in \mathbf{\Theta}_\infty(D^n, X),\ \eta((G\chi)\theta) = (F\chi)\eta^\star\theta$$

We write $\widehat{\mathcal{G}}(T_G\Delta, T_F\Gamma)_{\mathrm{alg}}$ the set of algebraic natural transformations between $T_G\Delta$ and $T_F\Gamma$

The algebraicity condition is a generalization of the defining equality of $\eta^\star$, required to hold on $\mathbf{\Theta}_\infty$ instead of only on $\mathbf{\Theta}_0$.

Algebraicity as a naturality condition. A natural transformation $\eta \in \widehat{\mathcal{G}}(T_G\Delta, T_F\Gamma)$ induces a natural transformation $\eta^\star \in \widehat{\mathbf{\Theta}_0}(N_G\Delta, N_F\Gamma)$. The following result shows that when η is algebraic, $\eta^\star$ satisfies a stronger naturality condition. This is the main motivation for the introduction of algebraicity.

Lemma 76. *If $\eta \in \widehat{\mathcal{G}}(T_G\Delta, T_F\Gamma)_{\mathrm{alg}}$ is an algebraic natural transformation, then $\eta^\star$ defines a natural transformation $\eta^\star \in \widehat{\mathbf{\Theta}_\infty}(N_G\Delta, N_F\Gamma)$.*

Proof. We have already defined, for every element γ of the presheaf $N_G\Delta$ an element $\eta^\star(\gamma)$ of the presheaf $N_F\Gamma$, and it is enough to verify that it induces a natural transformation between the presheaves $N_G\Delta$ and $N_F\Gamma$ over $\mathbf{\Theta}_\infty$. Consider an element γ of $N_G(\Delta)$, i.e., $\gamma \in \mathcal{D}(\Delta, GX)$, and recall that $\eta^\star(\gamma)$ is defined to be the transformation such that for every map $\chi \in \mathbf{\Theta}_0(D^n, X)$, we have $(F\chi)\eta^\star(\gamma) = \eta((G\chi)\gamma)$. Given a map $f \in X \to Y$ in $\mathbf{\Theta}_\infty$, we have

$$\begin{aligned}
(F\chi)(Ff)\eta^\star(\gamma) &= F(\chi f)\eta^\star(\gamma) & \\
&= \eta(G(\chi f)\gamma) & \text{by algebraicity} \\
&= \eta((G\chi)(Gf)\gamma) & \\
&= F\chi\eta^\star((Gf)\gamma) & \text{by definition of } \eta^\star
\end{aligned}$$

Hence, for every variable x derivable in the context Δ in the theory $\mathcal{S}_{\mathsf{GSeTT}}$, we have

$$(F\chi_x)(Ff)\eta^\star(\gamma) = (F\chi)\eta^\star((Gf)\gamma)$$

and thus we have the equality

$$(Ff)\eta^\star(\gamma) = \eta^\star(Gf\gamma)$$

This proves the commutativity of the following square

$$\begin{array}{ccc} \mathcal{D}(\Delta, GY) & \xrightarrow{\eta^\star_Y} & \mathcal{C}(\Gamma, FY) \\ {\scriptstyle Gf\circ\_}\downarrow & & \downarrow{\scriptstyle Ff\circ\_} \\ \mathcal{D}(\Delta, GX) & \xrightarrow[\eta^\star_X]{} & \mathcal{C}(\Gamma, FX) \end{array}$$

which concludes the proof □

Algebraicity of the nerve transformations. We now show the converse: Any given natural transformation $\eta \in \widehat{\mathbf{\Theta}_\infty}(N_G\Delta, N_F\Gamma)$ induces a natural transformation $\eta \in \widehat{\mathcal{G}}(T_G\Delta, T_F\Gamma)$ by restriction along D_P. We show that this transformation is algebraic.

Lemma 77. *Consider a natural transformation $\eta \in \widehat{\mathbf{\Theta}_\infty}(N_G\Delta, N_F\Gamma)$ along with its restriction $\overline{\eta} \in \widehat{\mathcal{G}}(T_G\Delta, T_F\Gamma)$. The induced natural transformation $\overline{\eta}^\star \in \widehat{\mathbf{\Theta}_0}(N_G\Delta, N_F\Gamma)$ coincides with η.*

Proof. Consider an element $\delta \in N_G\Delta$, i.e., δ is a map $\delta : \Delta \to GX$ in the category $\mathcal{D}$ for some object X of $\mathbf{\Theta}_0$. Then $\overline{\eta}^\star(\delta)$ is defined to be the unique map such that for every map $\chi : X \to D^n$ in $\mathbf{\Theta}_0^{\mathrm{op}}$ we have $(F\chi)\overline{\eta}^\star(\delta) = \overline{\eta}((G\chi)\delta)$. The naturality of the transformation η ensures that $(F\chi)\eta(\delta) = \eta((G\chi)\delta)$ for every map χ in the category $\mathbf{\Theta}_\infty$. In particular, this is satisfied for maps in $\mathbf{\Theta}_0$, and hence η satisfies the defining property of $\overline{\eta}^\star$, and hence $\eta = \overline{\eta}^\star$. □

Lemma 78. *For every natural transformation $\eta \in \widehat{\mathbf{\Theta}_\infty}(N_G\Delta, N_F\Gamma)$, the induced natural transformation $\eta \in \widehat{\mathcal{G}}(T_G\Delta, T_F\Gamma)$ is algebraic.*

Proof. By Lemma 77, the algebraicity condition rewrites as $\eta((Gf)\theta) = (Ff)\eta(\theta)$ for every map $\theta \in \mathcal{D}(\Delta, G\Theta)$ and every map $f \in \mathbf{\Theta}_\infty(D^n, \Theta)$. This is given by the naturality of η with respect to $\mathbf{\Theta}_\infty$. □

The equivalence. Combining Lemma 76 and Lemma 78, we have

Proposition 79. *The two operations defined above form a natural isomorphism*

$$\widehat{\mathcal{G}}(T_G\Delta, T_F\Gamma)_{\mathrm{alg}} \cong \widehat{\mathbf{\Theta}_\infty}(N_G\Delta, N_F\Gamma)$$

Proof. We have already proved in Lemmas 76 and 78 that these operations are well-defined, and moreover Lemma 77 shows that the induced transformation of a restriction is the transformation itself. So it suffices to show that restricting an induced algebraic natural transformation also yields the identity. Consider an algebraic natural transformation $\eta \in \widehat{\mathcal{G}}(T_G\Delta, T_F\Gamma)$. By definition, for every object X in $\mathbf{\Theta}_0$ and for all maps $\delta \in \mathcal{D}(\Delta, GX)$ and $\chi \in \mathbf{\Theta}_0(D^n, X)$, we have the equality $\eta((G\chi)\delta) = (F\chi)\eta^\star(\delta)$. In particular, taking $X = D^n$ and χ to be the identity yields $\eta(\delta) = \eta^\star(\delta)$. Hence $\eta^\star$ coincides with η on the presheaf $T_G\Delta$, and thus the induction and restriction operation are inverse operations. □

Algebraic natural transformations that agree on the variables. An important property of algebraic natural transformations, is that their value is entirely determined by their values on the variables of the theory. This is similar to substitutions, and indeed, we show later that this notion captures exactly the computation of the substitutions.

