Factorability Structures

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Homotopy in Concurrency and Rewriting

Overview

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Motivation

Idea (Bödigheimer, Visy)

Use appropriate normal forms to understand group homology.

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Homotopy

Construct out of normal form set a homotopy equivalence from BG to a smaller CW complex.

Factorability structure

- Set of geodesic normal forms with additional properties

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- Gives small chain complex for homology

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- Relates to quadratic normalisation

Word Length

Reminder: Word Length

G group, $\mathcal{E}$ generating system.

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$$N_{\mathcal{E}}(x) = \min\{n \,|\, x = a_n \ldots a_1, a_i \in \mathcal{E}\}$$

Factorability

Factorability: Idea

For a given group and generating system, prescribe a way to split off a generator.

Factorability

Definition

Let G be a group and $\mathcal{E}$ a generating set.

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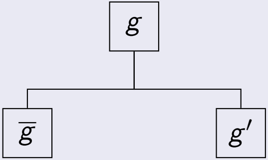
Let G be a group and $\mathcal{E}$ a generating set. A **factorability structure** is a map

$$\begin{aligned} \eta\colon G &\to G \times G \\ g &\mapsto (\overline{g}, g') \end{aligned}$$



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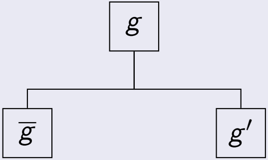

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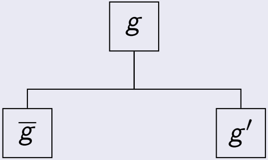

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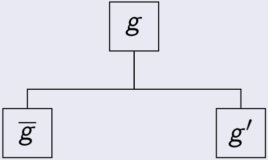

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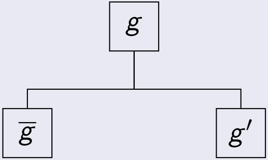

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- Compatibility with multiplication holds

Compatibility with Multiplication

$M \times \mathcal{E}$

Compatibility with Multiplication

$$\begin{array}{ccc} M \times \mathcal{E} & & \\ \Big\downarrow{\scriptstyle \mu} & & \\ M & \xrightarrow{\quad \eta \quad} & M \times \widetilde{\mathcal{E}} \end{array}$$

Compatibility with Multiplication

$$
\begin{array}{ccc}
M \times \mathcal{E} & \xrightarrow{\eta \times \mathrm{id}} & M \times \widetilde{\mathcal{E}} \times \mathcal{E} \\
\Big\downarrow \mu & & \\
M & \xrightarrow[\eta]{} & M \times \widetilde{\mathcal{E}}
\end{array}
$$

Compatibility with Multiplication

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\begin{array}{ccccc}
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\Big\downarrow \mu & & \Big\downarrow \mathrm{id} \times \mu & & \\
& & M \times M & \xrightarrow{\mathrm{id} \times \eta} & M \times \widetilde{\mathcal{E}} \times \widetilde{\mathcal{E}} \\
M & & \xrightarrow[\eta]{\hspace{4em}} & & M \times \widetilde{\mathcal{E}}
\end{array}
$$

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\begin{array}{ccccc}
M \times \mathcal{E} & \xrightarrow{\eta \times \mathrm{id}} & M \times \widetilde{\mathcal{E}} \times \mathcal{E} & & \\
 & & \downarrow{\scriptstyle \mathrm{id} \times \mu} & & \\
\downarrow{\scriptstyle \mu} & & M \times M & \xrightarrow{\mathrm{id} \times \eta} & M \times \widetilde{\mathcal{E}} \times \widetilde{\mathcal{E}} \\
 & & & & \downarrow{\scriptstyle \mu \times \mathrm{id}} \\
M & & \xrightarrow[\eta]{\hspace{6em}} & & M \times \widetilde{\mathcal{E}}
\end{array}
$$

Compatibility with Multiplication

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\begin{array}{ccccc}
M \times \mathcal{E} & \xrightarrow{\eta \times \mathrm{id}} & M \times \widetilde{\mathcal{E}} \times \mathcal{E} & & \\
\Big\downarrow \mu & & \Big\downarrow \mathrm{id} \times \mu & & \\
 & \color{red}{\text{graded} =} & M \times M & \xrightarrow{\mathrm{id} \times \eta} & M \times \widetilde{\mathcal{E}} \times \widetilde{\mathcal{E}} \\
 & & & & \Big\downarrow \mu \times \mathrm{id} \\
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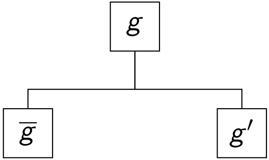
  Split off the last letter!
- Non-example: $\mathbb{Z}/k$ with generating system $\{+1, -1\}$ for $k > 3$.

