Rewriting in shuffle operads and resolutions of operads

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Motivations from algebra

Why algebraic rewriting?

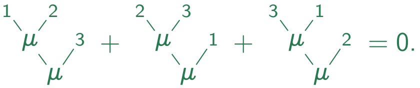
- Newman (1942) : rewriting is a **combinatorial** theory of equivalence
- Algebraic rewriting: a combinatorial theory of **congruence**
- In computer algebra: **ideal membership**, **resolutions**, **homological properties**
- In constructive mathematics: **cofibrant replacements**

Examples: monoids, commutative algebras, associative algebras [Shirshov 1962, Bergman 1978, Bokut' 1994, Mora 1994], higher categories [Street 1976, Burroni 1993], operads [Dotsenko-Khoroshkin 2010].

Our algebraic structure of interest is the structure of **symmetric operads** (May 1972, Loday 1996), which are abstractions of multilinear maps.

Example: symmetric operad Lie

The symmetric operad Lie is generated by one antisymmetric **operation** μ of **arity** 2, satisfying the **Jacobi relation**



Compare with

$$[[x_1, x_2], x_3] + [[x_2, x_3], x_1] + [[x_3, x_1], x_2] = 0.$$

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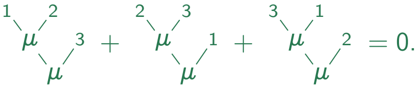

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$$[[x_1, x_2], x_3] + [[x_2, x_3], x_1] + [[x_3, x_1], x_2] = 0.$$

Due to the symmetric actions, there is no known way to do algebraic rewriting in symmetric operads: this motivates the study of **shuffle operads** [Dotsenko-Khoroshkin 2010].

Two ways of doing rewriting:

- with a monomial order and an algebraic formulation of confluence: **Gröbner bases**
- in a higher dimensional setting: **polygraphs**

Our goal is to mix the two approaches.

Shuffle operads and Gröbner bases

If associative algebras are a linear version of words, then shuffle operads are a linear version of planar trees.

Shuffle operads [Dotsenko-Khoroshkin 2010]

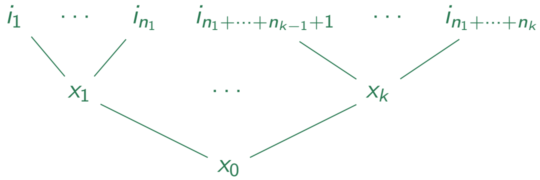
- The category Coll of **collections** is the presheaf category on Ord, the category of finite nonempty ordered sets with order-preserving bijections, with values in Vect, the category of vector spaces over **k**.
- A collection V is determined by $V(k) := V(\{1 < \cdots < k\})$ for $k \geq 1$. An element of $V(k)$ is of **arity** k.
- The **shuffle composition** of two collections V, W is

$$V \circ_{\text{Ш}} W(I) = \bigoplus_{k\geq 1} V(k) \otimes \left(\bigoplus_{f:I \twoheadrightarrow \{1,\ldots,k\}} W(f^{-1}\{1\}) \otimes \cdots \otimes W(f^{-1}\{k\}) \right)$$

  where $I \in \text{Ord}$ and f is a **shuffle surjection**, that is, $\min f^{-1}\{1\} < \cdots < \min f^{-1}\{k\}$. The unit for this composition is $\mathbb{1} := (\mathbf{k}, 0, \ldots)$.
- $(\text{Coll}, \circ_{\text{Ш}}, \mathbb{1})$ is a monoidal category. The category of **shuffle operads**, denoted by ШOp, is the category of internal monoids in $(\text{Coll}, \circ_{\text{Ш}}, \mathbb{1})$.

Tree monomials

- Let $X = (X(k))_{k\geq 1}$ such that $X(k)$ is a basis of $V(k)$ for every $k \geq 1$. In terms of planar trees, the collection $V \circ_{\text{Ш}} V$ has a basis of planar trees



- For $j \in \{1, \dots, k\}$, the **inputs** of x_j are $\{i_{n_1+\dots+n_{j-1}+1}, \dots < i_{n_1+\dots+n_j}\}$. The inputs of x_0 are $\{i_1 < i_{n_1+1} < \dots < i_{n_1+\dots+n_{k-1}+1}\}$. We always draw inputs in increasing order.
- By iterating this tree construction, we get the **free shuffle operad on X**, denoted by $X^{\text{Ш}}$ spanned by **tree monomials**. We refer to elements of $X^{\text{Ш}}(k)$ as **polynomials of arity k**.

