Combinatorics, Probability and Computing (2003) 00, 1–??.
DOI: 10.1017/S0963548301004989 Printed in the United Kingdom

Area and Inertial Moment of Dyck Paths

MICHEL NGUYỄN THẾ

Received 15 December 2002

In this paper, we investigate the limit law of the inertial moment of Dyck paths with respect to the x-axis, that is the sum of the squares of the altitudes. We find its Laplace transform using Louchard methodology, rediscovering a result which was in fact well known by probabilists. We give recurrence relations enabling to compute the moments of the joint limit law of the area and the inertial moments of both Dyck paths and Grand Dyck paths (bilateral Dyck paths).

CONTENTS

1. Introduction

The combinatorial school of Bordeaux rose quite naturally the problem of finding the inertial moment of polyominoes. In particular, Philippe Duchon [3] introduced the concept of parameter of rank k of polyominoes, which amounts to considering functionals of the type $h \mapsto \binom{h+1}{k-1}$, where h is the height of a column of a polyomino, or the altitude of a Dyck path. Recall that a Dyck path is a Bernouilli walk with steps -1 and $+1$, conditioned to start and end in zero, and to be nonnegative (*i.e.* always above the x-axis). A

Grand Dyck path is a Bernouilli walk with steps -1 and $+1$, conditioned to start and end in zero, and which can go above or below the x-axis.

We will here stick to functionals $h \mapsto |h|$ or $h \mapsto h^2$ of Dyck paths or of Grand Dyck paths, and we will stress the passage from Dyck paths to Grand Dyck paths, and will parallel it with the passage from excursions to Brownian bridges. The distribution $\int_0^1 e(t)^\beta\, dt$ for $\beta \in \mathbb{R}_+$ also arises in the study of the Wiener index of random trees [9].

While in the continuous real world, it is easier to make the study for Brownian bridges, and then to use it for excursions, we will see that it is easier, in the discrete real world with the generating function approach, to make the study first for Dyck paths, and then to elaborate from them to find the wanted results for great Dyck paths, as far as the functionals $h \mapsto |h|^k$ are concerned. (A similar consideration could be made concerning the passage from the standard Brownian motion starting from zero to Brownian bridges, and from the Grand Dyck paths to general paths starting from zero).

2. The Brownian approach

2.1. Known results on Brownian bridges and excursions

Let us first recall known results concerning Brownian bridges and excursions, that provide formulas for double Laplace transforms of functionals.

2.1.1. Brownian bridge In [18], can be found, up to a slight modification, the result below, which we state as a theorem for convenience with a proof:

Theorem 2.1. **(Shepp)** *Let V be a symmetric nonnegative piecewise continuous function and g a solution bounded at infinity of*

$$-\frac{1}{2}u''(x) + (\alpha + V(x)u(x)) = 0. \tag{1}$$

Then the function

$$\phi(t) = \mathbb{E}_0\left[e^{-\int_0^t V(W_s)\, ds} \mid W_t = 0\right],$$

where $\mathbb{E}_0$ means the expectation taken for processes W_t starting at 0, satisfies

$$\int_0^\infty e^{-\alpha t}\phi(t)\ \frac{dt}{\sqrt{t}} = -\sqrt{2\pi}\frac{g(0)}{g'(0)}.$$

Proof. Let g_1 and g_2 be two solutions of (1) such that g_1 is bounded at $+\infty$, g_2 is bounded at $-\infty$, and

$$g_1 g_2' - g_1' g_2 \equiv 2.$$

Then for every function bounded f with compact support, the unique bounded solution u of

$$-\frac{1}{2}u''(x) + (\alpha + V(x)u(x)) = f(x)$$

is given by the Green operator applied to f:

$$u(x) = g_1(x)\int_{-\infty}^x g_2(t)f(t)\ dt + g_2(x)\int_x^\infty g_1(t)f(t)\ dt,$$

and Feynman-Kac's formula [6, 8] allows to state that

$$u(x) = \mathbb{E}_x \left[\int_0^\infty e^{-\alpha t - \int_0^t V(W_s)\, ds} f(W_s)\ dt \right],$$

where W_s is a Brownian motion starting at x.

By making $x = 0$, one obtains

$$\mathbb{E}_0 \left[\int_0^\infty e^{-\alpha t - \int_0^t V(W_s)\, ds} f(W_s)\ dt \right] = g_1(0) \int_{-\infty}^0 g_2(t) f(t)\ dt + g_2(0) \int_0^\infty g_1(t) f(t)\ dt.$$

To get a Brownian bridge $\widetilde{W}$ from the standard Brownian motion W, we choose

$$f(x) = \frac{\sqrt{2\pi}}{2\varepsilon} \chi\left(|x| < \varepsilon\right),$$

where $\chi\left(|x| < \varepsilon\right)$ is equal to 1 if $|x| < \varepsilon$, and otherwise to zero, and we make $\varepsilon \to 0$.

For the left-hand side, we obtain

$$\lim_{\varepsilon \to 0} \int_0^\infty e^{-\alpha t} e^{-\int_0^t V(W_s)\, ds} \frac{\chi\left(|W_t| < \varepsilon\right)}{\Pr\left(|W_t| < \varepsilon\right)} \frac{\Pr\left(|W_t| < \varepsilon\right)}{\frac{2\varepsilon}{\sqrt{2\pi t}}} \frac{dt}{\sqrt{t}}.$$

As the ratio

$$\frac{\Pr\left(|W_t| < \varepsilon\right)}{\frac{2\varepsilon}{\sqrt{2\pi t}}} \leq 1$$

tends to 1 when $\varepsilon \to 0$, the bounded convergence theorem gives

$$\int_0^\infty e^{-\alpha t} \mathbb{E}_0 \left[e^{-\int_0^t V(W_s)\, ds} \middle|\ W_t = 0 \right] \frac{dt}{\sqrt{t}} = \sqrt{2\pi} g_1(0) g_2(0).$$

For reasons of symmetry, $g_1(x) \equiv g_2(-x)$, hence $g_1(0) = g_2(0)$ and $g_1'(0) = -g_2'(0)$. The equality $g_1 g_2' - g_1' g_2 \equiv 2$ written in zero then gives $-g_2(0) g_1'(0) = 1$, and

$$\int_0^\infty e^{-\alpha t} \mathbb{E}_0 \left[e^{-\int_0^t V(W_s)\, ds} \middle|\ W_t = 0 \right] \frac{dt}{\sqrt{t}} = -\sqrt{2\pi} \frac{g_1(0)}{g_1'(0)}.$$

If g is a solution bounded at $+\infty$ of (1), g is proportional to g_1, and we still have

$$\int_0^\infty e^{-\alpha t} \mathbb{E}_0 \left[e^{-\int_0^t V(W_s)\, ds} \middle|\ W_t = 0 \right] \frac{dt}{\sqrt{t}} = -\sqrt{2\pi} \frac{g(0)}{g'(0)}.$$