Lemma 80. *Consider a CaTT-category with families $F : \mathbf{\Theta}_\infty^{\mathrm{op}} \to \mathcal{C}$ along with a context Γ in $\mathcal{S}_{\mathsf{CaTT}}$ and an object Δ in $\mathcal{C}$. Two algebraic natural transformations $\eta, \eta' \in \widehat{\mathcal{G}}(T\Gamma, T_F\Delta)_{\mathrm{alg}}$ are equal if and only if we have $\eta(\chi_x) = \eta'(\chi_x)$ for every variable x in Γ.*

Proof. If two algebraic natural transformations are equal, then they necessarily agree on the characteristic maps of the variables, so it suffices to check the converse. Consider two algebraic natural transformations $\eta, \eta' \in \widehat{\mathcal{G}}(T\Gamma, T_F\Delta)_{\mathrm{alg}}$ that coincide on all the variables. We want to prove that they are equal. For this we show by induction on the depth of the term t that for any derivable term t in Γ, we have $\eta(\chi_t) = \eta'(\chi_t)$.

- Terms of depth 0 are simply variables, and for those, we have the equality by hypothesis.
- A term t of depth $d+1$ is of the form $\mathsf{op}_{\Theta,B}[\theta]$ or of the form $\mathsf{coh}_{\Theta,B}[\theta]$, with θ a substitution of depth at most d. In this case we consider the term t' to be $t' = \mathsf{op}_{\Theta,B}[\mathrm{id}_\Theta]$ or $t' = \mathsf{coh}_{\Theta,B}[\mathrm{id}_\Theta]$ respectively. This provides a factorization of the form $\chi_t = \chi_{t'}\theta$. Since η and η' are algebraic, we therefore have

  $$\eta(\chi_t) = (F(\chi_{t'}))\eta^\star(\theta) \qquad\qquad \eta'(\chi_t) = (F(\chi_{t'}))\eta'^\star(\theta)$$

  Moreover, for every variable y of Θ, since θ is of depth at most d, so is $y[\theta]$, and therefore, by induction, we have the following equalities:

  $$\begin{aligned}(F(\chi_y))\eta'^\star(\theta) &= \eta'(\chi_{y[\theta]})\\ &= \eta(\chi_{y[\theta]})\\ &= F(\chi_y)\eta^\star(\theta)\end{aligned}$$

  Thus $\eta'^\star(\theta)$ satisfies the defining property of $\eta^\star(\theta)$, and hence $\eta'^\star(\theta) = \eta^\star(\theta)$, which proves that $\eta(\chi_t) = \eta'(\chi_t)$. ☐

Algebraic transformations and pullbacks along display maps. We can compute the algebraic natural transformations mapping out of the nerve of a context extension. This is the main argument making algebraic natural transformation easy to compute with, is enough to compensate for the functor N not sending limits to colimits. We first present a construction needed to state the result. Consider a context Γ together with a derivable type $\Gamma \vdash A$ in CaTT, and a CaTT-category with families $F : \mathbf{\Theta}_\infty^{\mathrm{op}} \to \mathcal{C}$. By Lemma 57, A is classified by a substitution $\chi_A : \Gamma \to S^{n-1}$ in the category $\mathcal{S}_{\mathsf{CaTT}}$. By Lemma 24, this substitution gives rise to a natural transformation in $\widehat{\mathcal{G}}(VS^{n-1}, T\Gamma)$, and by precomposition, it induces a map $\widehat{\mathcal{G}}(T\Gamma, T_F\Delta) \to \widehat{\mathcal{G}}(VS^{n-1}, T_F\Delta)$. Applying Lemma 24 again allows us to rewrite this map as $f_A : \widehat{\mathcal{G}}(T\Gamma, T_F\Delta) \to \mathcal{C}(\Delta, FS^{n-1})$. Following the construction explicitly shows that for a natural transformation $\eta \in \widehat{\mathcal{G}}(T\Gamma, T_F\Delta)$ the map $f_A(\eta)$ is defined by the property following property

For any map $\chi \in \mathcal{S}_{\mathsf{GSeTT}}(S^{n-1}, D^k)$ (i.e., any variable in S^{n-1}), $(F\chi)f_A(\eta) = \eta(\chi\chi_A)$

In particular, considering a context of the form $(\Gamma, x : A)$, we have the type $\Gamma \vdash A$. The projection substitution $\pi : (\Gamma, x : A) \to \Gamma$ induces a morphism $T\pi : T\Gamma \to T(\Gamma, x : A)$ which can be thought of as a weakening. Using the commutation of the following pullback square

$$\begin{array}{ccc}(\Gamma, x : A) & \xrightarrow{\chi_x} & D^n\\ {\scriptstyle\pi}\downarrow & \lrcorner & \downarrow{\scriptstyle\partial_n}\\ \Gamma & \xrightarrow[\chi_A]{} & S^{n-1}\end{array}$$

The defining property of $f_A(\eta T\pi)$ rewrites as

For every map $\chi \in \mathcal{S}_{\mathsf{GSeTT}}(S^{n-1}, D^k)$, $(F\chi)f_A(\eta(T\pi)) = \eta(\chi\partial_n\chi_x)$

Lemma 81. *$\widehat{\mathcal{G}}(T(\Gamma, x : A), T_F\Delta)_{\mathrm{alg}}$ is obtained as the following pullback*

$$\begin{array}{ccc} \widehat{\mathcal{G}}(T(\Gamma, x : A), T_F\Delta)_{\mathrm{alg}} & \longrightarrow & (T_F\Delta)_n \\ \downarrow & \lrcorner & \downarrow \\ \widehat{\mathcal{G}}(T\Gamma, T_F\Delta)_{\mathrm{alg}} & \longrightarrow & \mathcal{C}(\Delta, FS^{n-1}) \end{array}$$