Normal Forms

g

Normal Forms



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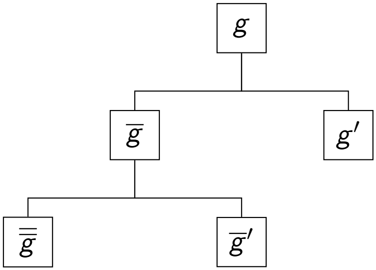

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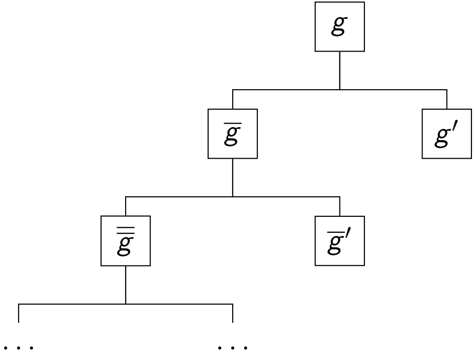

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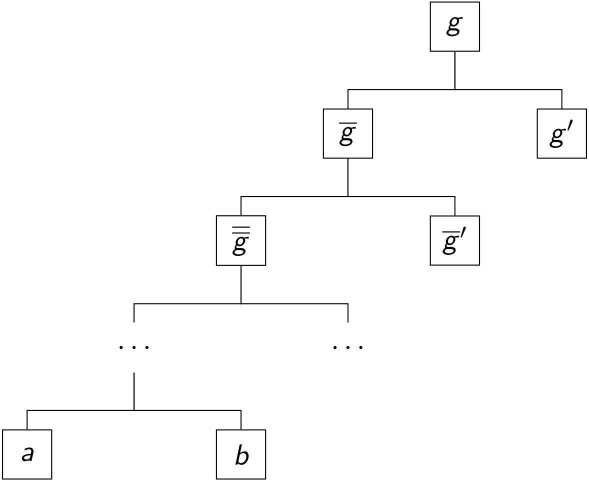

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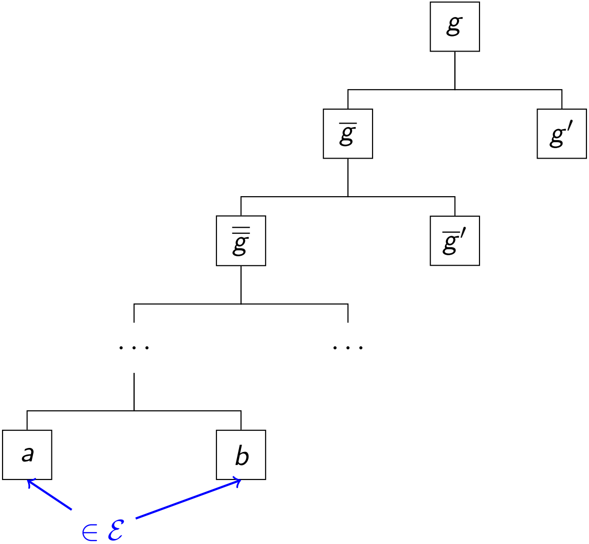

Visy Complex

Theorem (Visy, Wang, Heß)

For a factorable group G, there is a small chain complex computing the homology groups of G.

Modules: *Free with basis*

$$[a_n| \dots |a_1]$$

with $a_i \in \mathcal{E}$ and (a_{i+1}, a_i) unstable.

