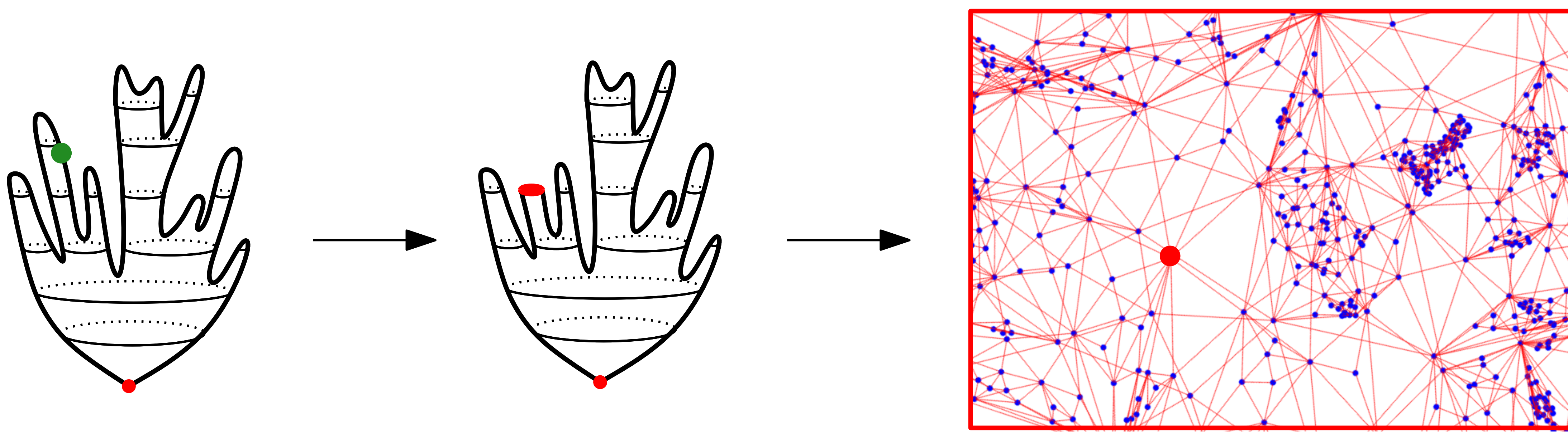
Triangulations with spins : algebraicity and local limit

Marie Albenque (CNRS and LIX)

joint work with **Laurent Ménard** (Paris Nanterre)
and **Gilles Schaeffer** (CNRS and LIX)

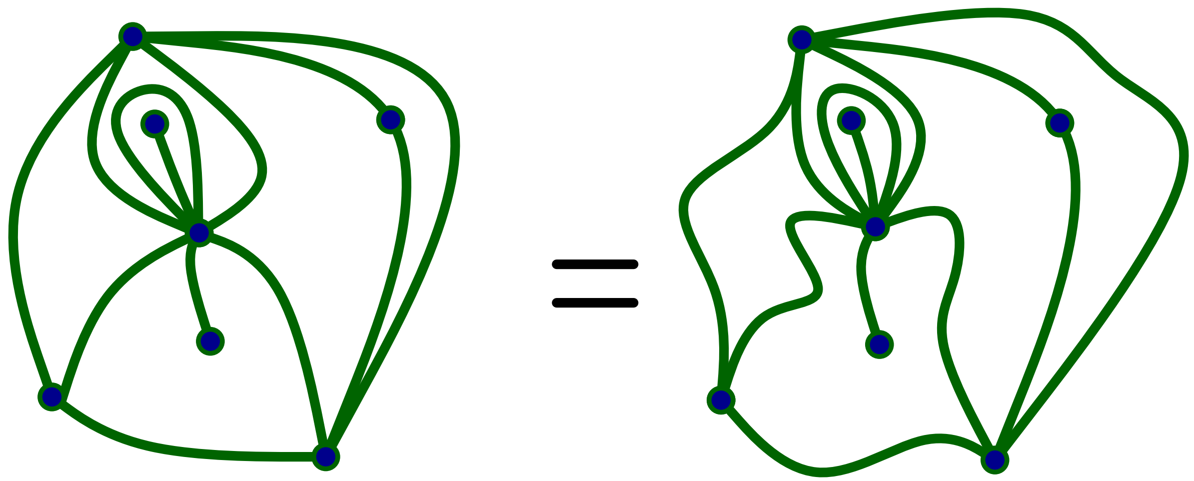


États de la recherche SMF, December 2018

I - Random maps without matter

Planar Maps as discrete planar metric spaces

A **planar map** is the proper embedding of a finite connected graph in the 2-dimensional sphere seen up to continuous deformations.



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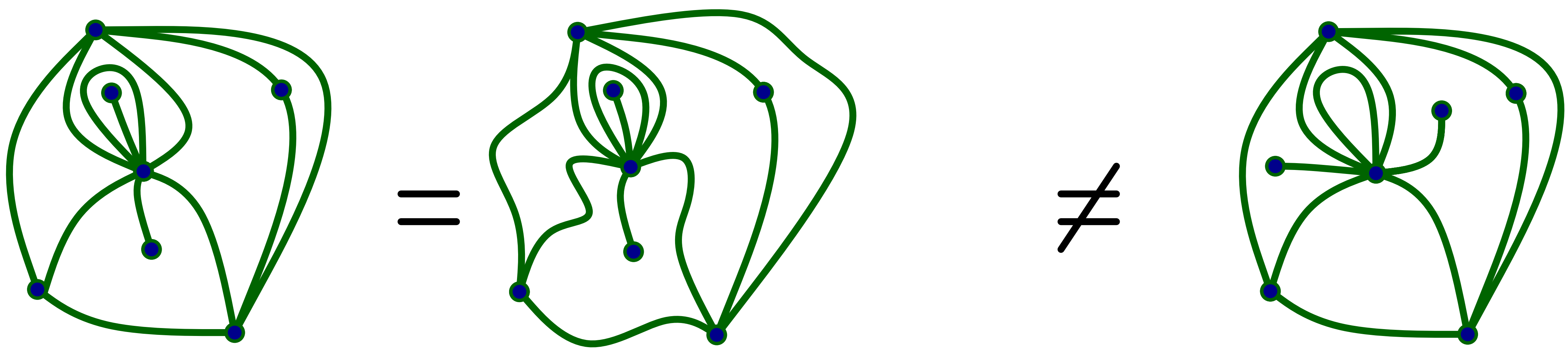

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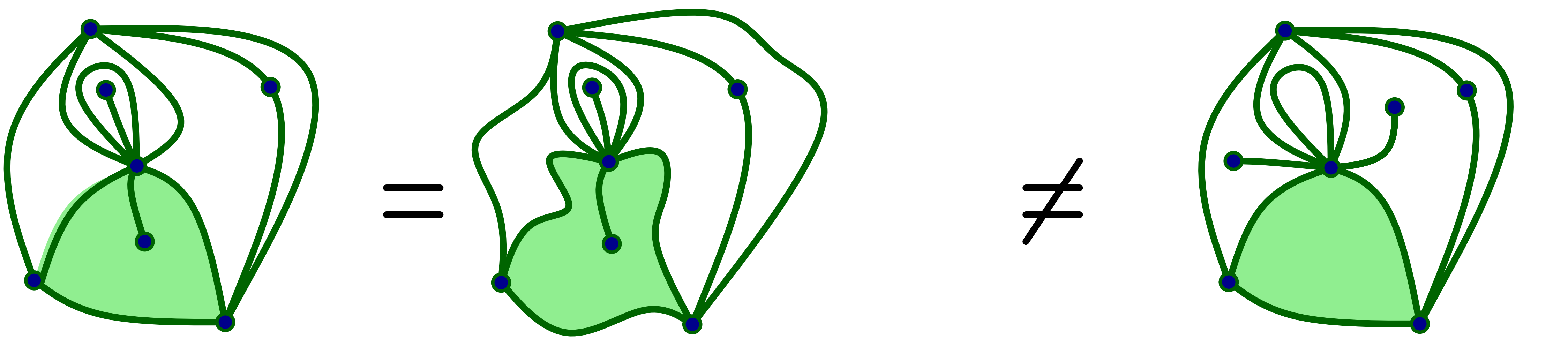

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p-angulation: each face is bounded by p edges

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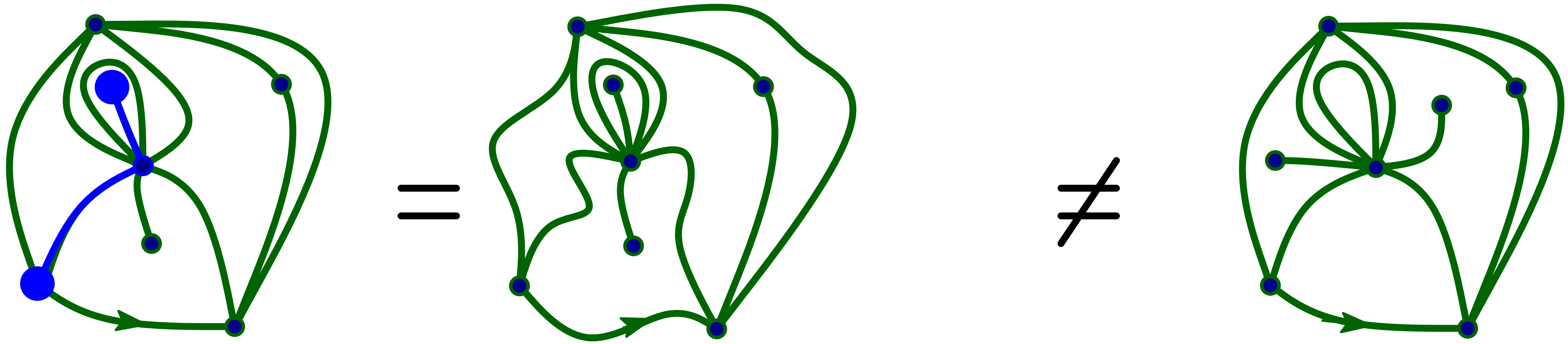

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Plane maps are **rooted** : by orienting an edge.

Distance between two vertices = number of edges between them.
Planar map = **Metric space**

"Classical" large random triangulations

Take a triangulation with n edges uniformly at random. What does it look like if n is large ?