□

2.1.2. Excursions Louchard proved the following result, enabling to go from Brownian bridges to excursions :

Theorem 2.2 (Louchard). *Let W_t denote the Brownian motion, and Y_t the Brownian excursion. If $V \geq 0$ is a symmetric piecewise continuous function, and if*

$$g(\alpha) := \int_0^\infty e^{-\alpha t} \mathbb{E}_0 \left[\exp\left(-\int_0^t V[W_s]\ ds \right) \middle| W_t = 0 \right] \frac{dt}{\sqrt{2\pi t}},$$

then, if L is the length of excursion Y,

$$\int_0^\infty (e^{-\alpha t}-1)\,\mathbb{E}\left[\exp\left(-\int_0^t V[Y_s]\,ds\right)\Big| L=t\right]\frac{dt}{\sqrt{2\pi t^3}} = -g^{-1}(\alpha)+\lim_{\alpha\to 0} g^{-1}(\alpha).$$

2.2. The absolute value and quadratic functionals

Let ϕ_{1B} and ϕ_{1E} denote the Laplace transforms of the integral of the absolute value of the Brownian bridge and of the excursion. Shepp [18] and Louchard [10] used their results to study the functional $x\mapsto |x|$ and obtain the formulas

$$\int_0^\infty e^{-us}\phi_B(\sqrt{2}s^{3/2})s^{-1/2}\,ds = \frac{Ai(u)}{Ai'(u)}$$

and

$$\int_0^\infty (e^{-\alpha t}-1)\mathbb{E}\left[e^{-t^{3/2}}\int_0^1 e(u)\,du\right]\frac{dt}{2\pi t^3} = \frac{2^{1/3}Ai'(2^{1/3}\alpha)}{Ai(2^{1/3}\alpha)} - 2^{1/3}\frac{2^{1/3}Ai'(0)}{Ai(0)}.$$

These two expressions prove quite useful to recursively compute the moments, as did Shepp in the same paper, or Louchard in [11], [12], by identifying the coefficients of the asymptotic expansion in α. These two distributions are still of interest today: see [20] and [4].

Shepp also studied the functional $x\mapsto x^2$, leading to the theorem 2.3 through formula

$$\int_0^\infty e^{-\alpha t}\phi_{2B}\left(\frac{1}{8}t^2\right)\frac{dt}{\sqrt{t}} = \sqrt{\pi}\frac{\Gamma(\alpha+1/4)}{\Gamma(\alpha+3/4)}, \tag{2}$$

where ϕ_{2B} is the Laplace transform of the integral of the square of Brownian bridges. Louchard did not, and this is the gap that we intend to fill, to obtain theorem 2.4.

Notice that in fact, Shepp found ϕ_{2B} from other methods and only checked that ϕ_{2B} satisfied formula (2), and did not find the expression of ϕ_{2B} from (2). In our new proof of theorem 2.4, we will perform a Laplace inversion with a methodology that would allow to recover the expression ϕ_{2B} from formula (2).

Theorem 2.3. *The Laplace transform ϕ_{2B} of the law of the quadratic functional of a normalized Brownian bridge is given by*

$$\phi_{2B}(x) = \left(\frac{\sqrt{2x}}{\sinh\left(\sqrt{2x}\right)}\right)^{1/2}.$$

Theorem 2.4. *The Laplace transform ϕ_{2E} of the law of the quadratic functional of a normalized excursion is given by*

$$\phi_{2E}(x) = \left(\frac{\sqrt{2x}}{\sinh\left(\sqrt{2x}\right)}\right)^{3/2}.$$

We must note nevertheless that both of those results can be embedded in one result only that was proved since by other probabilists. Here is briefly what it is about:

Proof of Theorems 2.3 and 2.4 In [22] p.18, one can find the formula, derived from Girsanov's theorem,

$$\mathbb{E}_0\left[\exp\left(-\frac{b^2}{2}\int_0^t ds\,|B_s|^2\right)\Big|B_t=0\right]=\left(\frac{bt}{\sinh(bt)}\right)^{\frac{\delta}{2}},$$

where B_t is a $\delta-$dimensional Brownian motion. By definition, a Brownian bridge is a 1-dimensional Brownian motion starting from zero and conditioned to end in zero, which proves Theorem 2.3, given the scaling property $B_s \overset{\mathcal{D}}{\equiv} \sqrt{\ell}B_{s/\ell}$. As the excursion process Y_t behaves like $|B_t|$ where B_t is a 3-dimensional Brownian motion [17], this also sets Theorem 2.4. □

Now let us give another proof of Theorem 2.4 by using formula (2).

Proof of Theorem 2.3 from Theorem 2.4. Given the computation

$$\int_0^\infty e^{-\alpha t}\phi_{2B}\left(\frac{1}{8}t^2\right)\frac{dt}{\sqrt{t}}=\int_0^\infty e^{-\alpha t}\phi_{2B}\left(\left(\frac{t}{\sqrt{8}}\right)^2\right)\frac{dt}{\sqrt{t}}=\int_0^\infty e^{-\alpha\sqrt{8}s}\phi_{2B}\left(s^2\right)8^{1/4}\frac{ds}{\sqrt{s}},$$

it proves convenient to define

$$g(\alpha)=\frac{8^{-1/4}}{\sqrt{2}}\frac{\Gamma(\alpha/\sqrt{8}+\frac14)}{\Gamma(\alpha/\sqrt{8}+\frac34)},$$

so that, for excursions,

$$\int_0^\infty\left(e^{-\alpha t}-1\right)\phi_{2E}(t^2)\frac{dt}{\sqrt{2\pi t^3}}=\sqrt{2}\cdot 8^{1/4}\left(-\frac{\Gamma(\alpha/\sqrt{8}+\frac34)}{\Gamma(\alpha/\sqrt{8}+\frac14)}+\frac{\Gamma(\frac34)}{\Gamma(\frac14)}\right). \tag{3}$$

We then get, after the change of variables $u=\alpha/\sqrt{8}$, $\sigma=t\sqrt{8}$, and $\sigma^3=t^3 8\sqrt{8}$,

$$\int_0^\infty\left(e^{-u\sigma}-1\right)\phi_{2E}\left(\frac{\sigma^2}{8}\right)8^{1/4}\frac{d\sigma}{\sqrt{2\pi\sigma^3}}=\sqrt{2}\cdot 8^{1/4}\left(-\frac{\Gamma(u+\frac34)}{\Gamma(u+\frac14)}+\frac{\Gamma(\frac34)}{\Gamma(\frac14)}\right).$$

After differentiation with respect to u, we obtain

$$\int_0^\infty e^{-u\sigma}\phi_{2E}\left(\frac{\sigma^2}{8}\right)\frac{d\sigma}{\sqrt{2\pi\sigma}}=\sqrt{2}\left(-\frac{\Gamma'(u+\frac34)}{\Gamma(u+\frac14)}+\frac{\Gamma(u+\frac34)\Gamma'(u+\frac14)}{\Gamma(u+\frac14)^2}\right),$$

which entails, by inversion of the Laplace transform,

$$\frac{\phi_{2E}\left(\frac{\sigma^2}{8}\right)}{\sqrt{2\pi\sigma}}=\frac{\sqrt{2}}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{\sigma s}\left(-\frac{\Gamma'(s+\frac34)}{\Gamma(s+\frac14)}+\frac{\Gamma(s+\frac34)\Gamma'(s+\frac14)}{\Gamma(s+\frac14)^2}\right)ds.$$

For $|\arg(z)|<\pi$, holds the expansion (computable by Maple)

$$-\frac{\Gamma'(z+\frac34)}{\Gamma(z+\frac14)}+\frac{\Gamma(z+\frac34)\Gamma'(z+\frac14)}{\Gamma(z+\frac14)^2}\underset{|z|\to\infty}{=}-\frac{1}{2z^{1/2}}+\frac{3}{128}\frac{1}{z^{5/2}}+O\left(\frac{1}{z^{7/2}}\right),$$

hence we can use the theorem of residues to compute the inverse of the Laplace transform.