More, explicitly, there is an isomorphism as follows

$$\widehat{\mathcal{G}}(T(\Gamma, x : A), T_F\Delta)_{\mathrm{alg}} \cong \left\{(\eta, \chi_t) \in \widehat{\mathcal{G}}(T\Gamma, T_F\Delta)_{\mathrm{alg}} \times \mathcal{C}(\Delta, FD^n) \mid \partial_n\chi_t = f_A(\eta)\right\}$$

Proof. Consider a context $(\Gamma, x : A)$ in the theory CaTT and, in order to simplify notations, define the set

$$X = \left\{(\eta, \chi) \in \widehat{\mathcal{G}}(T\Gamma, T_F\Delta)_{\mathrm{alg}} \times \mathcal{C}(\Delta, FD^n) \mid \partial_n\chi = f_A(\eta)\right\}$$

We consider the following map:

$$\begin{aligned} \widehat{\mathcal{G}}(T(\Gamma, x : A), T_F\Delta)_{\mathrm{alg}} &\to X \\ \eta &\mapsto (\eta(T\pi), \eta(\chi_x)) \end{aligned}$$

We show that this map is well defined, i.e., we show that for every $\eta \in \widehat{\mathcal{G}}(TD^n, T_F\Delta)_{\mathrm{alg}}$, we have $(\eta(T\pi), \eta(\chi_x)) \in X$. First note that η and $\eta(T\pi)$ act in the same way on every term on which they are both defined, but $\eta(T\pi)$ is defined on strictly fewer terms than η. Hence, since η is algebraic, $\eta(T\pi)$ is also necessarily so. Since, by definition, F is a CaTT-category with families, it provides $\mathcal{C}$ with a structure of globular category with families, and we have that $F\partial_n = \partial_n$. Hence, for every map $\chi \in \mathcal{S}_{\mathsf{GSeTT}}(S^{n-1}, D^k)$, we have, by naturality of η:

$$F\chi\partial_n\eta(\chi_x) = F(\chi\partial_n)\eta(\chi_x) = \eta(\chi\partial_n\chi_x)$$

Hence $\partial_n\eta(\chi_x)$ satisfies the defining property of $f_A(\eta(T\pi))$. Hence $(\eta(T\pi), \eta(\chi_x)) \in X$.

We prove that this mapping is a bijective. Consider a pair $(\eta, \chi) \in X$, an algebraic natural transformation η' mapping onto this pair has its action on the variables determined. Indeed, a variable in $(\Gamma, x : A)$ is either the variable x, or it is a variable of Γ. For the variable x, we have, by definition of η', that $\eta'(\chi_x) = \chi$, and for a variable y in Γ, then we have the factorisation $\chi_y = \chi_y\pi$ and so $\eta'(\chi_y) = \eta(\chi_y)$. By Lemma 80 this proves that the mapping is injective.

Conversely, we show that this mapping is surjective. We construct a natural transformation $\eta' \in \widehat{\mathcal{G}}(T(\Gamma, x : A), T_F\Delta)_{\mathrm{alg}}$ which extends the algebraic natural transformation η. First for any term t in $(\Gamma, x : A)$ which does not use the variable x, the term t is also definable in Γ, and we define $\eta'(\chi_t) = \eta(\chi_t)$. So it suffices to define the natural transformation η' on the terms in $(\Gamma, x : A)$ that contain the variable x, and to verify the naturality and algebraicity of η' on those terms. We proceed by induction on the coherence depth of the term.

- The term containing x of minimal coherence depth is necessarily the variable x itself, and in this case we define $\eta'(\chi_x) = \chi$. This assignment is natural on the variable x by definition of the set X.
- Suppose $\eta' \in \widehat{\mathcal{G}}(T_d\Gamma, T_F\Delta)$ to be defined and natural on all terms containing the variable x of coherence depth at most d, and consider a term t of depth $d + 1$. Then t is necessarily of the form $t = \mathsf{op}_{\Theta,B}[\theta]$ (resp. $t = \mathsf{coh}_{\Theta,B}[\theta]$), and we define $t' = \mathsf{op}_{\Theta,B}[\mathrm{id}_\Theta]$ (resp. $t' = \mathsf{coh}_{\Theta,B}[\mathrm{id}_\Theta]$), in such a way that $\chi_t = \chi'_t\theta$. Then note that θ is of coherence

depth at most d, and hence defines a natural transformation in $\mathcal{G}^{\mathrm{op}}(V\Theta, T_d\Gamma)$, hence by composition with η', this provides a natural transformation in $\widehat{\mathcal{G}}(V\Theta, T_F\Delta)$ which gives a morphism $\eta'^{\star}(\theta) : \Delta \to F\Theta$. We then define $\eta'(\chi_t) = F(\chi_{t'})(\eta'^{\star}(\theta))$. We check that the transformation defined this way is natural. Consider a variable y in the context D^n, corresponding to a morphism $\chi_y : D^k \to D^n$ in the category $\mathcal{G}$, and a term t of coherence depth $d+1$ in the context $(\Gamma, x : A)$ that uses the variable x, and denote t' and θ as above. We have the equalities

$$\begin{aligned} F(\chi_y)\eta'(\chi_t) &= F(\chi_y)F(\chi_{t'})\eta'^{\star}(\theta) \\ &= F(\chi_y\chi_{t'})\eta'^{\star}(\theta) \end{aligned}$$

If $\chi_y\chi_{t'} = \chi_z$, where z is a variable, then we have $\chi_y\chi_t = \chi_z\theta$, and by definition of $\eta'^{\star}(\theta)$, we have $F(\chi_z)\eta'^{\star}(\theta) = \eta'(\chi_z\theta)$. If $\chi_y\chi_{t'} = \chi_u$ where u is not a variable, it is again of the form $\mathsf{op}_{\Xi,C}[\xi]$ (resp. $\mathsf{coh}_{\Xi,C}[\xi]$), and we denote $u' = \mathsf{op}_{\Theta',C}[\mathrm{id}]$ (resp. $u' = \mathsf{coh}_{\Theta',C}[\mathrm{id}]$) in such a way that $\chi_u = \chi'_u\xi$. In this case, we have $\chi_y\chi_t = \chi_{u'}\xi\theta$, and thus we have

$$\begin{aligned} \eta'(\chi_y\chi_t) &= (F(\chi_{u'}))\eta'^{\star}(\xi\theta) & \\ &= F(\chi_{u'})F(\xi)\eta'^{\star}(\theta) & \text{by naturality of } \eta'^{\star} \\ &= F(\chi_y\chi_{t'})\eta'^{\star}(\theta) & \end{aligned}$$

In both cases, we have $\eta'(\chi_y\chi_t) = F(\chi_y)\eta'(\chi_t)$ which proves that η' is natural on χ_t.