Two points of view : global/local, continuous/discrete

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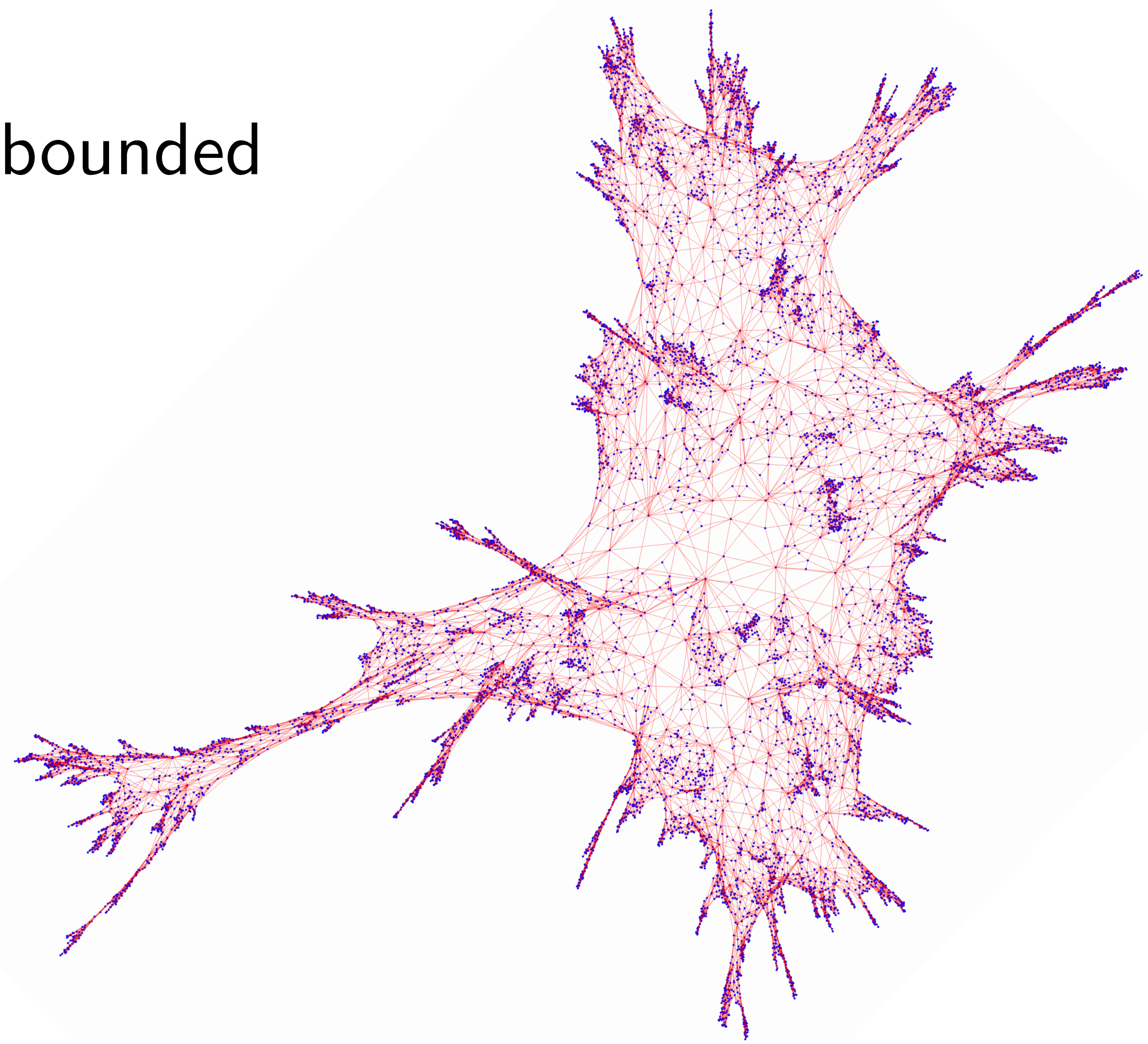
Global :
Rescale distances to keep diameter bounded

[Le Gall 13, Miermont 13] :
converges to the **Brownian map**



- Gromov-Hausdorff topology
- Continuous metric space
- Homeomorphic to the sphere
- Hausdorff dimension 4
- **Universality**

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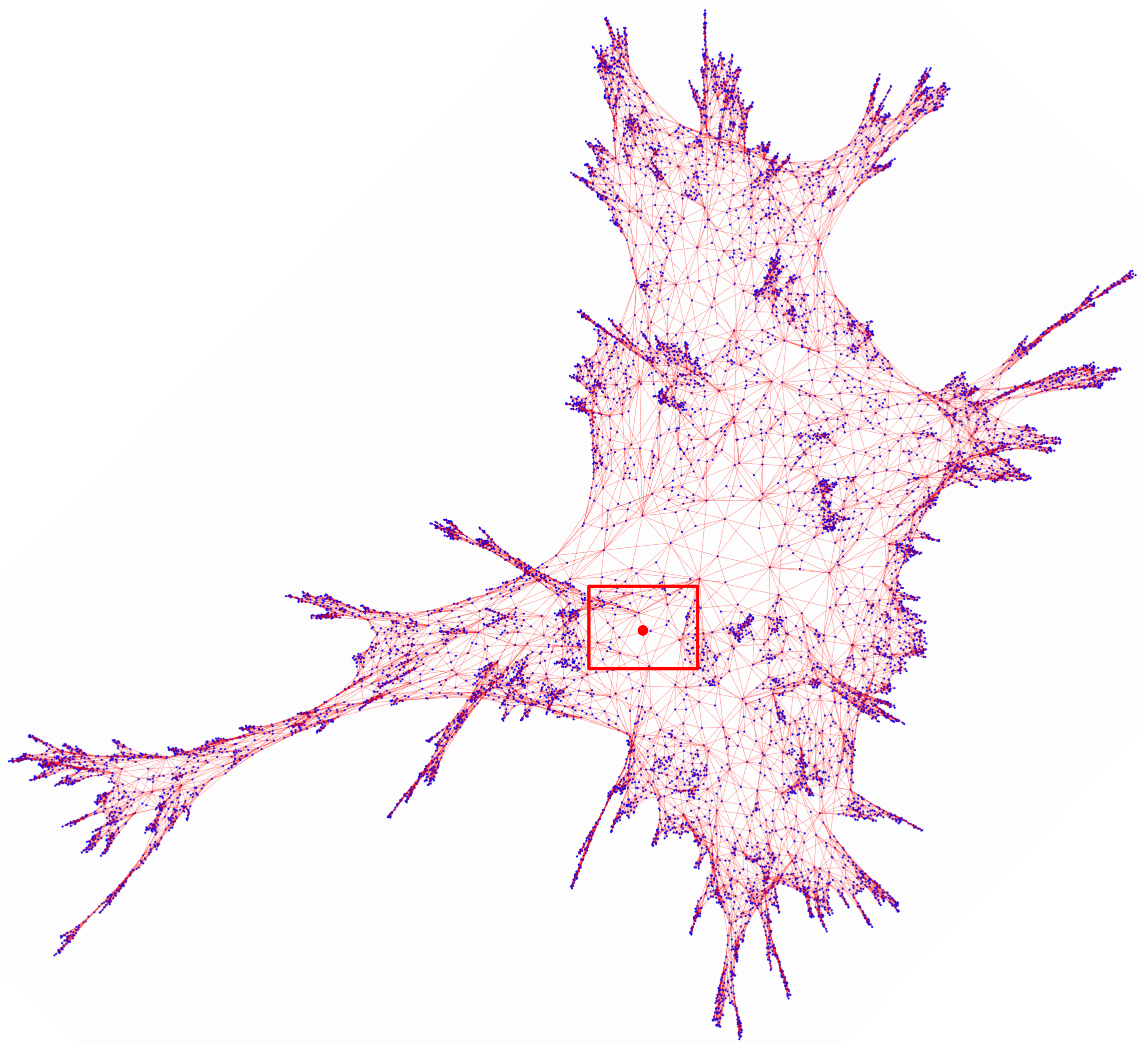
Local :
Don't rescale distances and look at neighborhoods of the root



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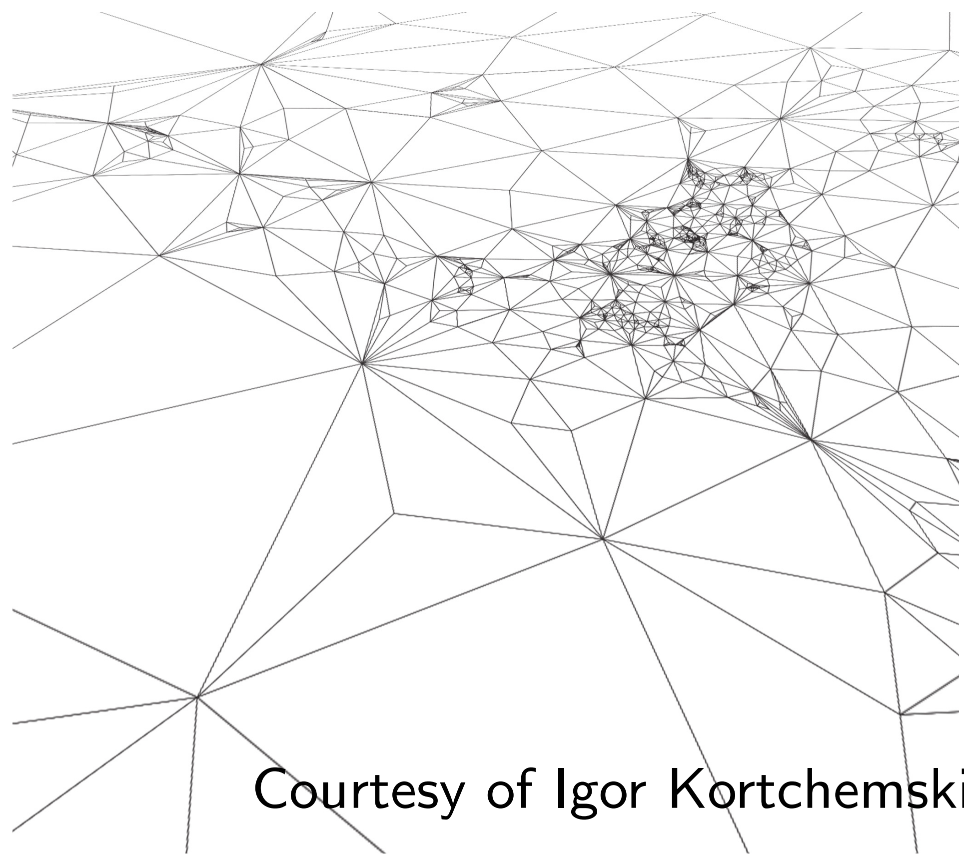
[Angel – Schramm 03, Krikun 05] :
Converges to the **Uniform Infinite Planar Triangulation**

- Local topology
- Volume of balls of radius R grow like R^4
- **"Universality"** of the exponent 4.