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Differentials: *Complicated but explicit.*

Unstable pairs

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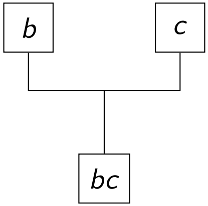

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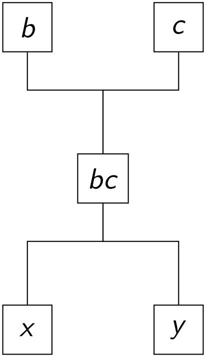

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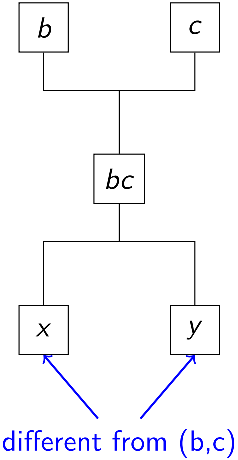

Symmetric Groups

Example (Visy)

Symmetric group $\mathfrak{S}_n$ with generating set of all transpositions.

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Factorization map

$$\begin{pmatrix} 1 & 2 & \dots & k & k+1 & \dots & n-1 & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(k) & \sigma(k+1) & \dots & \sigma(n-1) & \sigma(n) \end{pmatrix}$$

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$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 5 & 1 & 3 & 6 \end{pmatrix}$$

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$$\begin{pmatrix} 1 & 2 & 3 & 4 & \color{red}{5} & 6 \\ 2 & 4 & 5 & 1 & 3 & 6 \end{pmatrix}$$

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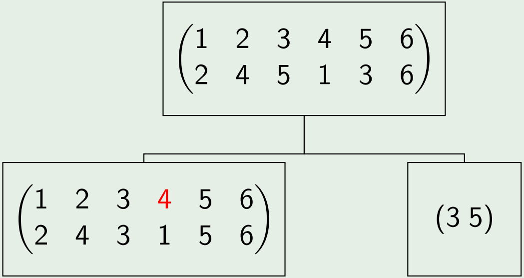

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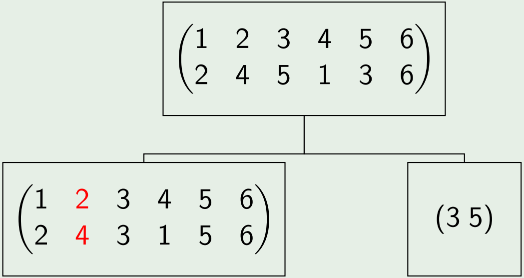

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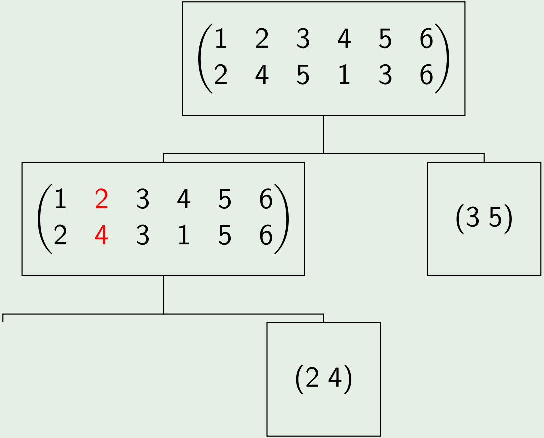

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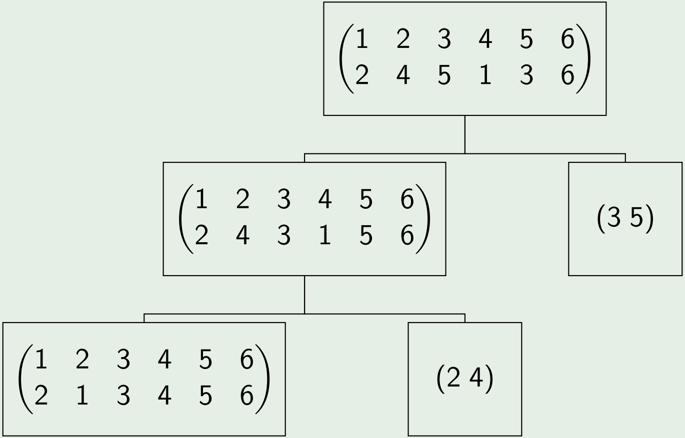

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Example (Bödigheimer, O.)