Example: shuffle operad $\mathrm{Lie}^\flat$

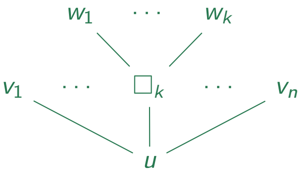
The shuffle operad $\mathrm{Lie}^\flat$ is generated by one operation μ of arity 2, and satisfies the **shuffle Jacobi relation**

$$\mu(\mu(1,2),3) - \mu(\mu(1,3),2) - \mu(1,\mu(2,3)) = 0.$$

With the planar tree interpretation, we can define contexts:

Contexts

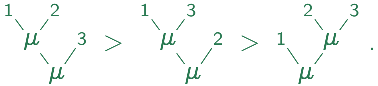
- A **context of inner arity** k is a tree monomial $C[-]$ of the form



  where $\square_k$ is a symbol of arity k and $u, \vec{v}, \vec{w}$ are tree monomials.
- Given a polynomial $f = \sum \lambda_i u_i$ of arity k, we define the polynomial $C[f] := \sum \lambda_i C[u_i]$.

Monomial orders

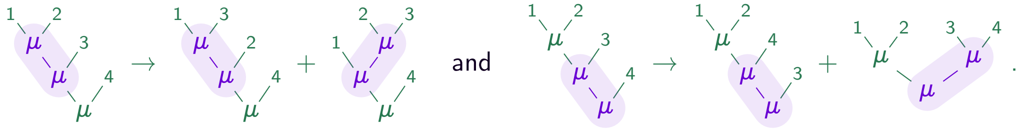
- A **monomial order** is a total order on tree monomials that is compatible with contexts. For a polynomial f,
  - its **leading monomial** $\mathrm{lm}(f)$ is the greatest tree monomial that occurs,
  - its **leading coefficient** $\mathrm{lc}(f)$ is the coefficient in front of the leading monomial,
- For example, there exists a monomial order called **path-lexicographic** such that



Gröbner bases for operads

- Given two polynomials f and g, if there exists a context C such that $C[\mathrm{lm}(g)] = \mathrm{lm}(f)$, then we define the **reduction** of f by g as the polynomial $f - \frac{\mathrm{lc}(f)}{\mathrm{lc}(g)} C[g]$.

- For example, the shuffle Jacobi relation induces the reductions



- A **Gröbner basis** of an ideal I of a free shuffle operad X^{III} is a generating set G such that every nonzero polynomial in I can be reduced by an element of G.

This approach allows us to obtain a homological result on operads:

Koszulness

- **Koszulness** is a property on operads that ensures the existence of a minimal model, given by: in particular, the **Koszul dual coooperad** of a Koszul operad is a minimal model of the operad.

Theorem [Dotsenko-Khoroshkin 2010]

A quadratic operad with a Gröbner basis is Koszul.

Polygraphic rewriting in shuffle operads

Shuffle 1-operads

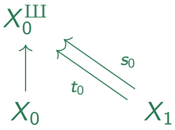
- A **shuffle 1-operad** is an internal category in the category $\mathbb{III}\mathrm{Op}$ of shuffle operads.

$$P_0 \underset{t_0}{\overset{s_0}{\leftleftarrows}} \xrightarrow{i_1} P_1$$

The elements of P_0 are called 0-**cells**, and those of P_1 are called 1-**cells**

Shuffle 1-polygraphs

A **shuffle 1-polygraph** is a diagram



where

- $X_0 = (X_0(k))_{k\geq 1}$ is the indexed set of **generators**
- $X_1 = (X_1(k))_{k\geq 1}$ is the indexed set of **rewriting rules**
- the **source** and **target** maps $s_0, t_0 : X_1 \to X_0^{\text{Ш}}$ are from rewriting rules to the free operad on the generators.

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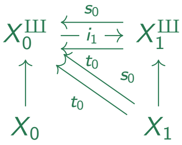

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- the **source** and **target** maps $s_0, t_0 : X_1 \to X_0^{\text{Ш}}$ are from rewriting rules to the free operad on the generators.
- $X^{\text{Ш}} = (X_0^{\text{Ш}}, X_1^{\text{Ш}})$ is the **free shuffle 1-operad** where $X_0^{\text{Ш}}$ is the shuffle operad of 0-cells and $X_1^{\text{Ш}}$ is the shuffle operad of 1-cells.
- The shuffle operad **presented** by X is the coequalizer $\overline{X}$ of $s_0, t_0 : X_1^{\text{Ш}} \rightrightarrows X_0^{\text{Ш}}$.