Courtesy of Igor Kortchemski

Local Topology for planar maps

$$\mathcal{M}_f := \{\text{finite rooted planar maps}\}.$$

Definition:
The **local topology** on $\mathcal{M}_f$ is induced by the distance:

$$d_{loc}(m, m') := \left(1 + \max\{r \geq 0 : B_r(m) = B_r(m')\}\right)^{-1}$$

where $B_r(m)$ is the graph made of all the vertices and edges of m which are within distance r from the root.

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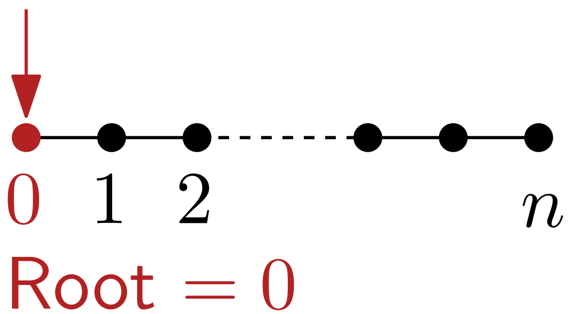
- $(\mathcal{M}, d_{loc})$: closure of $(\mathcal{M}_f, d_{loc})$. It is a **Polish** space (complete and separable).

- $\mathcal{M}_\infty := \mathcal{M} \setminus \mathcal{M}_f$ set of infinite planar maps.

Local convergence: simple examples



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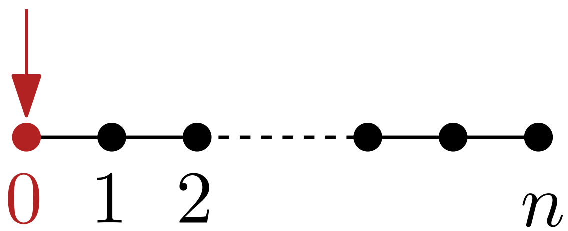

$\longrightarrow (\mathbb{Z}_+, 0)$

Root $= 0$

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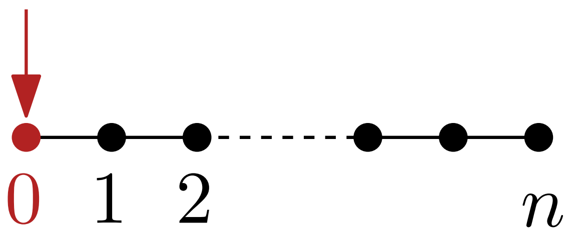

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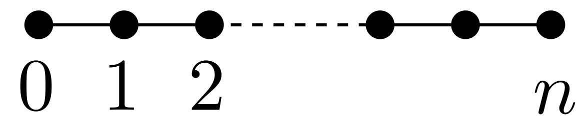

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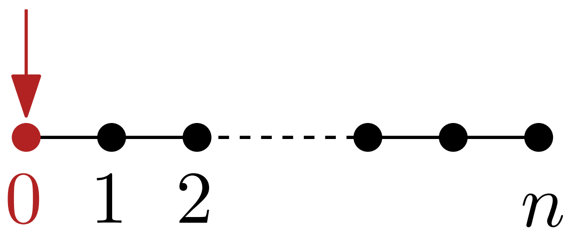

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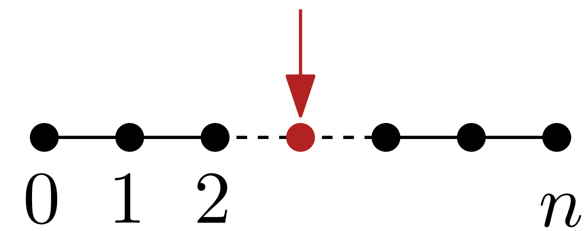

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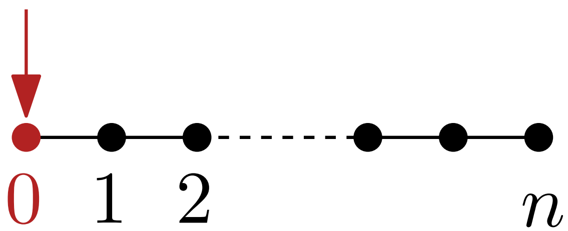

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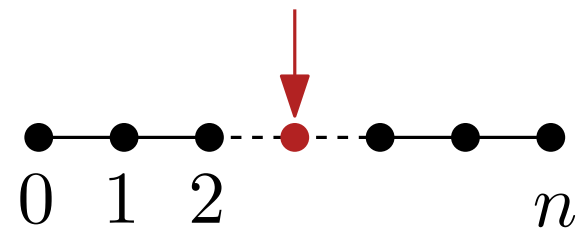

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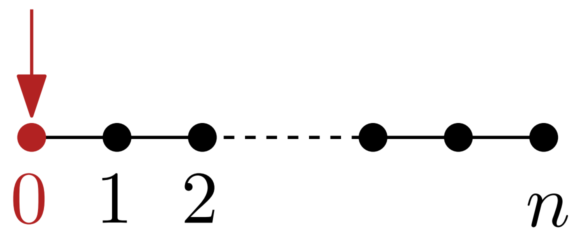

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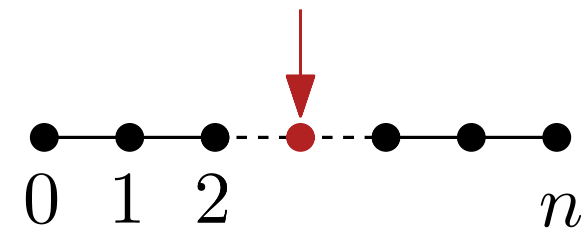

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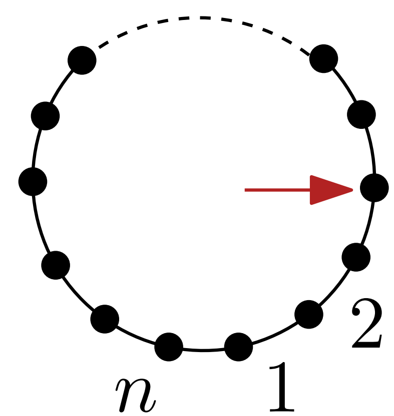

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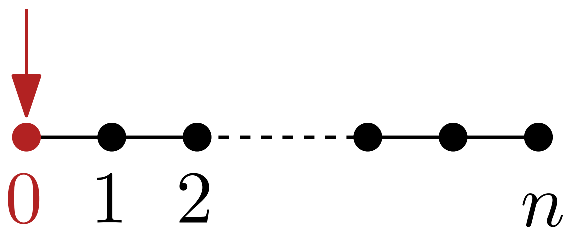

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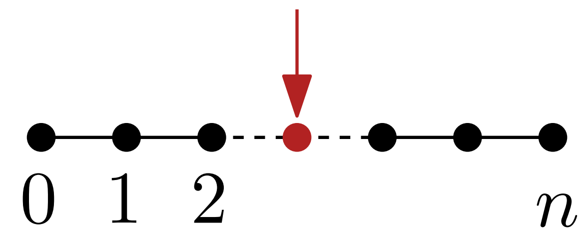

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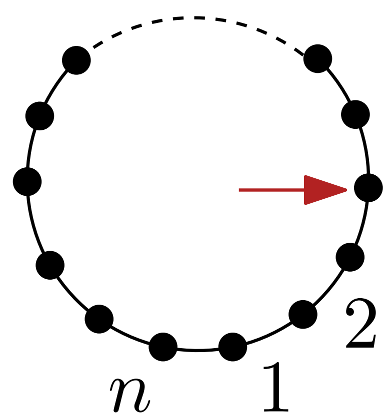

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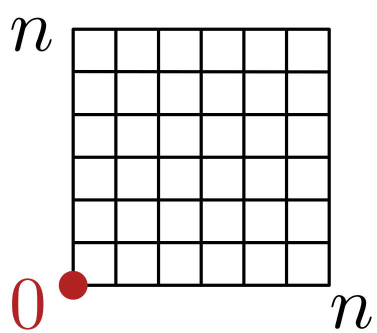

$\longrightarrow \left(\mathbb{Z}_+^2, 0\right)$

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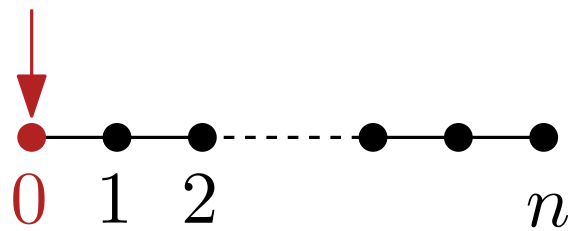

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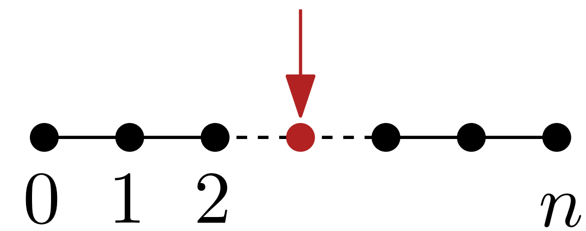

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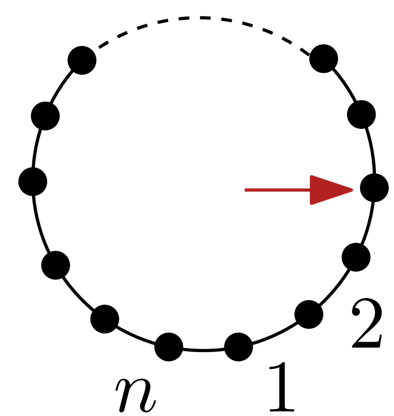