We now prove that the natural transformation we have just defined is a preimage of the couple (η, χ), and note that by definition, we have $\eta' T\pi = \eta$ and $\eta'(\chi_x) = \chi$, so it suffices to show that η' is algebraic. Consider a ps-context Θ together with a map $\theta : (\Gamma, x : A) \to \Theta$, and a map $\xi \in \mathbf{\Theta}_\infty(D^n, \Theta)$. The map ξ corresponds to a term in the ps-context Θ which is either a variable or of the form $\mathsf{op}_{\Theta',B}[\xi']$ (resp. $\mathsf{coh}_{\Theta',B}[\xi']$). If ξ defines a variable, the equality required for the algebraicity is implied by the naturality, so it suffices to verify it for the term constructors. We define $t' = \mathsf{op}_{\Theta',B}[\mathrm{id}_{\Theta'}]$ (resp. $\mathsf{coh}_{\Theta',B}[\mathrm{id}'_{\Theta'}]$), in such a way that we have $\xi = \chi_{t'}\xi'$. We then have the following equalities

$$\begin{aligned} \eta'(\xi\theta) &= F(t')\eta'^{\star}(\xi'\theta) & \\ &= F(t')F(\xi')\eta'^{\star}(\theta) & \text{by naturality of } \eta'^{\star} \\ &= F(\xi)\eta'^{\star}(\theta) & \end{aligned}$$

□

6.3 Kan extension of a CaTT-category with families.

Using algebraic natural transformations, we define and characterize the right Kan extension of a CaTT-category with family $F : \mathbf{\Theta}_\infty^{\mathrm{op}} \to \mathcal{C}$ along the functor $P_\infty : \mathbf{\Theta}_\infty^{\mathrm{op}} \to \mathcal{S}_{\mathsf{CaTT}}$. Like in Section 2.5, this right Kan extension is the key construction to prove the initiality of the syntactic category. In the case of CaTT, the presence of term constructors makes the existence harder to prove, as witnessed by the introduction of algebraic natural transformations.

Existence of the Kan extension. The previous results show that algebraic natural transformations can be built inductively following the structure of contexts, starting with the empty contexts and computed with a sequence of context comprehension operations. This lets us define and characterize the right Kan extension of any CaTT-category with families $\mathcal{C}$ along the functor P_∞, by proving that all the canonical diagrams of objects in $\mathcal{S}_{\mathsf{CaTT}}$ necessarily have a limit in $\mathcal{C}$.

Lemma 82. *Given a* CaTT*-category with families* $F : \mathbf{\Theta}_\infty^{\mathrm{op}} \to \mathcal{C}$*, there exists a pointwise right Kan extension* $\mathrm{Ran}_{P_\infty} F : \mathcal{S}_{\mathsf{CaTT}} \to \mathcal{C}$*. Moreover,* $(\mathrm{Ran}_{P_\infty} F)\varnothing$ *is the terminal object of* $\mathcal{C}$*, and* $\mathrm{Ran}_{P_\infty} F$ *satisfies the equation* $(\mathrm{Ran}_{P_\infty} F)(\Gamma, A) \cong ((\mathrm{Ran}_{P_\infty} F)\Gamma, \partial_n((\mathrm{Ran}_{P_\infty} F)\chi_A))$.

Proof. By Lemma 24 The existence of the pointwise Kan extension is equivalent to showing that for every context Γ in $\mathcal{S}_{\mathsf{CaTT}}$ there is a natural isomorphism $\mathcal{C}(\Delta, X) \cong \widehat{\mathbf{\Theta}_\infty}(N\Gamma, N_F\Delta)$, so it suffices to construct an object X which satisfies this property. Proposition 79 lets us rewrite as the above isomorphism

$$\mathcal{C}(\Delta, X) \cong \widehat{\mathcal{G}}(T\Gamma, T_F\Delta)_{\mathrm{alg}}$$

We proceed by induction on Γ to show that there exists an object X satisfying this property.

- For the context $\varnothing$, an element of the presheaf $T\varnothing$ is a substitution $\varnothing \vdash \gamma : D^n$, so by Lemma 57 it is necessarily of the form χ_t where t is a term in the context $\varnothing$. Since by Lemma 58 there is no such term, this implies that there is no element in $T\varnothing$, and thus it is the empty globular set, that is initial. Moreover the only natural transformation $! : T\varnothing \to T_F\Delta$ is vacuously algebraic. Hence $\widehat{\mathcal{G}}(T\Gamma, T_F\Delta)_{\mathrm{alg}} = \{\bullet\}$ is a singleton. So the limit of the diagram is the terminal object in $\mathcal{C}$, which exists by definition of a category with families.
- For a context of the form $(\Gamma, x : A)$, assume that there is an object Y in $\mathcal{C}$, together with a natural isomorphism $\mathcal{C}(\Delta, Y) \cong \widehat{\mathcal{G}}(T\Gamma, T_F\Delta)_{\mathrm{alg}}$. We can apply Lemma 81, which provides the following equalities

$$\begin{aligned}
\widehat{\mathcal{G}}(T(\Gamma, A), T_F\Delta)_{\mathrm{alg}} &\cong \lim\left(\widehat{\mathcal{G}}(T\Gamma, T_F\Delta)_{\mathrm{alg}} \longrightarrow \mathcal{C}(\Delta, FS^{n-1}) \xleftarrow{\mathcal{C}(\Delta,\partial_n)} \mathcal{C}(\Delta, Fn)\right) \\
&\cong \lim\left(\mathcal{C}(\Delta, Y) \longrightarrow \mathcal{C}(\Delta, FS^{n-1}) \xleftarrow{\mathcal{C}(\Delta,\partial_n)} \mathcal{C}(\Delta, Fn)\right) \\
&\cong \mathcal{C}\left(\Delta, \lim\left(Y \xrightarrow{f} FS^{n-1} \xleftarrow{\partial_n} Fn\right)\right)
\end{aligned}$$

  By definition of the structure of a globular category with families, the above limit in $\mathcal{C}$ exists and can be computed as $X = (Y, \partial_n(f))$. This choice of X by definition is an object such that $\mathcal{C}(\Delta, X) \cong \widehat{\mathcal{G}}(T\Gamma, T_F\Delta)_{\mathrm{alg}}$, hence it defines a limit for the canonical diagram associated to Γ.