Example: polygraphic presentation of $\mathsf{Lie}^\flat$

The shuffle operad $\mathsf{Lie}^\flat$ is presented by the shuffle 1-polygraph

$$X_{\mathsf{Lie}^\flat} := \left\langle \mu \in X_0(2) \;\middle|\; \alpha : \begin{array}{ccccc} 1 & & 2 & & \\ & \mu & & & 3 \\ & & & \mu & \end{array} \to \begin{array}{ccccc} 1 & & 3 & & \\ & \mu & & & 2 \\ & & & \mu & \end{array} + \begin{array}{ccccc} & & 2 & & 3 \\ 1 & & & \mu & \\ & \mu & & & \end{array} \right\rangle .$$

Rewriting systems from 1-polygraphs

Let X be a **left-monomial** 1-polygraph, that is, every source is a tree monomial.

- A **rewriting step** is a 1-cell

$$\lambda C[\alpha] + i_1(b) : \lambda C[u] + b \to \lambda C[a] + b$$

  of $X_1^{\text{Ш}}$, where $\alpha : u \to a$ is a rewriting rule, C is a context, λ is a nonzero scalar, and b is a polynomial of $X_0^{\text{Ш}}$ such that $C[u] \notin \text{supp}(b)$.

- X is **terminating** if there are no infinite rewriting paths.

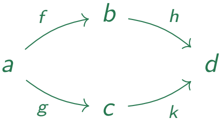
Branchings

- A **branching** is a pair of rewriting paths (f, g) with the same source.
- A **local branching** is a branching (f, g) where f and g are rewriting steps. We classify local branchings as:

additive **multiplicative** **intersecting** **critical**

Confluence

- The 1-polygraph X is (**locally**) **confluent** if, for every (local) branching (f, g), there exist rewriting paths h and k and the confluent diagram

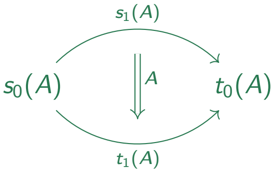


- The 1-polygraph X is **convergent** if it is confluent and terminating.
- A Gröbner basis is equivalent to a convergent 1-polygraph whose rewriting rules reduce the leading term to the rest.

Cellular extension

Let X be a 1-polygraph.

- A **cellular extension** is an indexed set of **generating 2-cells**



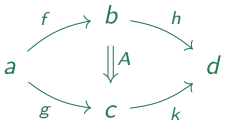
  where $s_0(A)$, $t_0(A)$ are 0-cells and $s_1(A), t_1(A) : a \to b$ are 1-cells of $X^{\text{Ш}}$.

- Let $\sim$ be the equivalence relation generated by $s_1(A) \sim t_1(A)$ for every element A of the cellular extension. The cellular extension is **acyclic** if the equivalence relation $\sim$ has one equivalence class.

The critical branchings theorem comes from [Knuth-Bendix 1971, Nivat 1972]. The coherent version comes from [Squier 1994, Guiraud-Hoffbeck-Malbos 2019, Malbos-R. 2020].

Theorem (coherent critical branchings)

Let X be a terminating 1-polygraph with a generating 2-cell for each critical branching (f, g):

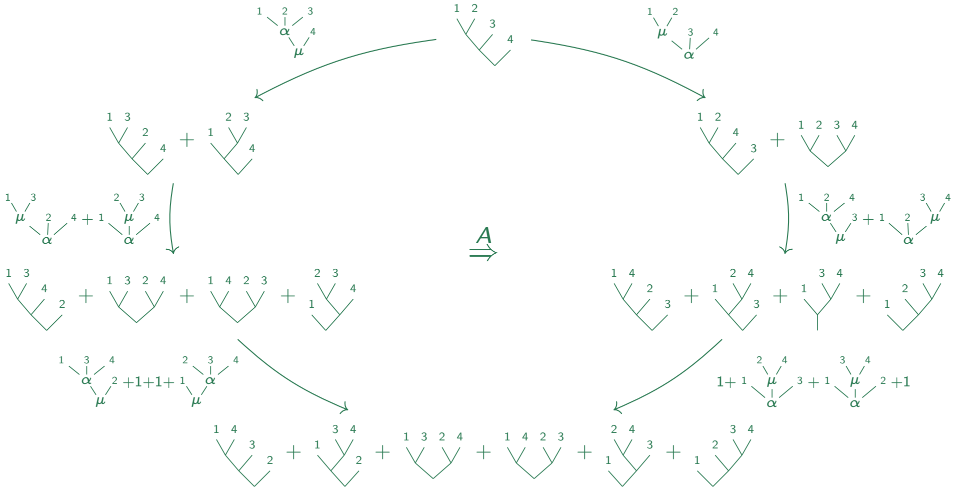


Then the cellular extension is acyclic.