As $\dfrac{\Gamma(z)}{\Gamma(z-\frac{1}{2})}$ has simple poles in $z=-n$ for $n\in\mathbb{N}$, its derivative has double poles in $z=-n$, so that

$$-\frac{\Gamma'(-n+h)}{\Gamma(-n+h-\frac{1}{2})}+\frac{\Gamma(-n+h)\Gamma'(-n+h-\frac{1}{2})}{\Gamma(-n+h-\frac{1}{2})^2}\underset{h\to 0}{=}-\frac{(-1)^n}{n!\Gamma(-n-\frac{1}{2})}\frac{1}{h^2}+O(1).$$

On the other hand,

$$e^{\sigma(-n+h-\frac{3}{4})}=e^{-(n+\frac{3}{4})\sigma}e^{\sigma h}\underset{h\to 0}{=}e^{-(n+\frac{3}{4})\sigma}(1+\sigma h+O(h^2)),$$

so that :

$$\operatorname{Res}\left(e^{\sigma(s-\frac{3}{4})}\left(-\frac{\Gamma'(s+\frac{3}{4})}{\Gamma(s+\frac{1}{4})}+\frac{\Gamma(s+\frac{3}{4})\Gamma'(s+\frac{1}{4})}{\Gamma(s+\frac{1}{4})^2}\right),s=-n\right)=-\frac{(-1)^n}{n!}\frac{\sigma e^{-\frac{3}{4}\sigma-n\sigma}\sigma}{\Gamma(-n-\frac{1}{2})}.$$

and, using $\Gamma(-1/2)=-2\sqrt{\pi}$,

$$\begin{aligned}\frac{\phi_{2E}\left(\frac{\sigma^2}{8}\right)}{\sqrt{2\pi\sigma}} &= -\sqrt{2}\sum_{n\in\mathbb{N}}\frac{(-1)^n}{n!}\frac{\sigma e^{-\frac{3}{4}\sigma-n\sigma}}{\Gamma(-n-\frac{1}{2})}=-\sqrt{2}\frac{\sigma e^{-\frac{3}{4}\sigma}}{\Gamma\left(-\frac{1}{2}\right)}\sum_{n\in\mathbb{N}}\binom{-\frac{1}{2}}{n}(e^{-\sigma})^n\\ &= \frac{\sqrt{2}\sigma e^{-\frac{3}{4}\sigma}}{2\sqrt{\pi}}(1-e^{-\sigma})^{-\frac{3}{2}}\\ \phi_{2E}(\sigma) &= \frac{(\sqrt{2\sigma})^{3/2}}{\sinh(\sqrt{2\sigma})^{3/2}}.\end{aligned}$$

□

Though formulas (2) and (3) do not allow to easily find recurrences to compute the moments (it is nevertheless possible through the asymptotic expansions given for example in [13]), the series expansion of the explicit expressions of ϕ_{2B} and ϕ_{2E} give such recurrences.

2.3. Joint area and quadratic laws

The functionals $x\mapsto|x|^\alpha$ are studied in general [7], but hoping to get explicit expressions seem to require to know an explicit solution of the equation $u''-2(\alpha+V)u=0$, where V is the considered functional. Explicit solutions seem to be known only in the cases $\alpha=-1,0,1,2$.

We will investigate here the double Laplace transform for the joint distribution of the absolute value and the quadratic functionals.

Proposition 2.1. *Let ϕ_{12B} be the Laplace transform of the joint law of the absolute value and the quadratic functional of the normalised Brownian bridge. Then*

$$\int_0^\infty e^{-\alpha t}\phi_{12B}\left(\frac{a}{2}t^{3/2},\frac{b^2}{8}t^2\right)\frac{dt}{\sqrt{2\pi t}}=-\frac{1}{b^{1/2}}\frac{D_{a^2b^{-3}-\frac{2\alpha}{b}-\frac{1}{2}}\left(2ab^{-3/2}\right)}{D'_{a^2b^{-3}-\frac{2\alpha}{b}-\frac{1}{2}}\left(2ab^{-3/2}\right)}.$$

Proof. Consider equation $u''-2(\alpha+V)u=0$ with $V(x)=\frac{b^2}{8}x^2+\frac{a}{2}|x|$, that is, for

$x \geq 0$, $u'' - \left(\frac{b^2}{4}x^2 + ax + 2\alpha\right) u = 0$, or

$$u'' - \left(\frac{1}{4}\left(bx + \frac{2a}{b}\right)^2 - \frac{a^2}{b^2} + 2\alpha\right) u = 0.$$

Let us look for a solution y of this equation. The parabolic cylinder function D_ν [1] is a solution bounded at $+\infty$ of

$$u'' = \left(\frac{1}{4}x^2 - \nu - \frac{1}{2}\right) u,$$

hence if we define $Y(x) = D_\nu(px+q)$, differentiation of order two with respect to x and identification give

$$u(x) = D_{a^2 b^{-3} - \frac{2\alpha}{b} - \frac{1}{2}}\left(b^{1/2}x + 2ab^{-3/2}\right).$$

If ϕ_{12B} is defined by

$$\phi_{12B}(a,b) = \mathrm{E}\left[e^{-\left(a\int_0^1 |\tilde{W}_s|\ ds + b\int_0^1 \tilde{W}_s^2\ ds\right)}\right]$$

then

$$\begin{aligned} \mathbb{E}\left[e^{-\int_0^t a|W_s| + b^2 W_s^2\ ds} \;\middle|\; W_t = 0\right] &= \mathrm{E}\left[e^{-\left(t^{3/2} a\int_0^1 |\tilde{W}_s|\ ds + b^2t^2\int_0^1 \tilde{W}_s^2\ ds\right)}\right] \\ &= \phi_{12B}\left(at^{3/2}, b^2t^2\right). \end{aligned}$$

Finally we have

$$\int_0^\infty e^{-\alpha t}\phi_{12B}\left(\frac{a}{2}t^{3/2}, \frac{b^2}{8}t^2\right)\frac{dt}{\sqrt{t}} = -\sqrt{2\pi}\frac{Y(0)}{Y'(0)},$$

or

$$\int_0^\infty e^{-\alpha t}\phi_{12B}\left(\frac{a}{2}t^{3/2}, \frac{b^2}{8}t^2\right)\frac{dt}{\sqrt{t}} = -\frac{\sqrt{2\pi}}{b^{1/2}}\frac{D_{a^2b^{-3}-\frac{2\alpha}{b}-\frac{1}{2}}\left(2ab^{-3/2}\right)}{D'_{a^2b^{-3}-\frac{2\alpha}{b}-\frac{1}{2}}\left(2ab^{-3/2}\right)}. \tag{4}$$

□

This formula would prove difficult to get an explicit expression for ϕ_{12B} and would at best lead to a sum of residues. The first terms of this sum could give a fair idea of what ϕ_{12B} looks like, but it will not be developed in this paper. As for moments, one should in principle be able to deduce them from (4), but this is already difficult to extract them from the limit case (2), even though it becomes easy with the explicit formula of ϕ_B. We will show in the next section how to compute those moments with the generating function approach.