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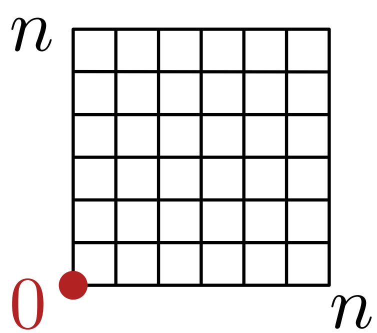

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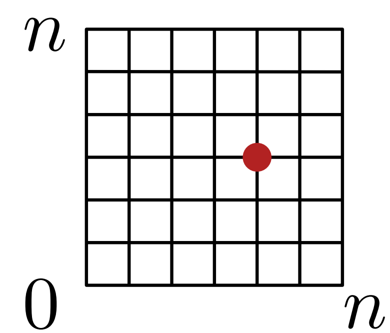

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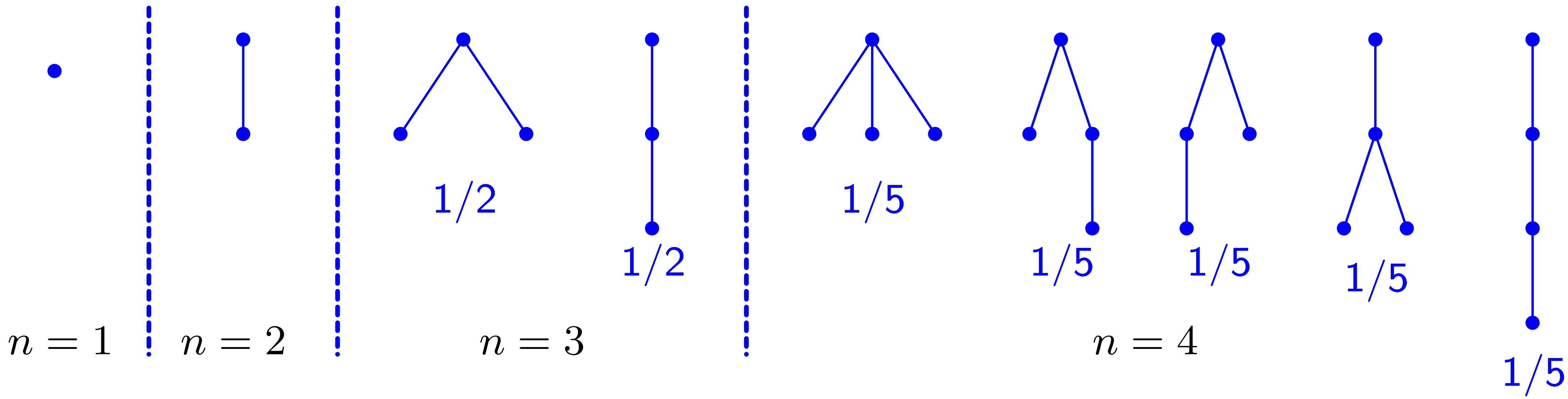
Local convergence: more complicated examples

Uniform plane trees with n vertices:



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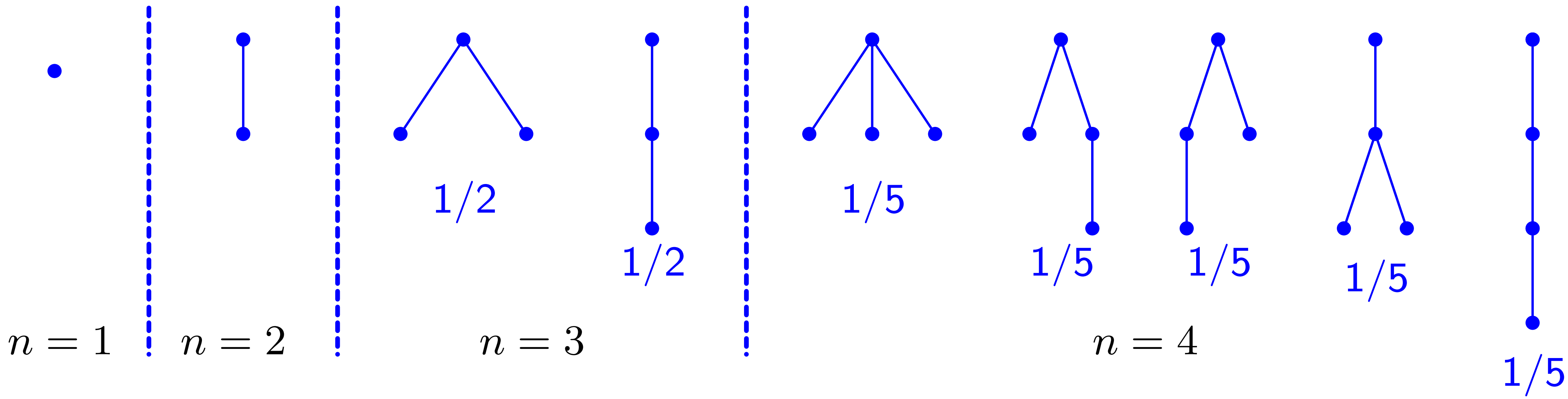