The construction of the object $X = \mathrm{Ran}_{P_\infty}(\Gamma)$ shows that the right Kan extension sends the terminal to the terminal and satisfies the required equation. ☐

Considering a CaTT-category with families $F : \mathbf{\Theta}_\infty^{\mathrm{op}} \to \mathcal{C}$, the right Kan extension $\mathrm{Ran}_{P_\infty} F$ exists, and since P_∞ is fully faithful, Lemma 26 shows that the universal natural transformation $\epsilon(F) : (\mathrm{Ran}_{P_\infty} F)P_\infty \Rightarrow F$ is a natural isomorphism.

Functors preserving pullbacks along display maps. We have proven that restricting the right Kan extension yields a functor equivalent to the one from which we started. Conversely, we now show that every functor preserving pullbacks along display maps is the Kan extension of its restriction.

Lemma 83. *Given a cartegory* $\mathcal{C}$ *and a functor* $F : \mathcal{S}_{\mathsf{CaTT}} \to \mathcal{C}$ *that preserves the terminal object and sends pullbacks along the projection maps* $\left\{\partial_n : D^n \to S^{n-1}\right\}$ *onto pullbacks, then* F *is the pointwise right Kan extension* $F \cong \mathrm{Ran}_{P_\infty} FP_\infty$*, and in particular this Kan extension exists.*

Proof. The proof is the same as the one of Lemma 29, but with $\widehat{\mathcal{G}}(T\_, T_{FP_\infty}\_)_{\mathrm{alg}}$ playing the role of $\widehat{\mathcal{G}}(V\_, T_F\_)$. If it exists, the right Kan extension $(\mathrm{Ran}_{P_\infty} FP_\infty)\Gamma$ is characterized by the fact that for all Δ, we have $\mathcal{C}(\Delta, (\mathrm{Ran}_{P_\infty} FP_\infty)\Gamma) \cong \widehat{\mathbf{\Theta}_\infty}(N\Gamma, N_{FP_\infty}\Delta)$. We prove that $F(\Gamma)$ satisfies this equation. First, it holds for the disk contexts D^n, since P_∞ is fully faithful, ND^n is the representable presheaf associated to $D^n \in \mathbf{\Theta}_\infty^{\mathrm{op}}$, and by the Yoneda lemma, $\widehat{\mathbf{\Theta}_\infty}(ND^n, N_{FP_\infty}(\Delta)) \cong N_{FP_\infty}(\Delta)_{D^n} = \mathcal{C}(\Delta, FD^n)$. We now prove this equation for the sphere contexts by induction

- The -1 sphere is the empty context and the property follows immediately from the preservation of terminal object.
- Assume that we have proved the property for S^{n-1}, then we have, using Lemma 81

$$
\begin{aligned}
\mathcal{C}(\Delta, S^n) &= \mathcal{C}(\Delta, \lim(D^n \to S^{n-1} \leftarrow D^n)) \\
&\cong \lim(\mathcal{C}(\Delta, D^n) \to \mathcal{C}(\Delta, S^{n-1}) \leftarrow \mathcal{C}(\Delta, D^n)) \\
&\cong \lim(\widehat{\mathbf{\Theta}_\infty}(ND^n, N_{FP_\infty}\Delta) \to \widehat{\mathbf{\Theta}_\infty}(NS^{n-1}, N_{FP_\infty}\Delta) \leftarrow \widehat{\mathbf{\Theta}_\infty}(ND^{n1}, N_{FP_\infty}\Delta)) \\
&\cong \lim(\widehat{\mathcal{G}}(TD^n, T_{FP_\infty}\Delta)_{\mathrm{alg}} \to \widehat{\mathcal{G}}(TS^{n-1}, T_{FP_\infty}\Delta)_{\mathrm{alg}} \leftarrow \widehat{\mathcal{G}}(TD^n, T_{FP_\infty}\Delta)_{\mathrm{alg}}) \\
&\cong \widehat{\mathcal{G}}(TS^n, T_{FP_\infty}\Delta)
\end{aligned}
$$

We now prove this same equation for any context by induction on the length.

- For the empty context, this is by preservation of the terminal object.
- For an extended context of the form Γ, A, a similar computation to the sphere case shows that

$$
\begin{aligned}
\mathcal{C}(\Delta, (\Gamma, A)) &= \mathcal{C}(\Delta, \lim(\Gamma \to S^{n-1} \leftarrow D^n)) \\
&\cong \lim(\widehat{\mathcal{G}}(T\Gamma, T_{FP_\infty}\Delta)_{\mathrm{alg}} \to \widehat{\mathcal{G}}(TS^{n-1}, T_{FP_\infty}\Delta)_{\mathrm{alg}} \leftarrow \widehat{\mathcal{G}}(TD^n, T_{FP_\infty}\Delta)_{\mathrm{alg}}) \\
&\cong \widehat{\mathcal{G}}(T(\Gamma, A), T_{FP_\infty}\Delta)
\end{aligned}
$$

□

6.4 Initiality and models of the theory CaTT.

From now on, the arguments that we give follow closely the structure of our study of the models of GSeTT given in Section 2.5.

Initiality of $\mathcal{S}_{\mathsf{CaTT}}$. We start by proving that the category $\mathcal{S}_{\mathsf{CaTT}}$ is initial among CaTT-categories with families, in the same sense that of Theorem 36.

Lemma 84. *For a CaTT-category with families $F : \mathbf{\Theta}_\infty^{\mathrm{op}} \to \mathcal{C}$, there is a morphism of categories with families*

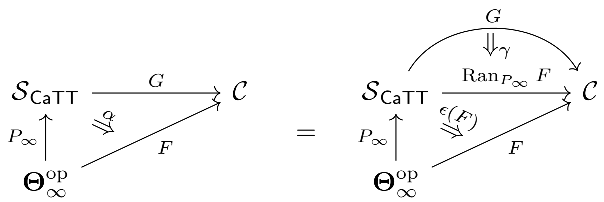
$$(\mathrm{Ran}_{P_\infty} F, \epsilon(F)) : \mathcal{S}_{\mathsf{CaTT}} \to \mathcal{C}$$

where the transformation $\epsilon(F) : (\mathrm{Ran}_{P_\infty} F)P_\infty \Rightarrow F$ is a natural isomorphism.