Proposition 2.2. *Let ϕ_{12E} be the Laplace transform of the joint law of the absolute* *value and the quadratic functional of the normalised excursion. Then*

$$\int_0^\infty (e^{-\alpha t}-1)\phi_{12E}\left(\frac{a}{2}t^{3/2},\frac{b^2}{8}t^2\right)\frac{dt}{\sqrt{2\pi t^3}} = \sqrt{b}\frac{D'_{a^2b^{-3}-\frac{2\alpha}{b}-\frac{1}{2}}\left(2ab^{-3/2}\right)}{D_{a^2b^{-3}-\frac{2\alpha}{b}-\frac{1}{2}}\left(2ab^{-3/2}\right)} -\sqrt{b}\frac{D'_{a^2b^{-3}-\frac{1}{2}}\left(2ab^{-3/2}\right)}{D_{a^2b^{-3}-\frac{1}{2}}\left(2ab^{-3/2}\right)}.$$

Proof. From the former proposition and Louchard's formula, we get the result. □

Inverting this modified Laplace transform would require to differentiate with respect to α, which does not lead to a friendly expression for the right hand side. Therefore, finding the residues of

$$\alpha \to e^{\sigma\alpha}\frac{\partial\left(D'_{a^2b^{-3}-\frac{2\alpha}{b}-\frac{1}{2}}\left(2ab^{-3/2}\right)/D_{a^2b^{-3}-\frac{2\alpha}{b}-\frac{1}{2}}\left(2ab^{-3/2}\right)\right)}{\partial\alpha}$$

for $\sigma>0$ should be deduced from the behaviour of

$$\frac{D'_{a^2b^{-3}-\frac{2\alpha}{b}-\frac{1}{2}}\left(2ab^{-3/2}\right)}{D_{a^2b^{-3}-\frac{2\alpha}{b}-\frac{1}{2}}\left(2ab^{-3/2}\right)}.$$

We will not investigate further in this paper.

3. The generating function approach

Thanks to Drmota's work ([2] Theorem 9), it is known that for functionals of polynomial growth, we have convergence of moments of these functionals for Dyck paths or Grand Dyck paths when the size of these paths tends to infinity, towards the moments of the same functionals of respectively the excursion and the Brownian bridge. Hence there is no any longer need to check for example Carleman's condition as did Takács in [19].

This section is devoted to find recursive relations to compute the moments of the joint distribution of the joint absolute value and quadratic functional of both the excursion and the Brownian bridge, identified as the limit distribution of the same functionals applied to Grand Dyck paths and Dyck paths. We proceed as did Takács in various papers among which [19], by finding a functional equation for the multivariate generating function of Dyck paths, and pumping the moments, which amounts to differentiate with respect to the wanted variables and make them equal to 1.

This method can be used to find the multivariate moments of the joint distribution of functionals of differents integer powers, and then give another proof of Louchard's theorem for the case of functionals of the type $x \mapsto |x|^M$.

3.1. Functional equation

Proposition 3.1. *Let $D(z,u,v)$ be the trivariate generating function enumerating Dyck paths according to their length (variable z), area S_1 (variable u), and quadratic functional S_2 (variable v). Then $D(z,u,v)$ satisfies:*

$$D(z,u,v) = 1 + z^2uvD(z,u,v)D(zuv,uv^2,v).$$

Proof. Let us consider a positive arch a of length $2n+2$, and let $h_0, h_1, \cdots, h_{2n+2}$ be the altitudes of its points. The integers $h'_1 = h_1 - 1, h'_2 = h_2 - 1, \cdots, h'_{2n+1} = h_{2n+1} - 1$ define a Dyck path d, and from

$$\sum_{i=0}^{2n+2} h_i^2 = \sum_{i=1}^{2n+1} (1+h_i)^2$$

that $S_1(a) = 2n+1+S_1(d)$ and $S_2(a) = 2n+1+2S_1(d)+S_2(d)$. Hence, the generating function of positive arches is given by $z^2uvD(zuv, uv^2, v)$. As a Dyck path is either an empty path or a Dyck path followed by a positive arch, we have the wanted functional equation. □

3.2. Joint moments for Dyck paths

In order to avoid the uselessly inextricate computations involved by the differentiation of z^2uv, we introduce a cosmetic change of variable by introducing the function $G(z,u,v) = zD(z,u,v)$ which satisfies equation

$$G(z,u,v) = z + z^2 G(z,u,v) G(zuv, uv^2, v).$$

Then we have the property :

Proposition 3.2. *Let $\epsilon_{r,s}$ denote the moment of order (r,s) of the joint distribution of the absolute value and the square functionals of the excursion. In other words, if $X = \int_0^1 e(s)\, ds$ and $Y = \int_0^1 e^2(s)\, ds$, where e is a normalized excursion, $\epsilon_{r,s} = \mathbb{E}[X^r Y^s]$. Then there exist constants $A_{r,s} \neq 0$ such that for every r, s,*

$$\epsilon_{r,s} = \frac{A_{r,s}}{2^{\frac{3r}{2}+2s}} \frac{\sqrt{\pi}\, r!s!}{\Gamma\left(\frac{3r+4s-1}{2}\right)}.$$

They can be computed by the following recurrences :

$$\begin{cases} A_{0,0} &= -\frac{1}{2}, \\ A_{r,s} &= (r+1)A_{r+1,s-1} + \sum_{j=0}^{s}\sum_{i=0}^{r} \mathbb{1}_{[i+j>0, r+s>i+j]} A_{i,j} A_{r-i,s-j} \\ & \quad + \mathbb{1}_{r>0} \frac{(3r+4s-4)}{2} A_{r-1,s}. \end{cases} \tag{3.2}$$

Before showing the proof, we need the following lemma:

Lemma 3.1. *For every $r, s \in \mathbb{N}$, $zG_{u^r v^s}(z,1,1)$ is a polynomial in $1/R$ where $R = \sqrt{1-4z^2}$. If we define $(A_{r,s}$ such that*

$$\frac{z}{r!s!} G_{u^r v^s}(z,1,1) \underset{z \to \frac{1}{2}}{\sim} A_{r,s}/R^{3r+4s-1},$$

then $(A_{r,s})$ is given by the recurrence (3.2).

Proof. Notice that, except for $r = s = 0$ for which $zG(z,1,1) = \frac{1-R}{2}$, $A_{r,s}$ is the dominant coefficient of $\frac{z}{r!s!} G_{u^r v^s}(z,1,1)$ seen as a polynomial in $1/R$.

We show the lemma by induction on $r+s$. The wanted property is true for $r = s = 0$.

Now fix r, s such that $r+s>0$ and suppose the result granted for r', s' such that $r'+s'<r+s$.