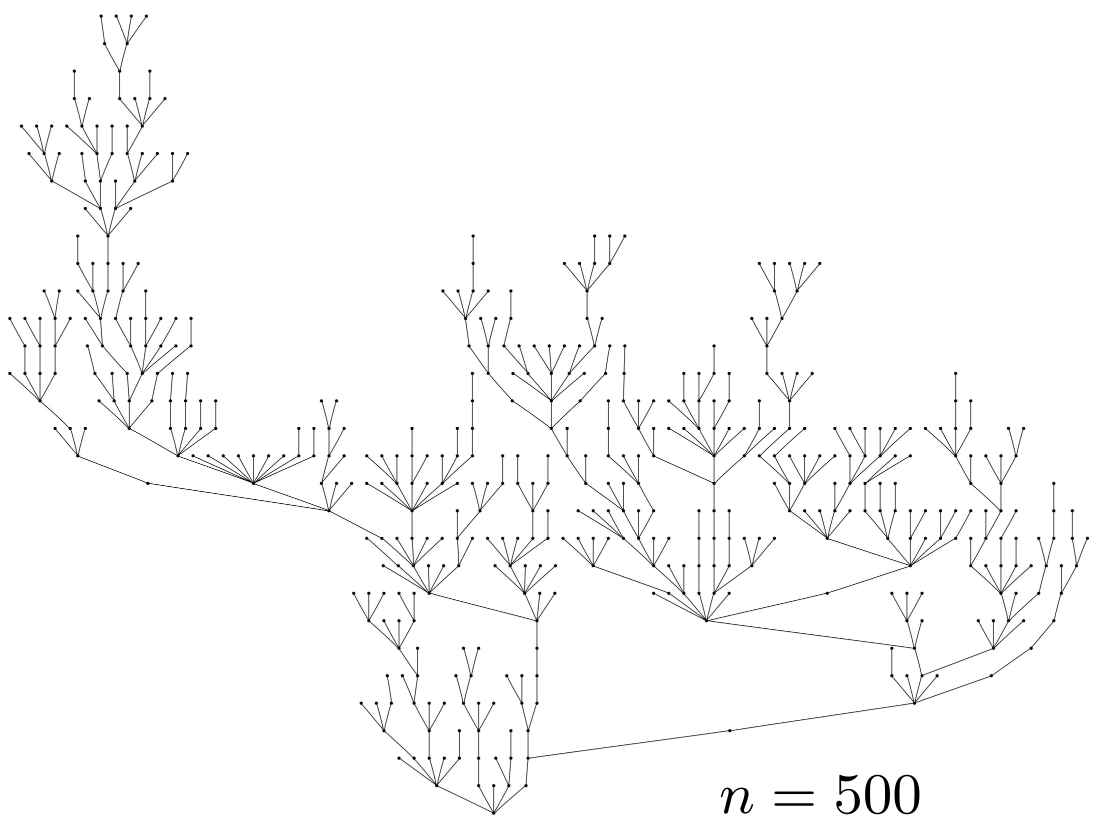

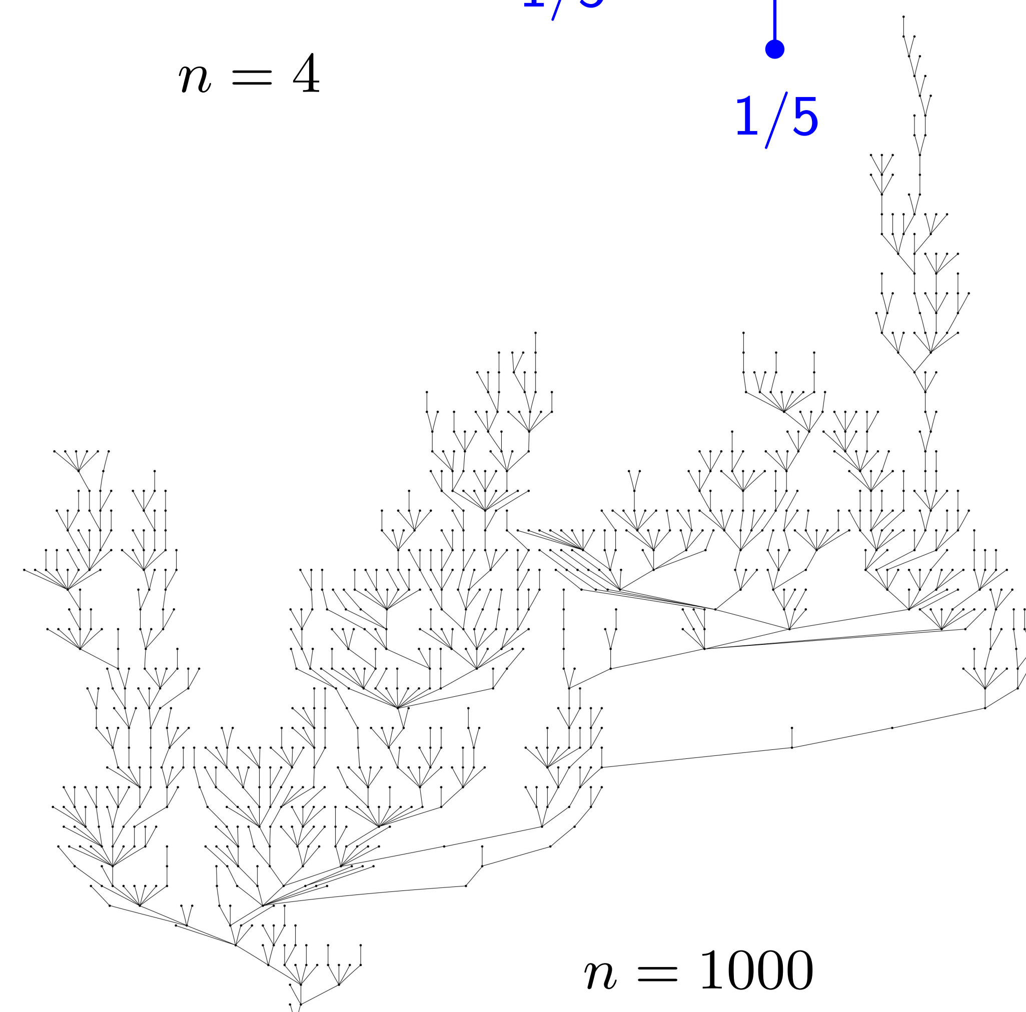

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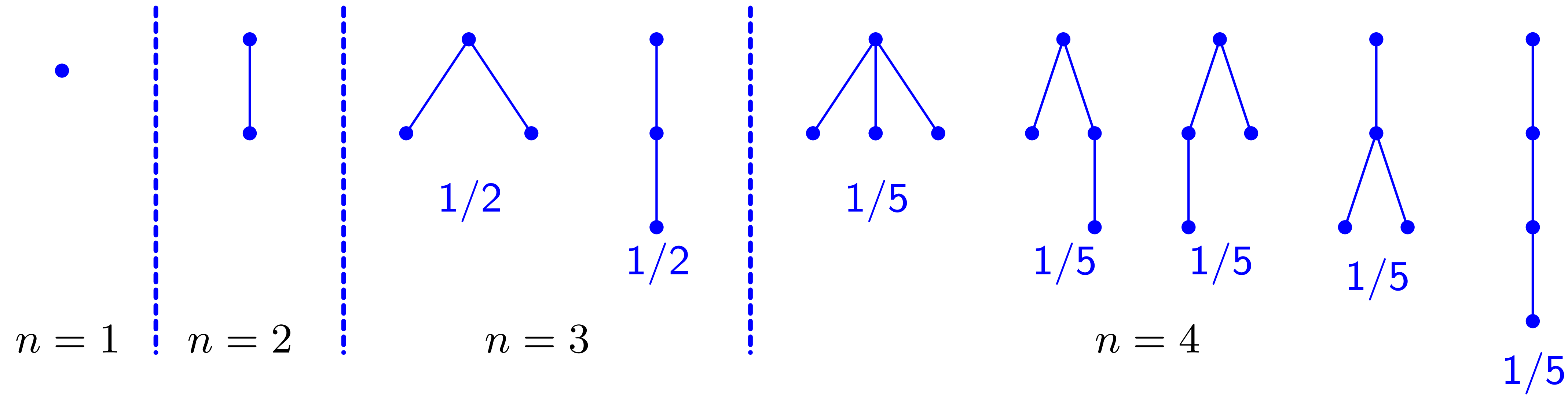
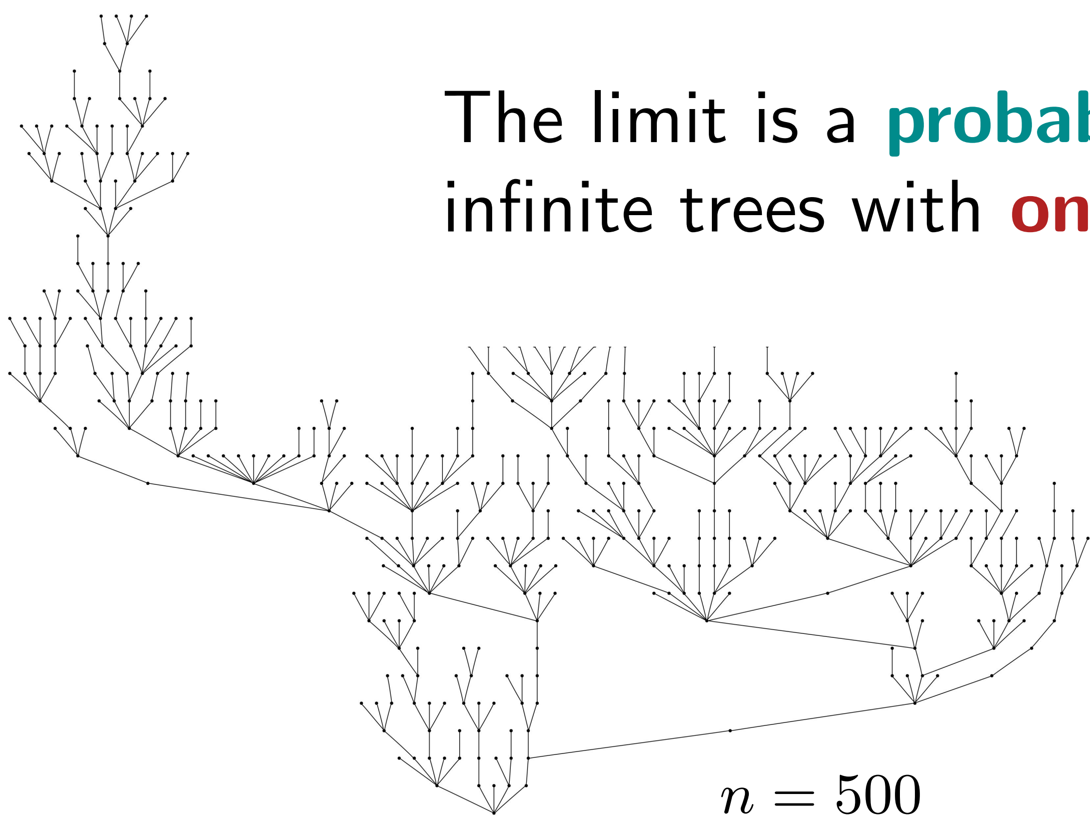
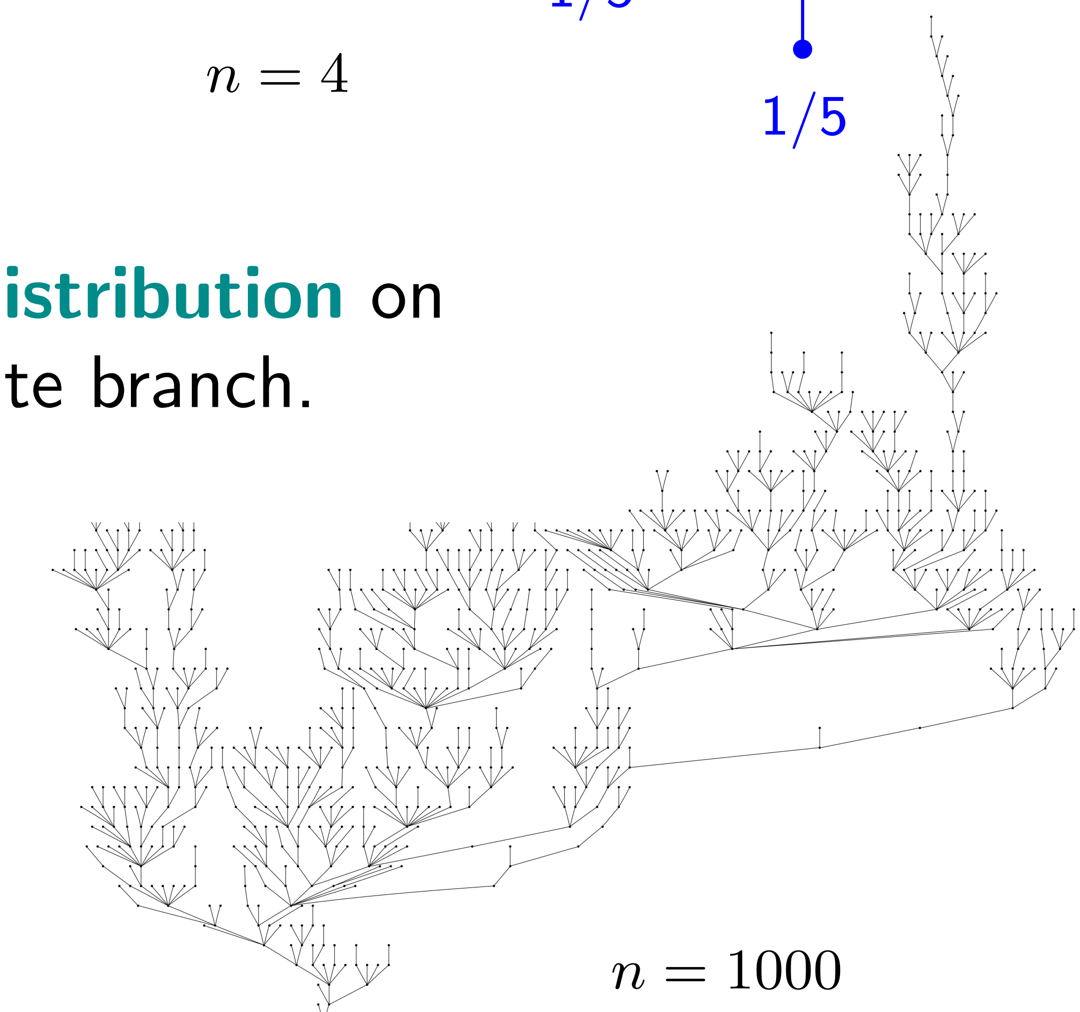


The limit is a **probability distribution** on infinite trees with **one** infinite branch.