Proof. Lemma 82 shows that the right Kan extension exists and preserves the terminal object and the pullbacks along display maps (provided by the equation concerning the context comprehension operation). Hence $\mathrm{Ran}_{P_\infty} F$ can be chosen uniquely to be a morphism of categories with families. Denote $\epsilon(F) : (\mathrm{Ran}_{P_\infty} F)P_\infty \Rightarrow F$ the universal natural transformation obtained as part of the Kan extension. Then, by definition, $(\mathrm{Ran}_{P_\infty} F, \epsilon(F))$ defines a morphism of CaTT-categories with families, and Lemma 26 shows that $\epsilon(F)$ is a natural isomorphism. □

Theorem 85 (local initiality of the syntactic category)**.** *The morphism of* CaTT*-categories with families* $(\mathrm{Ran}_{p_\infty} F, \epsilon(F)) : \mathcal{S}_{\mathsf{CaTT}} \to \mathcal{C}$ *is a terminal object in the category* $\underline{\mathbf{cCwF}}(\mathcal{S}_{\mathsf{CaTT}}, \mathcal{C})$*.*

Proof. This is exactly the universal property of the right Kan extension. Consider a morphism of CaTT-category with families $(G, \alpha) : \mathcal{S}_{\mathsf{CaTT}} \to \mathcal{C}$. The universal property of the right Kan extension lets us construct a natural transformation $\gamma : G \Rightarrow \mathrm{Ran}_{P_\infty} F$ such that we have the following equality:



Thus, γ is a natural transformation satisfying $\gamma : (G, \alpha) \to (\mathrm{Ran}_{P_\infty} F, \epsilon(F))$. □

Models of the theory CaTT. We use initiality (Theorem 85) to characterize the models of the theory CaTT. Denote $\mathcal{U} : \mathbf{gCwF} \to \mathbf{CwF}$ the forgetful functor, and consider, for a given category with families $\mathcal{C}$, the category $\mathcal{U}_{\mathrm{id}}^{-1}(\mathcal{C})$ whose objects are the CaTT-categories with families $\mathcal{D}$ such that $\mathcal{U}(\mathcal{D}) = \mathcal{C}$, morphisms are the morphisms of CaTT-categories with families which project onto $\mathrm{id}_{\mathcal{C}}$ by $\mathcal{U}$.

Proposition 86. *Consider a category with families* $\mathcal{C}$*, there is an equivalence of categories* $\underline{\mathbf{CwF}}(\mathcal{S}_{\mathsf{CaTT}}, \mathcal{C}) \simeq \mathcal{U}_{\mathrm{id}}^{-1}(\mathcal{C})$*.*

Proof. We build a pair of functors

$$\underline{\mathbf{CwF}}(\mathcal{S}_{\mathsf{CaTT}}, \mathcal{C}) \underset{\mathrm{Ran}_{P_\infty}}{\overset{P_\infty^*}{\rightleftarrows}} \mathcal{U}_{\mathrm{id}}^{-1}(\mathcal{C})$$

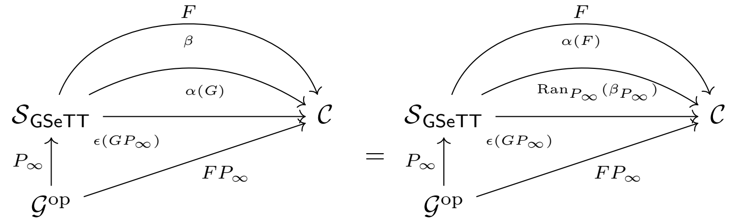
and show that they define an equivalence of categories.

Definition of the functor P_∞^*. This functor is given by precomposition. Given a morphism of categories with families $F : \mathcal{S}_{\mathsf{CaTT}} \to \mathcal{C}$, we define $P_\infty^*(F) = FP_\infty$, with the structure of globular category with families on $\mathcal{C}$ given by $D^*(FI)$ (where $I : \mathcal{S}_{\mathsf{GSeTT}} \to \mathcal{S}_{\mathsf{CaTT}}$ is the embedding). The pair (F, id) defines a morphism of CaTT-categories with families $\mathsf{CaTT} \to P_\infty^*(F)$.

Definition of the functor Ran_{P_∞}. This functor is given by the right Kan extension. Given a CaTT-category with families $G : \mathbf{\Theta}_\infty^{\mathrm{op}} \to \mathcal{C}$, it associates the morphism of categories with families $\mathrm{Ran}_{P_\infty} F$ which exists and is a morphism of categories with families by Lemma 84. Given a morphism $\alpha : G \Rightarrow G'$ in $\mathcal{U}_{\mathrm{id}}^{-1}(\mathcal{C})$, we have $\alpha\epsilon(G) : (\mathrm{Ran}_{P_\infty} G)P_\infty \Rightarrow G'$. By universal property of the Kan extension, there is a unique $\mathrm{Ran}_{P_\infty}(\alpha) : \mathrm{Ran}_{P_\infty} G \Rightarrow \mathrm{Ran}_{P_\infty} G'$ such that $\mathrm{Ran}_{P_\infty}(\alpha)\epsilon(G) = \epsilon(G')\alpha$.

Equivalence $P_\infty^* \mathrm{Ran}_{P_\infty} \simeq \mathrm{id}$. Consider a CaTT-category with families $G : \mathbf{\Theta}_\infty^{\mathrm{op}} \to \mathcal{C}$, then we have the map $\epsilon(G) : (\mathrm{Ran}_{P_\infty} G)P_\infty \Rightarrow G$, which is an isomorphism in $\mathcal{C}$. Given a map α in $\mathcal{U}_{\mathrm{id}}^{-1}(\mathcal{C})(G, G')$, by definition, $\mathrm{Ran}_{P_\infty}(\alpha)\epsilon(G) = \epsilon(G')\alpha$. This is exactly the naturality of ϵ.