Then, the formula

$$\frac{\partial^{r+s}G(z,u,v)}{\partial u^r v^s} = \delta_{r0}\delta_{s0}z + z\sum_{j=0}^{s}\binom{s}{j}\sum_{i=0}^{r}\binom{r}{i}\frac{\partial^{i+j}G(z,u,v)}{\partial u^i\partial v^j}\frac{\partial^{r-i+s-j}G(zuv,uv^2,v)}{\partial u^{r-i}\partial v^{s-j}}$$

entails

$$\begin{aligned}
&(1-2zG(z,1,1))z\frac{\partial^{r+s}G(z,u,v)}{r!s!\partial u^r\partial v^s}\Big|_{u=v=1}\\
=\ & z\sum_{\substack{0\le i\le r, 0\le j\le s\\ 0<i+j<r+s}}\binom{s}{j}\binom{r}{i}\frac{\partial^{i+j}G(z,u,v)}{\partial u^i\partial v^j}\Big|_{u=v=1}\frac{\partial^{r-i+s-j}G(zuv,uv^2,v)}{\partial u^{r-i}\partial v^{s-j}}\Big|_{u=v=1}
\end{aligned}$$

with $1-2z^2G(z,1,1)=R$ and $3i+4j-1+3(r-i)+4(s-i)-1=3r+4s-2$ for all i,j, which shows that $zG_{u^rv^s}(z,1,1)$ is a polynomial in $1/R$ of degree $3r+4s-1$. We can define $r!s!A_{r,s}$ to be its dominant coefficient.

From

$$z\frac{\partial^r G(zuv,uv^2,v)}{r!\partial u^r}\Big|_{u=v=1} = zG_{u^r}(z,1,1)+rz\cdot zG_{zu^{r-1}}(z,1,1)+\cdots,$$

it is easy to see that

$$zG_{zu^{r-1}}/(r-1)! = \frac{1}{2}8z^2A_{r-1,0}/R^{3(r-1)}+\cdots = A_{r-1,0}/R^{3(r-1)}+\cdots,$$

and similarly, using Faa di Bruno's formula [14], retaining only the dominant coefficients in

$$\begin{aligned}
&z\frac{\partial^{r+s}G(zuv,uv^2,v)}{\partial u^r\partial v^s}\\
=\ &\sum_{k_1+2k_2=s}\frac{s!}{k_2!}\sum_{q_1+q_2+q_3=k_1}\frac{2^{q_2}}{q_1!q_2!q_3!}\cdot z^{q_1}\frac{\partial^r}{\partial u^r}u^{q_1+q_2+k_2}G_{z^{q_1}u^{q_2+k_2}v^{q_3}}(zuv,uv^2,v),
\end{aligned}$$

which correspond to $(k_2=0, k_1=s, q_1=0)$ and $((q_3=s, q_2=0)$ or $(q_3=s-1, q_2=1))$, that

$$\begin{aligned}
z\frac{\partial^{r+s}G(zuv,uv^2,v)}{r!s!\partial u^r\partial v^s}\Big|_{u=v=1} &= A_{r,s}/R^{3r+4s-1}+2s\frac{(r+1)!(s-1)!}{r!s!}A_{r+1,s-1}/R^{3r+4s-2}\\
&\quad +r\frac{(r-1)!s!(3r+4s-4)8z}{r!s!2}A_{r-1,s}/R^{3r+4s-2}+\cdots\\
&= A_{r,s}/R^{3r+4s-1}+2(r+1)A_{r+1,s-1}/R^{3r+4s-2}\\
&\quad +2(3r+4s-4)A_{r-1,s}/R^{3r+4s-2}+\cdots.
\end{aligned}$$

Table 1 Values of $A_{r,s}$.

r,s	0	1	2	3	4	5
0	-1/2	12	13/4	63	37297/16	280371/2
1	1/4	3/2	239/8	4515/4	2199119/32	24733149/4
2	5/16	107/16	8461/32	66045/4	192844087/128	3045954135/16
3	15/16	159/4	2585	482475/2	3951804863/128	83537034105/16
4	$\frac{1105}{256}$	$\frac{75783}{256}$	$\frac{14599815}{512}$	$\frac{475670055}{128}$	$\frac{1300934713135}{2048}$	$\frac{35461554388785}{256}$
5	$\frac{1695}{64}$	$\frac{84843}{32}$	$\frac{11323097}{32}$	$\frac{3926382201}{64}$	$\frac{3455175997157}{256}$	$\frac{235919056678557}{64}$

Finally, by identification

$$\begin{aligned}
A_{r,s} &= (r+1)A_{r+1,s-1} + \sum_{j=0}^{s}\sum_{i=0}^{r} \mathbb{I}_{[i+j>0,r+s>i+j]} A_{i,j} A_{r-i,s-j} \\
&\quad + \mathbb{I}_{r>0} A_{0,0} z 2(3r+4s-4) A_{r-1,s}/R^{3r+4s-2} \\
&= (r+1)A_{r+1,s-1} + \sum_{j=0}^{s}\sum_{i=0}^{r} \mathbb{I}_{[i+j>0,r+s>i+j]} A_{i,j} A_{r-i,s-j} \\
&\quad + \mathbb{I}_{r>0} \frac{(3r+4s-4)}{2} A_{r-1,s}/R^{3r+4s-2},
\end{aligned}$$

which is the announced formula. □

Proof of Proposition 3.3. As $[z^n]G_{u^p v^q}(z,1,1)$ (the coefficient of z^n in the power series expansion of $G_{u^p v^q}(z,1,1)$) is neglectable compared to $[z^n]G_{u^r v^s}(z,1,1)$ as soon as $p+q<r+s$, the joint factorial moments are the same as the moments themselves. Using the well-known asymptotics of Catalan numbers [5], we get the wanted result. □

It is now easy to compute the joint moments with a few lines of Maple, which leads to the tables 1 and 2, where we put the first values of $A_{r,s}$ and $\epsilon_{r,s}$. For $r=0$ and $s=0$ we recover in table 2 the values found respectively by Takács [19] or the series expansion of ϕ_{2B}.

3.3. Joint moments for Grand Dyck paths

Let us now prove similar results for the Brownian bridge. We will now focus our attention on Grand Dyck paths.

Proposition 3.3. *Let $\beta_{r,s}$ denote the moment of order (r,s) of the joint distribution of the absolute value and the square functionals of the excursion. In other words, if $X=\int_0^1 b(s)\,ds$ and $Y=\int_0^1 b^2(s)\,ds$, where b is a normalized excursion, $\beta_{r,s}=\mathbb{E}[X^rY^s]$.*

Table 2 Moments of joint absolute value and square of excursion.

r,s	0	1	2	3	4	5
0	1	1/2	$\frac{19}{60}$	$\frac{631}{2520}$	$\frac{1219}{5040}$	$\frac{92723}{332640}$
1	$\frac{\sqrt{2\pi}}{4}$	$\frac{9\sqrt{2\pi}}{64}$	$\frac{51\sqrt{2\pi}}{512}$	$\frac{7143\sqrt{2\pi}}{81920}$	$\frac{422781\sqrt{2\pi}}{4587520}$	$\frac{25311493\sqrt{2\pi}}{220200960}$
2	$\frac{5}{12}$	$\frac{221}{840}$	$\frac{1643}{7920}$	$\frac{4321753}{21621600}$	$\frac{291886913}{1269777600}$	$\frac{881725807}{2855588736}$
3	$\frac{15\sqrt{2\pi}}{128}$	$\frac{339\sqrt{2\pi}}{4096}$	$\frac{16573\sqrt{2\pi}}{229376}$	$\frac{838799\sqrt{2\pi}}{11010048}$	$\frac{13125751\sqrt{2\pi}}{138412032}$	$\frac{330256199219\sqrt{2\pi}}{2418335023104}$
4	$\frac{221}{1008}$	$\frac{16565}{96096}$	$\frac{40594139}{245044800}$	$\frac{5310071279}{27935107200}$	$\frac{181927010953}{713897184000}$	$\frac{87609843192007}{223592598028800}$
5	$\frac{565\sqrt{2\pi}}{8192}$	$\frac{3935\sqrt{2\pi}}{65536}$	$\frac{463551\sqrt{2\pi}}{7340032}$	$\frac{3648007367\sqrt{2\pi}}{46506442752}$	$\frac{3817396540159\sqrt{2\pi}}{33856690323456}$	$\frac{199793478812293\sqrt{2\pi}}{1083414090350592}$