We can then consider compositions of generating 2-cells by gluing confluent diagrams: this leads to the notion of **higher dimensional rewriting**.

Example: coherent convergence of $X_{\mathrm{Lie}^\flat}$

The 1-polygraph $X_{\mathrm{Lie}^\flat}$ only has one critical pair and is convergent. The cellular extension will have only one generating 2-cell:



Higher dimensional rewriting in shuffle operads

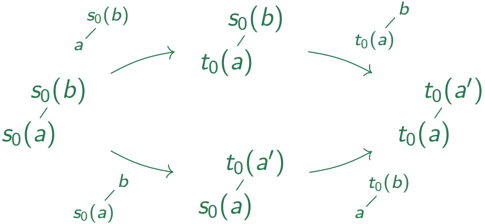
Shuffle ω-operads

- A **shuffle ω-operad** is an internal (strict) ω-category in ШOp, that is, an object

$$P_0 \mathrel{\substack{\xleftarrow{s_0} \\ \xrightarrow{i_1} \\ \xleftarrow[t_0]{}}} P_1 \mathrel{\substack{\xleftarrow{s_1} \\ \xrightarrow{i_2} \\ \xleftarrow[t_1]{}}} P_2 \mathrel{\substack{\xleftarrow{s_1} \\ \xrightarrow{i_3} \\ \xleftarrow[t_1]{}}} \cdots \mathrel{\substack{\xleftarrow{s_{n-1}} \\ \xrightarrow{i_n} \\ \xleftarrow[t_{n-1}]{}}} P_n \mathrel{\substack{\xleftarrow{s_n} \\ \xrightarrow{i_{n+1}} \\ \xleftarrow[t_n]{}}} \cdots$$

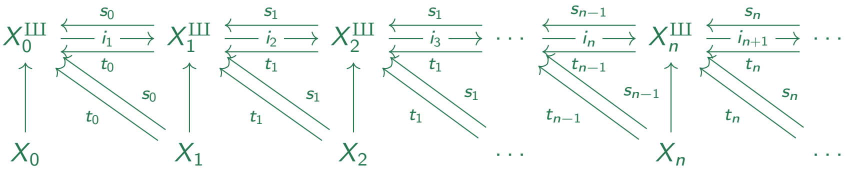
  satisfying globularity, associativity, and identity axioms.

- The interaction between the ω-category structure and the operad structure gives the **linear exchange relation**: for any n-cells a and b, the two paths below are equal:



Shuffle ω-polygraphs

- The definition of 1-polygraphs extends to that of **shuffle ω-polygraphs**:



- An ω-polygraph is a **polygraphic resolution** if each cellular extension X_{n+1} is **acyclic**.

Overlapping polygraphic resolution

Let X be a convergent 1-polygraph. We can construct the **overlapping polygraphic resolution** $\mathrm{Ov}(X)$ on X, where the elements of $\mathrm{Ov}(X)_n$ correspond to certain overlappings of n rewriting rules:

and so on...

Theorem [Malbos-R. 2020]

A operad P with a convergent quadratic polygraphic presentation X is Koszul.

Idea of proof.

- Extend the 1-polygraph X to the overlapping polygraphic resolution $\mathrm{Ov}(X)$.
- Study the induced P-**bimodule resolution** $(P\langle \mathrm{Ov}(X)_n \rangle)_n$, whose generators are concentrated on the superdiagonal.
- Calculate the **Quillen homology** of the operad P, which is concentrated on the diagonal, which gives a sufficient condition for Koszulness.

And now...

We have defined the notion of **polygraphic resolution** of an operad.

- How to construct a resolution/cofibrant replacement in the category of **differential graded operads**?
- Does the overlapping resolution give a **minimal** cofibrant replacement?
- Can shuffle operadic rewriting be generalized to **shuffle properads**?