Equivalence $\mathrm{id} \simeq \mathrm{Ran}_{P_\infty} P_\infty^*$. A morphism of category with families $F : \mathcal{S}_{\mathsf{CaTT}} \to \mathcal{C}$, defines a morphism of CaTT-categories with families $(F, \mathrm{id}) : \mathcal{S}_{\mathsf{GSeTT}} \to D^*(F)$. By Theorem 85, we have a natural transformation $\alpha(F) : F \to \mathrm{Ran}_{P_\infty}(FP_\infty)$, obtained by universal property of the right Kan extension. It is thus uniquely characterized by $\epsilon(F)\alpha(F)_{P_\infty} = \mathrm{id}$.Since $\epsilon(D^*(F))$ is an isomorphism, so is $\alpha(F)_{P_\infty}$. Consider the isomorphism $\gamma : F \cong \mathrm{Ran}_{P_\infty}(FP_\infty)$ obtained by Lemma 83. Then $(\alpha(F)\gamma^{-1})_{P_\infty}$ is an isomorphism, so by Lemma 27, so is $\alpha(F)\gamma^{-1}$, and thus so is $\alpha(F)$. We now show that the family $\alpha(F)$ is natural in F: Given two morphisms of categories with families $F, G : \mathcal{S}_{\mathsf{GSeTT}} \to \mathcal{C}$, we consider the two following diagram, whose compositions are both equal to β_{P_∞}, using the equations that characterize ϵ and $\mathrm{Ran}_{P_\infty}(\beta_{P_\infty})$.



By universality of the Kan extension, this shows the equation $\alpha(G)\beta = \mathrm{Ran}_{P_\infty}(\beta_{P_\infty})\alpha(F)$, which is exactly the naturality of α. $\square$

Set-theoretic models of CaTT. In the case where we consider the models in **Set**, this theory lets us obtain an equivalence with the Grothendieck-Maltsiniotis weak ω-categories.

Proposition 87. *There is an equivalence of categories* $\mathcal{U}_{\mathrm{id}}^{-1}(\mathbf{Set}) \simeq \omega\text{-}\mathbf{Cat}$.

Proof. In a CaTT-structure on **Set** given by $F : \mathbf{\Theta}_\infty^{\mathrm{op}} \to \mathbf{Set}$, the functor F preserves the globular products, and thus defines a Grothendieck-Maltsiniotis weak ω-category. A morphism of CaTT-category with families is exactly a morphism of weak ω-categories between them, thus this defines a fully faithful functor $\mathcal{U}_{\mathrm{id}}^{-1}(\mathbf{Set}) \to \omega\text{-}\mathbf{Cat}$. Moreover, every weak ω-category is obtained this way, the CaTT-category with families structure is obtained by considering the globular structure induced by FD_P. Hence this functor is essentially surjective so it is an equivalence. $\square$

Theorem 88. *There is an equivalence of categories* $\mathbf{Mod}(\mathcal{S}_{\mathsf{CaTT}}) \simeq \omega\text{-}\mathbf{Cat}$.

Proof. By Proposition 86 and Proposition 87 we have the following equivalences of categories

$$\mathbf{Mod}(\mathcal{S}_{\mathsf{GSeTT}}) = \underline{\mathbf{CwF}}(\mathcal{S}_{\mathsf{CaTT}}, \mathbf{Set}) \simeq \mathcal{U}_{\mathrm{id}}^{-1}(\mathbf{Set}) \simeq \omega\text{-}\mathbf{Cat}$$

$\square$

6.5 Interpretation of the proof.

We have now proven the main result of this article. Yet the proof involves some constructions that may seem *ad-hoc* or unnatural. the aim of this section is to show that these construction can be interpreted syntactically and correspond meaningful properties about the behavior of the dependent type theory CaTT.

Substitutions to a globular context. We have proved with Theorem 5 that a substitution is completely determined by its action on variables of a context, we can extract from Theorem 36 a partial answer to the converse question: considering a given action on variables, is there a substitution acting this way? We already know that the action cannot be completely free, since the substitution must respect the typing, and hence the source and target. In fact, in the case where the target context is a context in $\mathcal{S}_{\mathsf{GSeTT}}$, this theorem shows that this is the only obstruction. Consider $\mathcal{S}_{\mathsf{CaTT}}$ as a globular category with families, and consider, in the category $\mathcal{S}_{\mathsf{CaTT}}$, a context Γ which comes from the theory GSeTT: Γ is constructed only from variables and term constructors do not appear in it. Then for an arbitrary context Δ, one can apply Theorem 36 to characterize the substitutions $\Delta \to \Gamma$ as being equivalent to the natural transformations $\widehat{\mathcal{G}}(V\Gamma, T\Delta)$. In other words, in this case, a substitution γ is nothing else than the data of, for every variable x in Γ, a term t in Δ with the intent that $t = x[\gamma]$ in a way that is compatible with the source and target relations.

Substitutions to an arbitrary context. We can interpret Theorem 85 as a generalization of the previous discussion, where we characterize substitutions with an arbitrary context Γ as target (as opposed to one built from variables only). We cannot generalize it naively, by requiring that we associate a term to any variable of Γ. Indeed, if we consider the context

$$\Gamma = \texttt{(x : *) (f : id x -> id x)}$$

then the source of the variable `f` is the term `id x`, which is not itself a variable, hence the compatibility of the source and target cannot be expressed as a naturality condition. Categorically, this means that the set of variables $V\Gamma$ is not equipped with a structure of a globular set given by the source and target: in our example, we would indeed have $(V\Gamma)_2 = \{\texttt{f}\}$, but the source of this term is `id x`, which is not an element of $V(\Gamma)_1$. The solution we have chosen is to associate not only a term to any variable of Γ, but also to any term of Γ, in a way that respects the source and target, and thus we now represent substitutions $\gamma : \Delta \to \Gamma$ as natural transformations in $\mathcal{G}^{\mathrm{op}}(T\Gamma, T\Delta)$. However, this gives too much freedom, and there are such natural transformations that are ill-defined transformations. For instance, consider the contexts

$$\Gamma = \texttt{(x : *) (f : x -> x)} \qquad\qquad \Delta = \texttt{(x : *)}$$

together with a natural transformation $\eta : T\Delta \Rightarrow T\Gamma$ such that $\eta(\texttt{id x}) = \texttt{f}$. This can never be the action of a substitution, since $(\texttt{id } x)[\gamma] = \texttt{id } (x[\gamma])$. The problem is that, in this representation, we do not account for the fact that a substitution must respect the term constructors. The notion of algebraic natural transformation achieves exactly this: an algebraic natural transformation is a transformation that respects term constructors, and Theorem 85 ensures that, by considering only the algebraic natural transformations, we recover exactly the data of the substitutions.