Then there exist constants $B_{r,s} \neq 0$ such that for every r, s,

$$\beta_{r,s} = \frac{B_{r,s}}{2^{\frac{3r}{2}+2s}} \frac{\sqrt{\pi} r! s!}{\Gamma\left(\frac{3r+4s+1}{2}\right)}.$$

They can be computed by the following recurrences :

$$\left\{ \begin{array}{rcl} B_{0,0} & = & -\frac{1}{2}, \\ B_{r,s} & = & (r+1)B_{r+1,s-1} + \sum_{j=0}^{s} \sum_{i=0}^{r} \mathbb{1}_{[i+j>0, r+s>i+j]} B_{i,j} B_{r-i,s-j} \\ & & + \mathbb{1}_{r>0} \frac{(3r+4s-4)}{2} B_{r-1,s}. \end{array} \right. \tag{3.3}$$

Proof. Let us introduce the generating function $P(z,u,v)$ of nonvoid Grand Dyck paths. A nonvoid Grand Dyck path is a sequence of nonvoid positive arches, hence

$$(1 - 2z^2 uvD(zuv, uv^2, v))P(z,u,v) = 1$$

From it we deduce that for every r, s, $P_{u^r v^s}(z,u,v)$ is a Laurent series in $1/R$ of degree $3r+4s+1$. Define the double sequence $(B_{r,s})$ such that

$$P_{u^r v^s}(z,1,1) \sim r!s!B_{r,s}/R^{3r+4s+1}.$$

Now define $H(z,u,v) := 1 - 2z^2 uvD(zuv, uv^2, v) = 1 - 2zG(zuv, uv^2, v)$. From $HP = 1$, we deduce

$$\sum_{j=0}^{s} \sum_{i=0}^{r} \binom{s}{j} \binom{r}{i} H_{u^i v^j} P_{u^{r-i} v^{s-j}} = \delta_{r0} \delta_{s0}.$$

By identifying coefficients, one obtains

$$\sum_{j=0}^{s} \sum_{i=0}^{r} A_{i,j} B_{r-i,s-j} = \delta_{r0} \delta_{s0} A_{0,0} B_{0,0},$$

that gives a fairly convenient recurrence formula to define and compute the $B_{r,s}$ for a start, that is

$$A_{0,0} = -1/2, \quad B_{0,0} = 1, \quad B_{r,s} = 2 \sum_{j=0}^{s} \sum_{i=0}^{r} \mathbb{1}_{i+j>0, r+s>i+j} A_{i,j} B_{r-i,s-j}.$$

Nevertheless, we can in fact devise a recurrence depending only on $B_{r,s}$ in the following way.

First, if we define $A(u,v) = \sum A_{r+s}(-1)^{r+s}u^r v^s$, the recurrence on the A_{r+s} translates in

$$vA_u + A^2 - A(0,0)^2 - \frac{3}{2}u^2 A_u + 2uvA_v + \frac{u}{2}A = 0. \quad (3.3)$$

From $HP = 1$, we deduce that, if we define $B(u,v) = \sum B_{r+s}(-1)^{r+s}u^r v^s$, then BA is a constant, that is $BA = (BA)(0,0) = -1/2$, hence $A = -1/(2B)$, $A_u = B_u/(2B^2)$, $A_v = B_v/(2B^2)$, and formula (3.3) becomes

$$v\frac{B_u}{2B^2} + \frac{1}{4B^2} - \frac{1}{4} - \frac{3}{2}u^2\frac{B_u}{2B^2} + uv\frac{B_v}{B^2} - \frac{u}{4B} = 0$$

or

$$2vB_u + 1 - B^2 - 3u^2 B_u + 4uvB_v - uB = 0. \quad (5)$$

Using

$$\begin{aligned} B(u,v) &= \sum(-1)^{r+s}B_{rs}u^r v^s \\ vB_u &= \sum(-1)^{r+s}rB_{rs}u^{r-1}v^{s+1} \\ u^2B_u &= \sum(-1)^{r+s}rB_{rs}u^{r+1}v^s \\ uvB_v &= \sum(-1)^{r+s}sB_{rs}u^{r+1}v^s \\ uB &= \sum(-1)^{r+s}sB_{rs}u^{r+1}v^s. \end{aligned}$$

we then obtain, for $r+s>0$,

$$2(r+1)B_{r+1,s-1} - \sum_{i=0}^{r}\sum_{j=0}^{s} B_{i,j}B_{r-i,s-j} + (3r+4s-2)B_{r-1,s} = 0.$$

Again, singularity analysis [5] delivers $\beta_{r,s} = \frac{B_{r,s}}{2^{\frac{3r}{2}+2s}}\frac{\sqrt{\pi}r!s!}{\Gamma\left(\frac{3r+4s+1}{2}\right)}$. □

We can easily compute the moments $\beta_{r,s}$, of which we can see the first values in table 4. For $r=0$ and $s=0$, we recover the values found by Shepp [18] or Tolmatz [20, 21].

It does not seem possible to find a friendly expression of B (or A). Nevertheless, if we define

$$g(\alpha,a,b) = \int_0^\infty e^{-\alpha t}\phi_{12B}\left(at^{3/2}, b^2t^2\right)\,\frac{dt}{2\pi t},$$

a straightforward computation gives

$$g(\alpha,a,b) = \frac{1}{\sqrt{2\alpha}}B\left(\frac{a}{(2\alpha)^{3/2}}, \frac{b^2}{(2\alpha)^2}\right),$$

or

$$B(u,v) = g(1/2, u, \sqrt{v}),$$

Table 3 Values of $B_{r,s}$.

r,s	0	1	2	3	4	5
0	1	$1/2$	$21/8$	$671/16$	$180323/128$	$20898423/256$
1	$1/2$	$11/4$	$173/4$	$22931/16$	$1319183/16$	$233526463/32$
2	$\frac{7}{8}$	$\frac{273}{16}$	$\frac{38733}{64}$	$\frac{4615949}{128}$	$\frac{3344400297}{1024}$	$\frac{856393361595}{2048}$
3	$\frac{21}{8}$	$\frac{2037}{16}$	$\frac{275373}{32}$	$\frac{1672797}{2}$	$\frac{7175989065}{64}$	$\frac{5097582269451}{256}$
4	$\frac{1463}{128}$	$\frac{289555}{256}$	$\frac{133182175}{1024}$	$\frac{39357656121}{2048}$	$\frac{59615450866005}{16384}$	$\frac{28408170184726485}{32768}$
5	$\frac{2121}{32}$	$\frac{187677}{16}$	$\frac{67703769}{32}$	$\frac{57697769535}{128}$	$\frac{59486822349255}{512}$	$\frac{2313457431011811}{64}$

Table 4 Moments of joint absolute value and square of Brownian bridge.