$n = 500$



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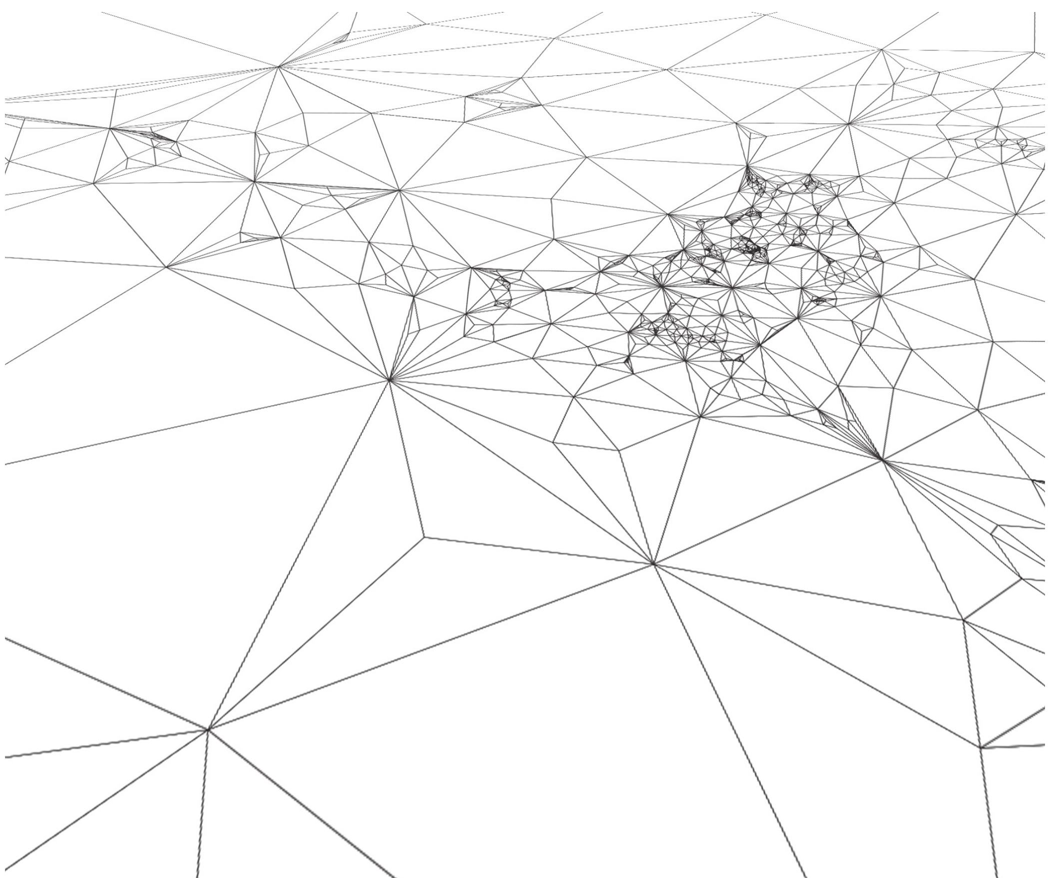
Local convergence of uniform triangulations

Theorem [Angel – Schramm, '03]
As $n \to \infty$, the uniform distribution on triangulations of size n converges weakly to a probability measure called the Uniform Infinite Planar Triangulation (or **UIPT**) for the **local topology**.



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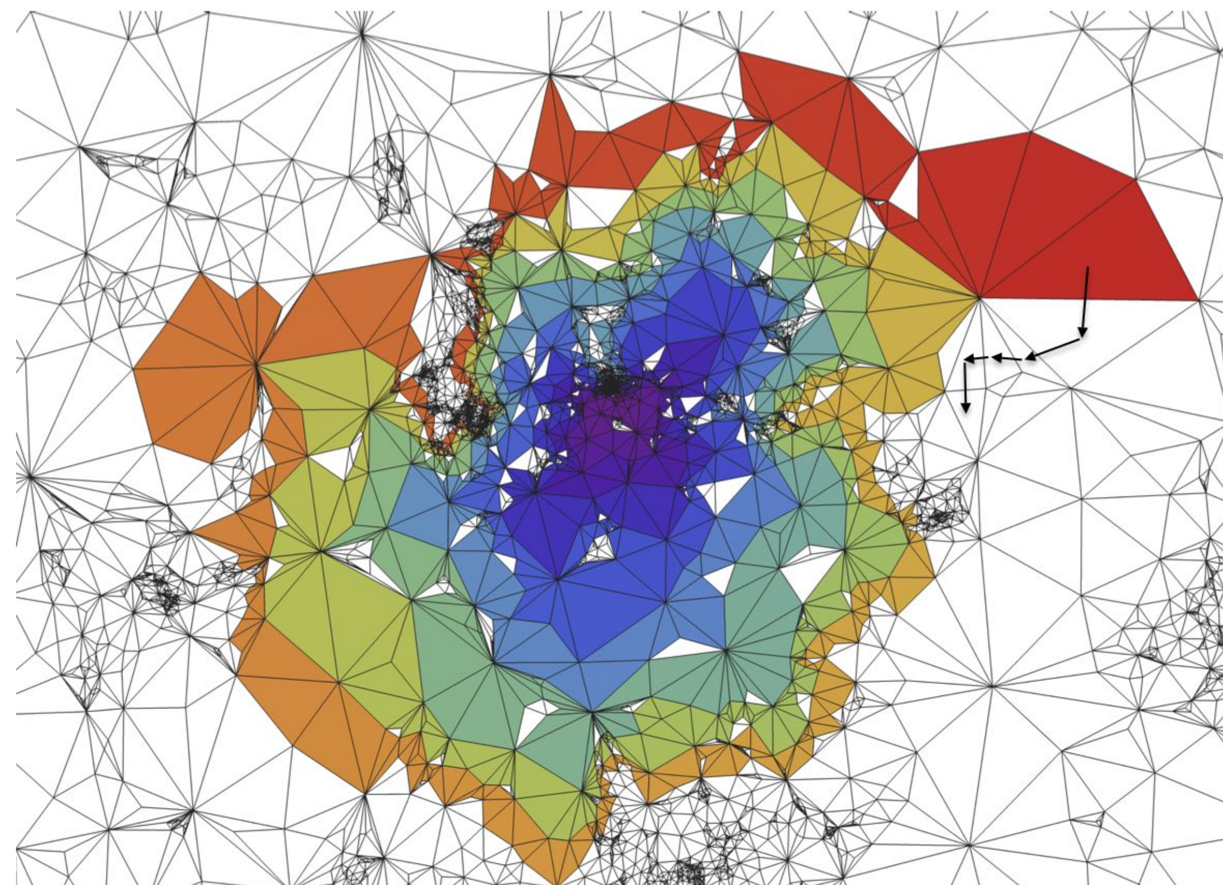

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Some properties of the UIPT:

- The UIPT has almost surely one end [Angel – Schramm, '03]
- Volume (nb. of vertices) and perimeters of balls known to some extent.

For example $\mathbb{E}\left[|B_r(\mathbf{T}_\infty)|\right] \sim \frac{2}{7} r^4$ [Angel '04, Curien – Le Gall '12]

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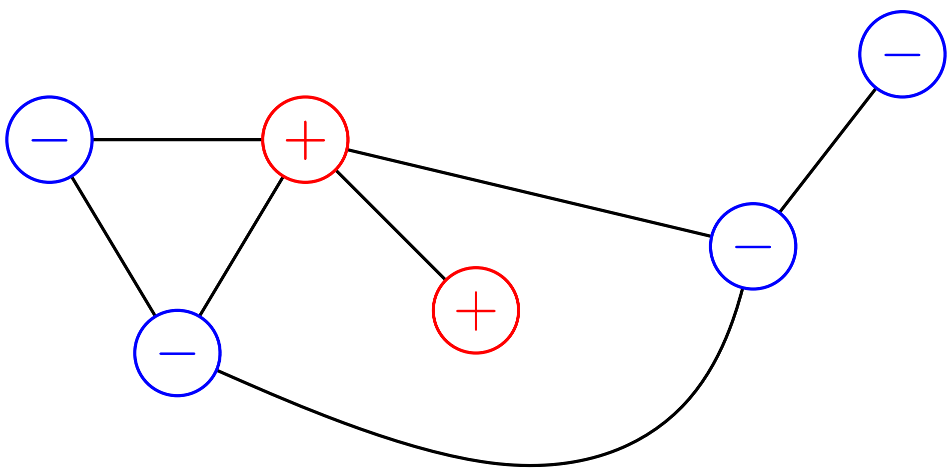
Universality: we expect the **same behavior** for slightly different models (e.g. quadrangulations, triangulations without loops, ...)

II - Ising model on random maps

Adding matter: Ising model on triangulations

First, Ising model on a finite deterministic graph:

$G = (V, E)$ finite graph



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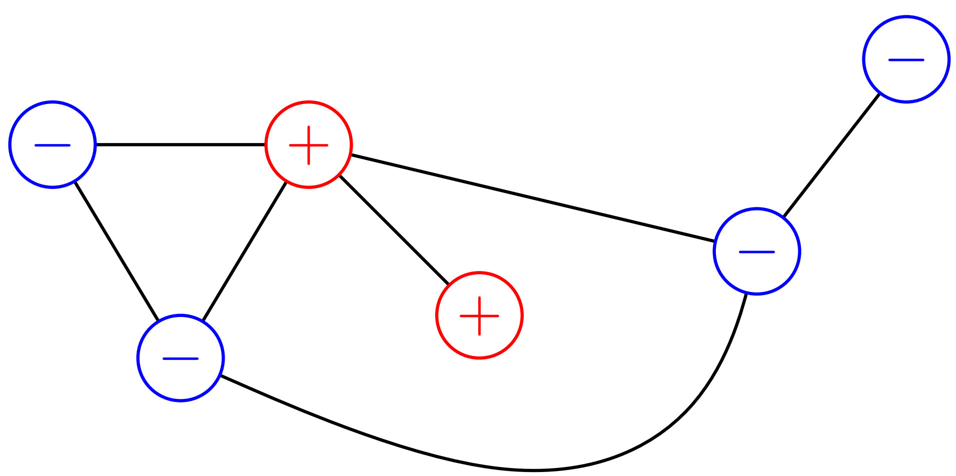

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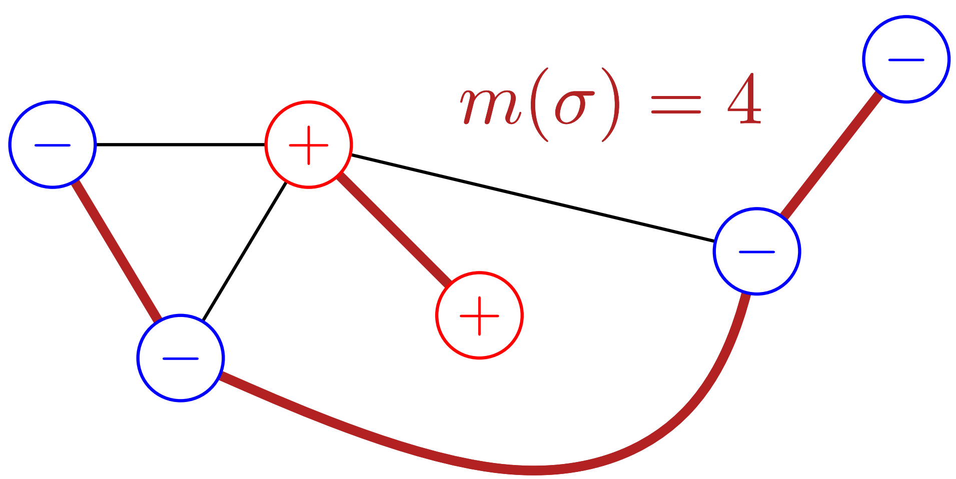

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Combinatorial formulation: $P(\sigma) \propto \nu^{m(\sigma)}$
with $m(\sigma) =$ number of monochromatic edges and $\nu = e^{\beta}$.