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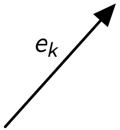

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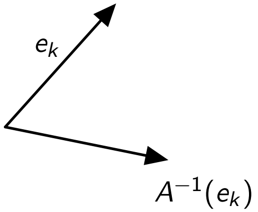

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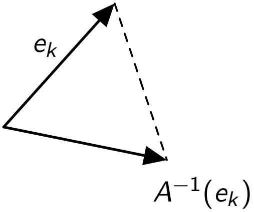

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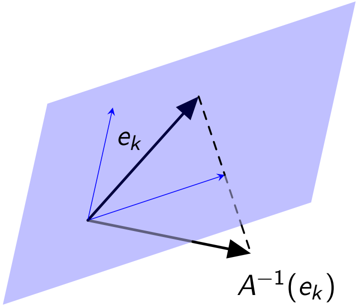

Orthogonal Groups: Remarks

Remarks on the proof

- First considered by Brady and Watt

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Orthogonal Groups: Remarks

Remarks on the proof

- First considered by Brady and Watt
- Rely on their results
- Crucial for $A \in O(n)$ (Brady-Watt):

$$N_{\mathcal{R}}(A) = \dim \operatorname{im}(A - \mathbb{1}_n)$$

Finite Coxeter Groups

Question

What about other finite Coxeter groups?

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Proposition (O.)

- *The factorability structure on $O(n)$ restricts to a factorability structure on $B_n \subseteq O(n)$.*

Finite Coxeter Groups

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What about other finite Coxeter groups?

Proposition (O.)

- *The factorability structure on $O(n)$ restricts to a factorability structure on $B_n \subseteq O(n)$.*
- *The factorability structure on $O(4)$ does not descend to a factorability structure on D_4.*

Finite Coxeter Groups II

Coxeter Generators

What about Coxeter generators?

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Coxeter Generators

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Theorem (Rodenhausen)

A factorable monoid $(M, \mathcal{E}, \eta)$ admits a presentation of the form

$$M \cong \langle \mathcal{E} | (a, b) = \eta(ab) \text{ for all } a, b \in \mathcal{E} \rangle .$$

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Observation

One cannot break the braid relations in the symmetric group without introducing new generators.

Braid Groups

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A **braid** on n **strands** is an embedding

$$\{1, \ldots, n\} \times [0,1] \to \mathbb{R}^2 \times [0,1]$$

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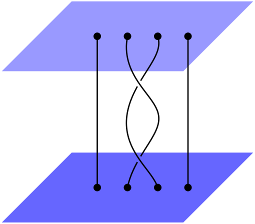

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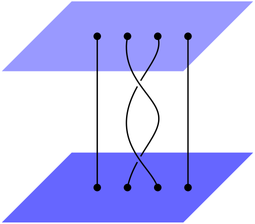

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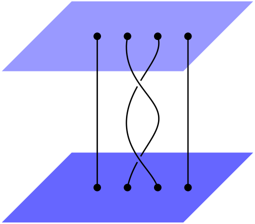
- Strands start and end in marked points



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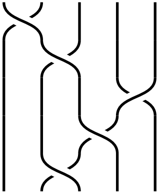

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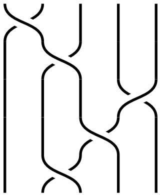

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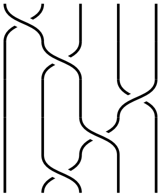

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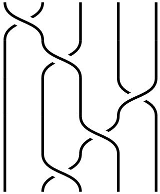

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Factorability

Are braid groups factorable?

Braid Groups and Factorability

First Answer: No

The braid group B_n is not factorable w.r.t. $\{\sigma_1^{\pm 1}, \ldots, \sigma_{n-1}^{\pm 1}\}$.

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Idea

Use Garside theory by Garside, Dehornoy, Lafont, ...

Monoids

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Often easier to consider monoids!

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Group of fractions

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Ore condition

If M satisfies the *Ore condition*, then $H_*(M) \cong H_*(G)$ holds.

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Divisibility in Monoids

$a \in M$ is right-divisible by $b \in M$ if there is a $c \in M$ such that $a = cb$.

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Example

In a free monoid, x_1 and x_2 do not have common multiples.