r, s	0	1	2	3	4	5
0	1	$1/6$	$1/20$	$\frac{61}{2520}$	$\frac{1261}{75600}$	$\frac{79}{5280}$
1	$\frac{\sqrt{2\pi}}{8}$	$\frac{11\sqrt{2\pi}}{384}$	$\frac{173\sqrt{2\pi}}{15360}$	$\frac{22931\sqrt{2\pi}}{3440640}$	$\frac{1319183\sqrt{2\pi}}{247726080}$	$\frac{233526463\sqrt{2\pi}}{43599790080}$
2	$\frac{7}{60}$	$\frac{13}{360}$	$\frac{12911}{720720}$	$\frac{355073}{28274400}$	$\frac{53085719}{4655851200}$	$\frac{19030963591}{1499184086400}$
3	$\frac{21\sqrt{2\pi}}{1024}$	$\frac{679\sqrt{2\pi}}{81920}$	$\frac{13113\sqrt{2\pi}}{2621440}$	$\frac{79657\sqrt{2\pi}}{19660800}$	$\frac{68342753\sqrt{2\pi}}{16609443840}$	$\frac{26971334759\sqrt{2\pi}}{5374077829120}$
4	$\frac{19}{720}$	$\frac{8273}{617760}$	$\frac{761041}{79814592}$	$\frac{1874174101}{214169155200}$	$\frac{3974363391067}{404779703328000}$	$\frac{1893878012315099}{145558781316748800}$
5	$\frac{101\sqrt{2\pi}}{16384}$	$\frac{993\sqrt{2\pi}}{262144}$	$\frac{358221\sqrt{2\pi}}{115343360}$	$\frac{549502567\sqrt{2\pi}}{172738215936}$	$\frac{62949018359\sqrt{2\pi}}{16122233487360}$	$\frac{12240515507999\sqrt{2\pi}}{2192623754280960}$

and formula (4) of the former section gives the following expression of B :

$$B(u,v) = -\frac{1}{(v/8)^{1/4}} \frac{D_{u^2 v^{-3/2} 2^{-1/2} - \sqrt{2b}/2 - 1/2}(2^{7/4} u v^{-3/4})}{D'_{u^2 v^{-3/2} 2^{-1/2} - \sqrt{2b}/2 - 1/2}(2^{7/4} u v^{-3/4})}.$$

We did not manage to show or check directly that this expression is a solution of (5): being able to do that would give another proof of theorem 2.3 without resorting to Brownian motion at all.

3.4. New proof of Louchard theorem for monomial functionals

Using the same framework as before would allow to compute the joint moments of any integer functionals of both Brownian bridges and excursions. In addition to this practical consequence, it also allows us to give another proof of Louchard's theorem for those functionals.

Theorem 3.1. *Let V be a function of the form $V(x) = |x|^M$ with $M \in \mathbb{N}$. Then if ϕ_B is the Laplace transform of the distribution of the functional V of the normalized Brownian bridge, and ϕ_E is the Laplace transform of the distribution of the functional V*

of the normalized excursion, then if :

$$g(\alpha) := \int_0^\infty e^{-\alpha t}\phi_B\left(t^{1+\frac{M}{2}}\right)\,\frac{dt}{\sqrt{2\pi t}},$$

then

$$\int_0^\infty \left(e^{-\alpha t}-1\right)\phi_B\left(t^{1+\frac{M}{2}}\right)\,\frac{dt}{\sqrt{2\pi t^3}} = -g^{-1}(\alpha) + \lim_{\alpha\to 0} g^{-1}(\alpha).$$

Proof. For a path $p = (h_0, h_1, \cdots, h_{2n})$, we will denote the parameter of interest by

$$S_M(p) = \sum_{i=0}^{2n} |h_i|^M.$$

If $a = (h_0, h_1, \cdots, h_{2n+2})$ is a positive arch, and $d = (h_1, \cdots, h_{2n+1})$ the associated Dyck path, then $S_M(a) = \sum_{j=0}^M \binom{M}{j} S_j(d)$. If $D(z, u_1, \cdots, u_M)$ enumerates the Dyck paths, where each u_j corresponds to the parameter f_j, $G(z,u,v) = zD(z,u,v)$ satisfies some functional equation of the type :

$$G(z,u,v) = z + zG(z,u,v)G(zu_1\cdots u_M, \cdots, u_M),$$

where $G(zu_1\cdots u_M, \cdots, u_M)/z$ enumerates the positive arches — its $(j+1)$-th component is given by

$$u_j u_{j+1}^2 \cdots u_{M-1}^{\binom{M-1}{j}} u_M^{\binom{M}{j}} = \prod_{q=j}^{q=M} u_q^{\binom{q}{j}}.$$

Using the same framework as before, one can show by induction that for every $M+1$-uple $(\alpha_0, \alpha_1, \cdots, \alpha_M)$, there exists some constant $A_{\alpha_0,\alpha_1,\cdots,\alpha_M}$ such that, in the vicinity of $z = \frac{1}{2}$,

$$z\frac{\partial^{\sum \alpha_i} G(z, u_1, \cdots, u_M)}{\partial z^{\alpha_0} u_1^{\alpha_1}\cdots u_M^{\alpha_M}} = \alpha_0!\alpha_1!\cdots,\alpha_M!\frac{A_{\alpha_0,\alpha_1,\cdots,\alpha_M}}{R^{-1+\sum_{i=0}^M (2+M)\alpha_i}} + O\left(\frac{1}{R^{-2+\sum_{i=0}^M (2+M)\alpha_i}}\right).$$

In particular, if we denote $A_{0,\cdots,0,s} = A_s$,

$$z\frac{\partial^s G(z, u_1, \cdots, u_M)}{\partial u_M^s} = s!\frac{A_s}{R^{(2+M)s-1}} + O\left(\frac{1}{R^{(2+M)s-2}}\right).$$

Define $G_+(z, u_1, \cdots, u_M) = G(zu_1\cdots u_M, \cdots, u_M)$. We have $zG_+ = \frac{G-z}{G}$, and the generating function P of non-null arches satisfies

$$P = \frac{1}{1-2zG_+} = \frac{G}{2z-G}.$$

Hence we have

$$zG = \frac{2z^2P}{1+P} = 2z^2 - \frac{2z^2}{1+P},$$

which becomes, after a differentiation with respect to v :

$$zG_v = \frac{2z^2P_v}{1+P} - \frac{2z^2PP_v}{(1+P)^2},$$

or

$$(1+P)^2 z G_v = 2z^2(1+P)P_v - 2z^2 P P_v = 2z^2 P_v.$$

Define B_s such that $P_{v^s} \sim s! B_s / R^{(2+M)s+1}$. After differentiating s times with respect to u_M, and only retaining the dominant monomials in $1/R$, we obtain:

$$\sum_{i+j+k=s} \frac{s!}{i!j!k!} \frac{i!B_i}{R^{(2+M)i+1}} \frac{j!B_j}{R^{(2+M)j+1}} \frac{(k+1)!A_{k+1}}{R^{(2+M)(k+1)-1}} = \frac{(s+1)!B_{s+1}}{2R^{(2+M)(s+1)+1}}.$$

If we define $A(u) = \sum A_r(-u)^r$ and $B(u) = \sum B_r(-u)^r$, then we have $B^2 A' = B'/2$, which means that there exists a constant K such that $A(u) = -1/(2B(u)) + K$. From $A(0) = -1/2$ and $B(0) = 1$, we deduce that $K = 0$ and that AB is constant, equal to $-1/2$.