Codensity of the functor P_∞. Theorem 85 states that for any CaTT-category with families $\mathcal{C}$, the right Kan extension along P_∞ gives the unique morphism of CaTT-category with families from the syntactic category to $\mathcal{C}$. In particular, applying this theorem to the CaTT-category $\mathcal{S}_{\mathsf{CaTT}}$ with the structure given by P_∞ shows that $\mathrm{Ran}_{P_\infty} P_\infty$ is this unique morphism. Since the identity functor $\mathrm{id}_{\mathcal{S}_{\mathsf{CaTT}}}$ is also a morphism, this shows in particular that $\mathrm{id}_{\mathcal{S}_{\mathsf{CaTT}}} = \mathrm{Ran}_{P_\infty} P_\infty$: in other words, the functor P_∞ is codense. More concretely, this proves that every context in the category $\mathcal{S}_{\mathsf{CaTT}}$ is canonically obtained as a limit of ps-contexts.

Developing the limits. There is also an interesting interpretation of Proposition 79, which establishes the equivalence between algebraic natural transformations $\widehat{\mathcal{G}}(T\Gamma, T\Delta)_{\mathrm{alg}}$ and natural transformations $\widehat{\mathbf{\Theta}_\infty}(N\Gamma, N\Delta)$. Indeed, consider that the natural transformations $\widehat{\mathbf{\Theta}_\infty}(N\Gamma, N\Delta)$ are maps of cones, between a cone of apex Γ and a cone of apex Δ, over the canonical diagram of Γ. Recall that the only objects that appear in the canonical diagram of Γ are ps-contexts, which are themselves globular products of disks. Hence one can "develop" this diagram, and obtain from the above map of cones, a new map of cones, with same apex, but over a diagram only made out of disks. Proposition 79 shows that algebraic natural transformations are exactly those maps of cones between diagrams over disks that can be obtained by such an operation. This theorem can thus be seen as a way to develop a canonical limit of ps-contexts into a non-canonical limit of disks. This matches the syntactic construction of context as a succession of context comprehension, which exhibits each context as a succession of pullback of disks. In that respect, the contexts in the theory are analogous to the CW-complexes in topology.

Finitely generated polygraphs. Recall that the nerve functor is defined by

$$N : \mathcal{S}_{\mathsf{CaTT}} \to \widehat{\mathbf{\Theta}_\infty}$$
$$\Gamma \mapsto \mathcal{S}_{\mathsf{CaTT}}(\Gamma, \_)$$

A colimit in $\mathbf{\Theta}_\infty$, which is thus a limit in $\mathcal{S}_{\mathsf{ps},\infty}$, is preserved by the functor $N\Gamma$, by continuity of the hom-functor. Hence, $N\Gamma$ defines a weak ω-category in the sense of Grothendieck-Maltsiniotis. By Theorem 88, $N\Gamma$ thus defines a model of the theory CaTT. In fact, one can describe the corresponding model, given by $\mathcal{S}_{\mathsf{CaTT}}(\Gamma, \_) : \mathcal{S}_{\mathsf{CaTT}} \to \mathbf{Set}$, which by continuity of the hom-functor preserves all the limits, and hence is a model. This shows that we have a functor $\mathcal{S}_{\mathsf{CaTT}} \to \mathbf{Mod}(\mathsf{CaTT})$, given by the coYoneda embedding, which is fully faithful and thus exhibits $\mathcal{S}_{\mathsf{CaTT}}$ as a full subcategory of the weak ω-categories. We call *finitely generated polygraphs* (or *computads*) the weak ω-categories that come from an object of $\mathcal{S}_{\mathsf{CaTT}}$, which generalize similar notions studied in higher category theory [12, 30], in particular the polygraphs play an important role in the theory of strict ω-categories as they are the cofibrant objects for the folk model structure [24]. This remark draws an analogy between our presentation of weak ω-categories and the Gabriel-Ülmer duality [18] in which the syntactic category sits inside the models as the opposite of the free finitely generated objects.

Further Work

In this article, we have presented a construction allowing us to characterize the semantics of the theory CaTT. The method that we use roughly breaks down in two steps. First we identified a class of contexts playing a particular role in the theory, that we called ps-contexts and showed they formed a subcategory of the syntactic category that is equivalent to the coherator of the Grothendieck-Maltsiniotis definition of weak ω-category. Second, we characterized the models of CaTT as presheaves over this coherator satisfying additional conditions. While in the first step the characterization of the coherator is inherently very specific to the theory CaTT, the second step is fairly generic, and we believe that it points towards a more general framework connecting dependent type theory with notions algebraic theories. This idea has been present since the early days of dependent type theory and was explored by Cartmell [13] through the notion of generalized algebraic theory. However, we believe that our work brings the formulation closer to that of monads with arities [9], and we believe that there is a strong connection between the method we use and the nerve theorem in this framework, as investigated in [33].

We believe that our work gives a promising approach to tackling the initiality conjecture, which could be solved in the aforementioned particular case of dependent type theories which entertain a close enough connection with the category CaTT. We however point out this theory is much simpler than Martin-Löf type theory and its variations, particularly since it does not have any computation rules, and we believe that the presence of those rules strongly increases the difficulty for proving this conjecture for such theories.

We believe that it would be valuable to establish a connection between our interpretation of the contexts as finitely generated polygraphs and the notion of polygraphs usually defined for strict ω-categories [24, 31]. In particular, in the strict case, polygraphs are used to characterized cofibrant objects for a model structure, and we would like to investigate whether such a model structure, or a weaker version of it in the form of a weak factorization system, would make sense for weak ω-categories [20].

The approach that we have presented in this article generalizes to other related higher structures, allowing for a type theoretic presentation of these structures. In particular, similar methods have been conducted in order to study monoidal weak ω-categories [6], cubical weak ω-

categories [4] and strictly unital weak ω-categories [17]. Further work along the lines of this article includes generalizing the methods that we have presented in order to study the semantics of such theories. In the case of monoidal weak ω-categories, the theories are close enough that a transfer of the semantics can be done [6], and understanding the semantics of CaTT is enough to settle the semantics of the other theory. The theory for cubical ω-categories, can be presented in a very similar fashion to the theory CaTT and we believe that similar methods can be used to study its semantics. The theory for strictly unital ω-categories is more complicated since it contains rewriting rules. Understanding the semantics is a relevant challenge left for future work and we believe that either one could achieve it by relating this theory to CaTT or by adapting the methods that we have presented to account for the rewriting rules.

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