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$\mathcal{T}_n = \{$rooted planar triangulations with $3n$ edges$\}$.

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$$\mathbb{P}_n^\nu\Big(\{(T,\sigma)\}\Big) = \frac{\nu^{m(T,\sigma)}\delta_{|e(T)|=3n}}{[t^{3n}]Q(\nu,t)}.$$

where $Q(\nu,t) =$ generating series of **Ising-weighted triangulations**:

$$Q(\nu,t) = \sum_{T\in\mathcal{T}_f}\ \sum_{\sigma:V(T)\to\{-1,+1\}} \nu^{m(T,\sigma)} t^{e(T)}.$$

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Remark: This is a probability distribution on triangulations **with** spins. But, forgetting the spins gives a probability a distribution on triangulations **without** spins **different from the uniform distribution**.

Adding matter: New asymptotic behavior

Counting exponent for undecorated maps:

coeff $[t^n]$ of generating series of (undecorated) maps
(e.g.: triangulations, quadrangulations, general maps, simple maps,...)
$\sim \kappa\rho^{-n}n^{-\mathbf{5/2}}$

Note : κ and ρ depend on the combinatorics of the model.

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Theorem [Bernardi – Bousquet-Mélou 11]
For every ν the series $Q(\nu,t)$ is algebraic, has $\rho_\nu > 0$ as unique dominant singularity and satisfies

$$[t^{3n}]Q(\nu,t) \underset{n\to\infty}{\sim} \begin{cases} \kappa\,\rho_{\nu_c}^{-n}\,n^{-7/3} & \text{if } \nu = \nu_c = 1+\frac{1}{\sqrt{7}}, \\ \kappa\,\rho_{\nu}^{-n}\,n^{-5/2} & \text{if } \nu \neq \nu_c. \end{cases}$$

This suggests an unusual behavior of the underlying maps for $\nu = \nu_c$.
See also [Boulatov – Kazakov 1987], [Bousquet-Melou – Schaeffer 03] and [Bouttier – Di Francesco – Guitter 04].

Adding matter: link with Liouville Quantum Gravity

Maps without matter "converge" to $\sqrt{\frac{8}{3}}$-LQG

[Miermont'13],[Le Gall'13], [Miller,Sheffield '15],

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The critical Ising model is *believed* to converge to $\sqrt{3}$-LQG.

Similar statements for other models of decorated maps
(with a spanning subtree, with a bipolar orientation,...)
but no proofs.

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YES, in some cases [Gwynne, Holden, Sun '17], [Ding, Gwynne '18]

Unknown for Ising, but $d_{\sqrt{3}}$ is a good candidate for the volume growth exponent.

What is $d_{\sqrt{3}}$?

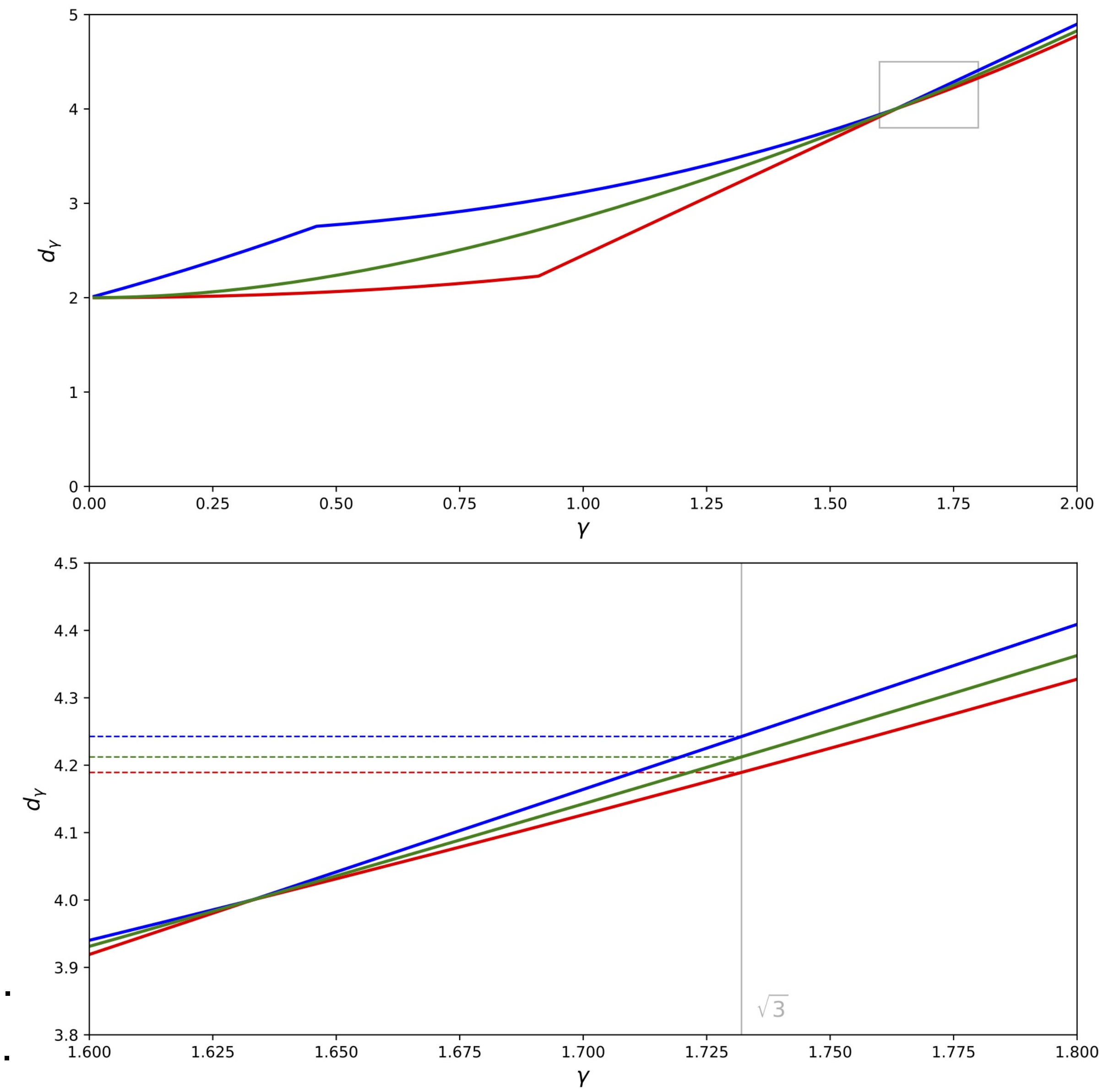
Adding matter: link with Liouville Quantum Gravity

Watabiki's prediction:

$$d_\gamma = 1 + \frac{\gamma^2}{4} + \frac{1}{4}\sqrt{(4+\gamma^2)^2 + 16\gamma^2}$$ gives $d_{\sqrt{3}} \approx 4.21...$

[Ding, Gwynne '18]
Bounds for d_γ which give:
$4.18 \leq d_{\sqrt{3}} \leq 4.25$.

In particular $d_{\sqrt{3}} \neq 4$ and growth volume would then be different than the uniform model.



Green = Watabiki.
Blue and Red = bounds by Ding and Gwynne.

III - Results and idea of proofs

Local convergence of triangulations with spins

Probability measure on triangulations of $\mathcal{T}_n$ with a spin configuration:

$$\mathbb{P}_n^\nu\Big(\{(T,\sigma)\}\Big) = \frac{\nu^{m(T,\sigma)}}{[t^{3n}]Q(\nu,t)}.$$

Theorem [AMS]
As $n \to \infty$, the sequence $\mathbb{P}_n^\nu$ converges weakly to a probability measure $\mathbb{P}^\nu$ for the **local topology**.
The measure $\mathbb{P}^\nu$ is supported on infinite triangulations with one end.