Let us now denote $\phi_B(x) = \sum_{n\geq 0} \frac{\beta_n}{n!}(-x)^n$ and $\phi_E(x) = \sum_{n\geq 0} \frac{\epsilon_n}{n!}(-x)^n$. We have $B_s = \frac{\beta_s}{\sqrt{\pi} s!} 2^{(1+\frac{M}{2})s} \Gamma\left(\frac{(2+M)s+1}{2}\right)$ and $A_s = \frac{\epsilon_s}{\sqrt{\pi} s!} 2^{(1+\frac{M}{2})s} \Gamma\left(\frac{(2+M)s-1}{2}\right)$. The function

$$g(\alpha) = \int_0^\infty e^{-\alpha t} \phi_B\left(t^{1+\frac{M}{2}}\right) \frac{dt}{\sqrt{2\pi t}}.$$

satisfies, on the one hand,

$$\begin{aligned}
g(\alpha) &= \int_0^\infty e^{-\alpha t} \sum_{n\geq 0} \frac{\beta_n}{n!} \left(-t^{1+\frac{M}{2}}\right)^n \frac{dt}{\sqrt{2\pi t}} \\
&= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\alpha t} \sum_{n\geq 0} \frac{\beta_n}{n!} (-1)^n t^{\frac{2n+Mn-1}{2}} \, dt \\
&= \frac{1}{\sqrt{2\pi}} \sum_{n\geq 0} \frac{\beta_n}{n!} (-1)^n \alpha^{-\frac{(2+M)n+1}{2}} \Gamma\left(\frac{(2+M)n+1}{2}\right) \\
&= \frac{1}{\sqrt{2}} \sum_{n\geq 0} \alpha^{-\frac{(2+M)n+1}{2}} B_n 2^{-(1+\frac{M}{2})n} \\
&= \frac{1}{\sqrt{2\alpha}} B\left(\left(\frac{1}{2\alpha}\right)^{1+\frac{M}{2}}\right)
\end{aligned}$$

and on the other hand, if we define

$$f(\alpha) = \int_0^\infty (e^{-\alpha t} - 1) \phi_E\left(t^{1+\frac{M}{2}}\right) \frac{dt}{\sqrt{2\pi t^3}},$$

similar computations give

$$\begin{aligned}
f'(\alpha) &= -\int_0^\infty e^{-\alpha t} \phi_E\left(t^{1+\frac{M}{2}}\right) \frac{dt}{\sqrt{2\pi t}} \\
&= \cdots \\
&= \left(2\sqrt{2\alpha} A\left(\left(\frac{1}{2\alpha}\right)^{1+\frac{M}{2}}\right)\right)' \\
&= \left(-\frac{1}{g(\alpha)}\right)'
\end{aligned}$$

which entails, given $f(0) = 0$, the wanted relation. $\square$

4. Open questions and conclusion

Natural questions arising from the previous results consists in asking if one could find the limit distribution of the quadratic functionals of Dyck paths without resorting to Brownian motion, and in particular how to get rid of the variable u in $D(z, u, v)$.

The recurrence relations on $A_{r,s}$ and $B_{r,s}$ give rise to a differential equation on the double Laplace transforms of the limit joint distribution, of which we know the expression thanks to the study of the first section. Is it possible to solve this differential equation directly and to find this expression?

Another way of development would consist in pursuing the study in terms of generating functions towards functionals of the standard Brownian motions. A good start would be to recover the distribution of the absolute value and the quadratic functionals of the standard Brownian motion, which are also known. This would require the study of meanders.

The philosophy of this paper is to seek how to unify the point of view of generating functions and the more probabilistic point of view of Brownian motion. For example, can we find an analogue of Feynman-Kac formula in terms of generating functions? It would also be interesting to confront those tools with Prellberg's method [15, 16], which was discarded in this paper, though it also gives rise to double Laplace transforms and Airy functions for the area of polyominoes.

5. Acknowledgements

The author thanks Philippe Duchon for introducing this problem, and Philippe Flajolet for fruitful discussions, among which the passage from Dyck paths to Grand Dyck paths in terms of generating functions. He also thanks Nico Temme for sharing his insight on special functions.

References

[1] Abramowitz, M. and Stegun, I. (1973) *Handbook of Mathematical Functions. A Discipline of Programming*. Dover.

[2] Drmota, M. (2002) Stochastic Analysis of Tree-like Structures.

[3] Duchon, P. (1999) Q-grammars and wall polyominoes. *Annals of Combinatorics* **3** 311–321.

[4] Flajolet, P. and Louchard, G. (2001) Analytic Variations on the Airy Distribution. *Algorithmica* **31** 361–377.

[5] Flajolet, P. and Sedgewick, R. (1992) The Average Case Analysis of Algorithms: Complex Asymptotics and Generating Functions. INRIA Research Report **2026**. 100 pages.

[6] Ito, K., McKean, H. (1965) *Diffusion processes and their sample paths*. Academic Press, New York.

[7] Jeanblanc, M., Pitman, J. and Yor, M. (1997) The Feynman-Kac formula and decomposition of Brownian paths. *Computational and Applied MathematicsAlgorithmica* **16** 27–52.

[8] Karatzas, I., Shreve, S.E. (1997) *Brownian Motion and Stochastic Calculus, 2nd ed.*. Springer-Verlag, New York.

[9] Janson, S. (2001) The Wiener Index of Simply Generated Random Trees. Tech. Report 2001:42, Uppsala.

[10] Louchard, G. (1984) Kac's Formula, Lévy's Local Time and Brownian Excursion. *Journal of Applied Probability* **21** 479–499.

[11] Louchard, G. (1984) The Brownian Excursion Area: a Numerical Analysis. *Computers and Mathematics with Applications* **10** 413–417.

[12] Louchard, G. (1986) Erratum. *Computers and Mathematics with Applications* **12** 375.

[13] Luke, Y. L. (1976) *The Special Functions and Their Approximations.* Vol. 1. Academic Press, New York.

[14] Mishkov, R.L. (1999) Generalisation of the formula of Faa di Bruno for a composite function with a vector argument. *Internat. J. Math. & Math. Sci.* **24**(7) 481–491.

[15] Prellberg, T. (1995) Uniform q-Series Asymptotics for Staircase Polygons. *J. Phys. A: Math. Gen.* **28** 1289-1304.

[16] Prellberg, T. and Brak, R. (1995) Critical Exponents from Non-Linear Functional Equations for Partially Directed Cluster Models. *J. Stat. Phys.* **78** 701-730.

[17] Revuz, D., Yor, M. (1998) *Continuous Martingales and Brownian Motion, 3rd edition.* Springer-Verlag, New York.

[18] Shepp L. A. (1982) On the Integral of the Absolute Value of the Pinned Wiener Process. *The Annals of Probability* **10**(1) 234–239.

[19] Takács, L. (1991) A Bernoulli Excursion and its Various Applications. *Advances of Applied Probability* **23** 557–585.

[20] Tolmatz, L. (2000) Asymptotics of the distribution of the integral of the absolute value of the Brownian bridge for large arguments. *The Annals of Probability* **28** 132–139.

[21] Tolmatz, L. (2002) On the distribution of the square integral of the Brownian bridge. *The Annals of Probability* **30**(1) 253–269.

[22] Yor, Marc. (1992) *Some Aspects of Brownian Motion. Part I.* Lecture Notes in Math., ETH Birkhäuser Verlag.