Ore condition

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- Element in a group of fractions of an arbitrary monoid M: $a_1^{-1} a_2 a_3^{-1} \dots a_{k-1}^{-1} a_k$, $a_i \in M$.

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- In an Ore monoid, cd^{-1}, $c, d \in M$, suffices

Ore monoids

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Fact (Folklore)

If M is Ore and $G(M)$ its group of fractions, $BM \to BG(M)$ is homotopy equivalence.

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Braid monoids

Consider positive braids: Braids with only over-crossings.

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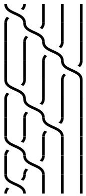

Factorability Structure on Braid Monoid

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There is a factorability structure on $(B_n^+, \mathcal{D})$ *and on* $(B_n, \mathcal{D} \cup \mathcal{D}^{-1})$.

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Generalization

One can use similar arguments for any Garside group.

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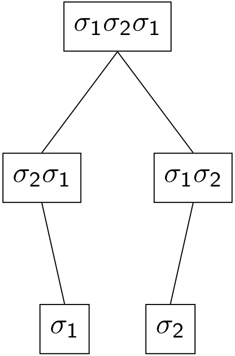

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Give a direction to defining relations of a monoid.

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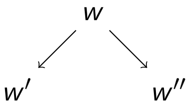

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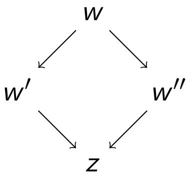

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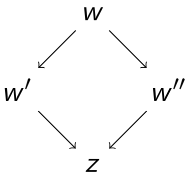

Normal Forms

Complete rewriting systems yield nice normal forms.

Homology and String Rewriting

Theorem (Brown)

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Corollary (Brown)

If M has a finite complete rewriting system, then BM is homotopy equivalent to a complex with finitely many cells in each dimension.

Factorability and String Rewriting

Theorem (Rodenhausen)

A factorable monoid $(M, \mathcal{E}, \eta)$ *admits a presentation of the form*

$$M \cong \langle \mathcal{E} | (a, b) = \eta(ab) \text{ for all } a, b \in \mathcal{E} \rangle .$$

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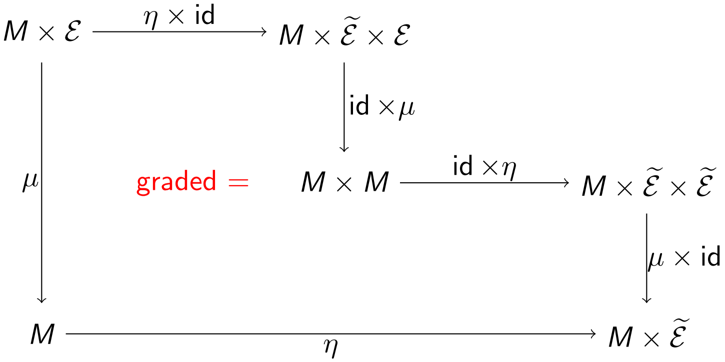
- This rewriting system is not always noetherian.
- Can strengthen compatibility with multiplication to establish noetherianity.

Compatibility with Multiplication



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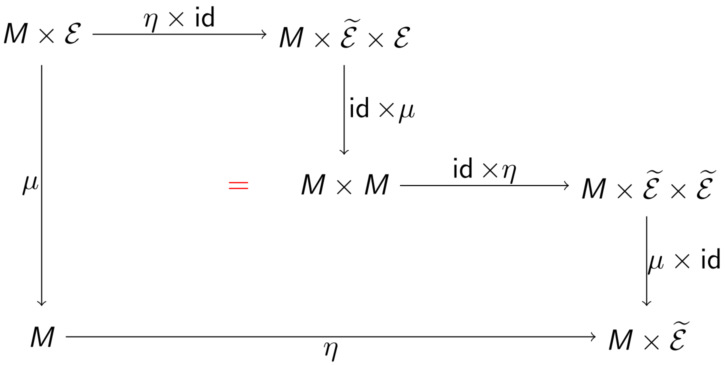

Quadratic Normalisation

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Relation to Factorability

Strengthened factorability seems to be a special case of quadratic normalisation.

Thank you

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