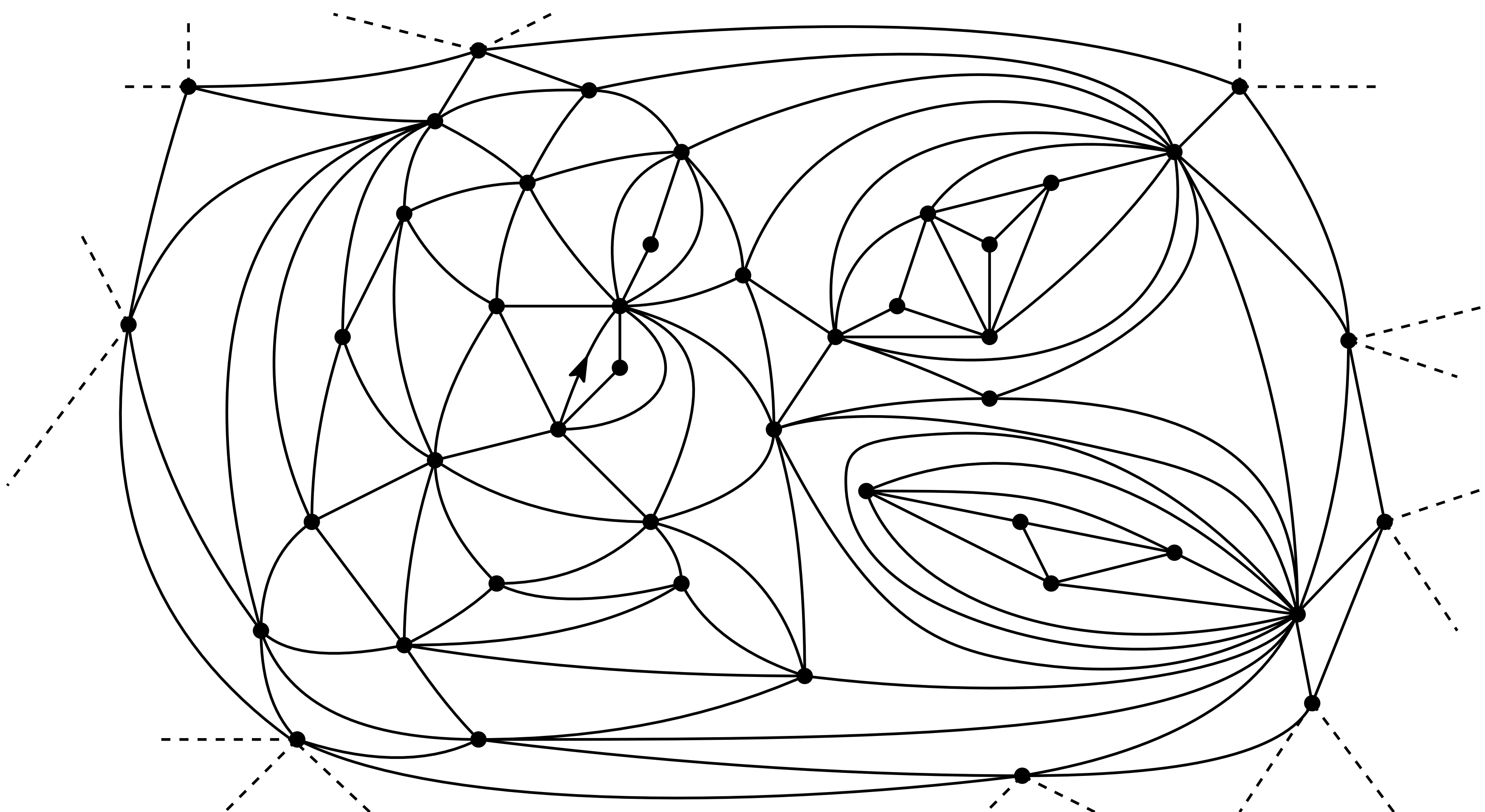
Local Topology for planar maps : balls

Definition:

The **local topology** on $\mathcal{M}_f$ is induced by the distance:

$$d_{loc}(m, m') := (1 + \max\{r \geq 0 : B_r(m) = B_r(m')\})^{-1}$$

where $B_r(m)$ is the graph made of all the faces of m with at least one vertex at distance $r - 1$ from the root.



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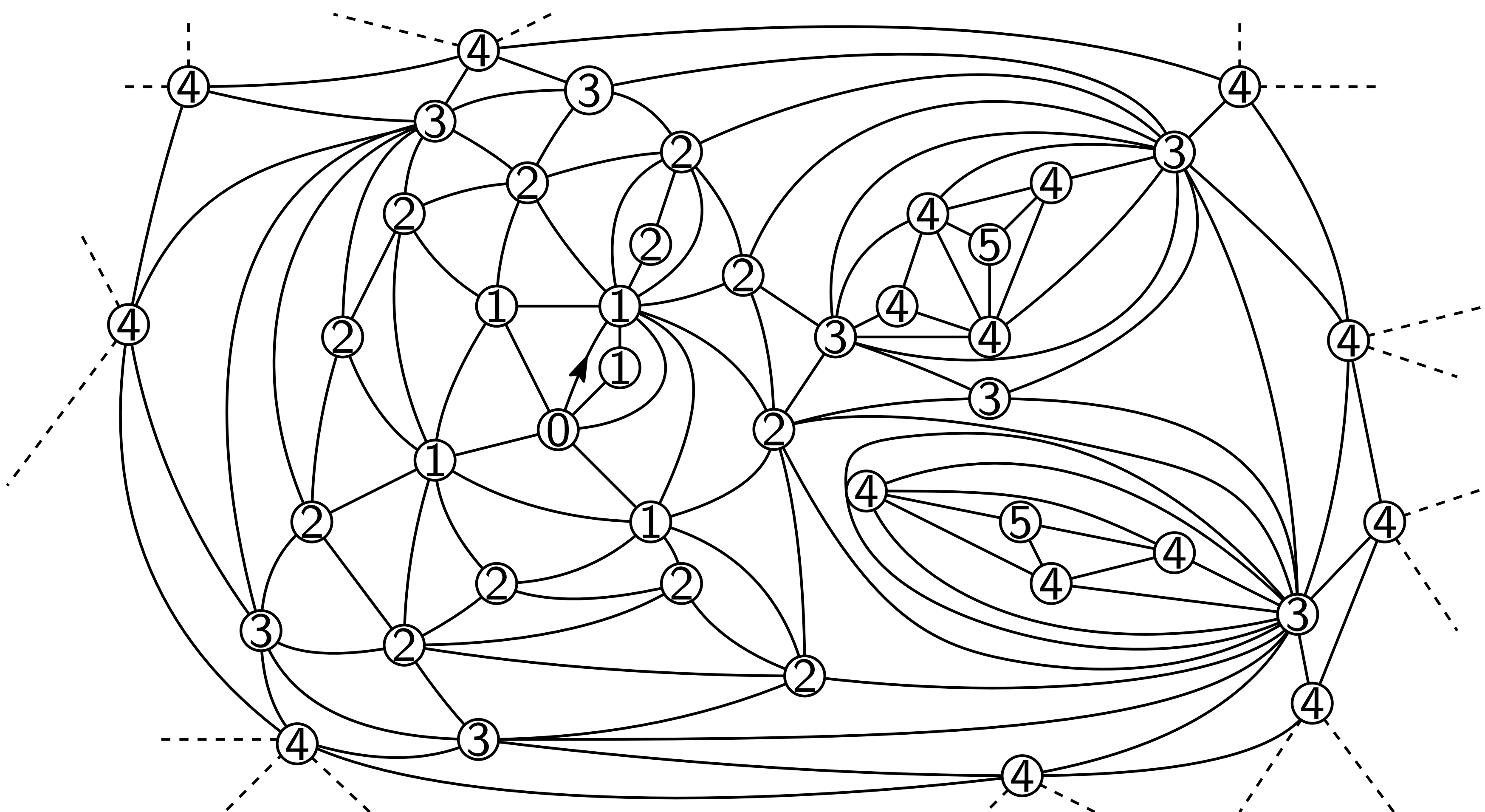

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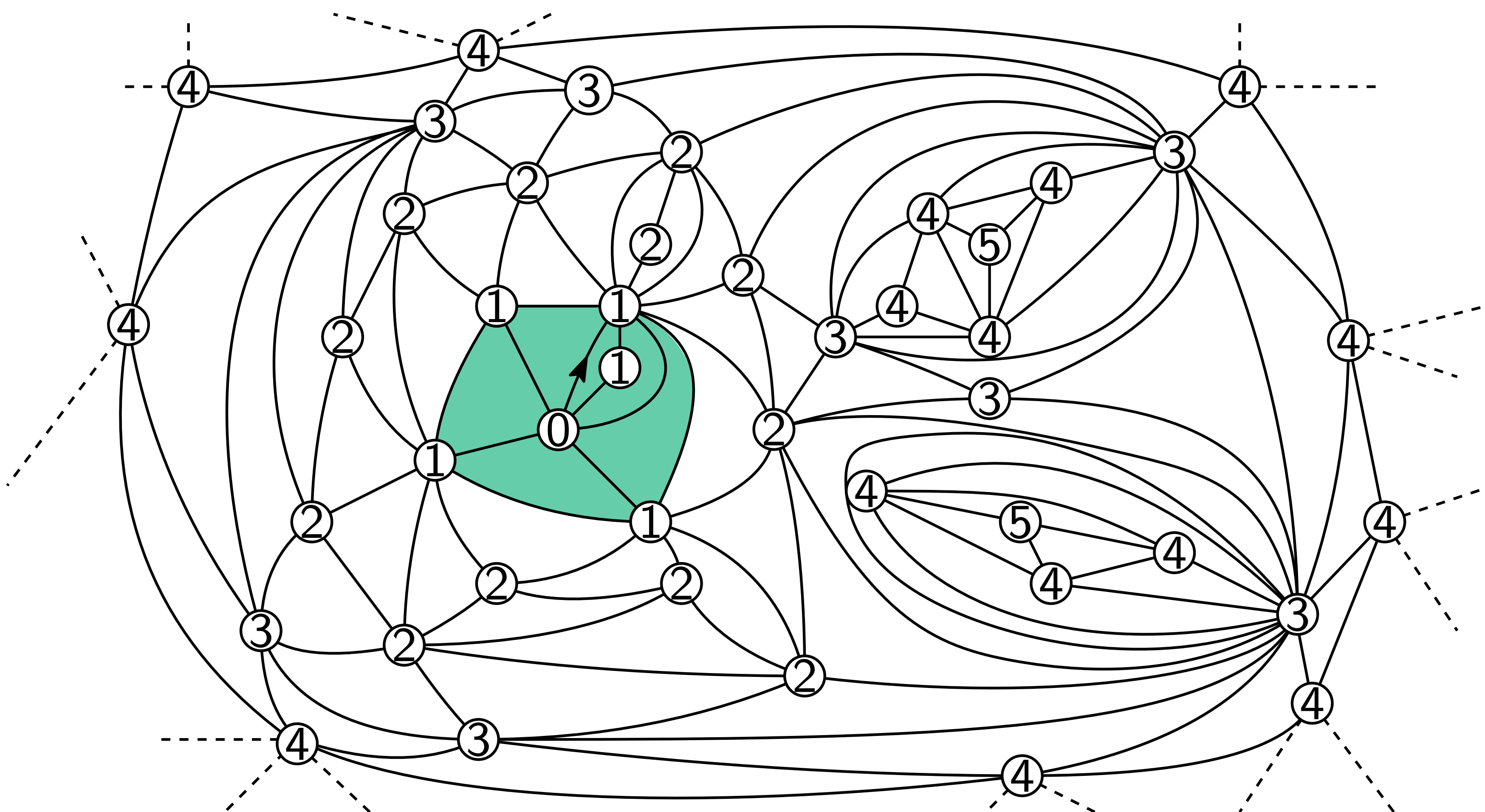

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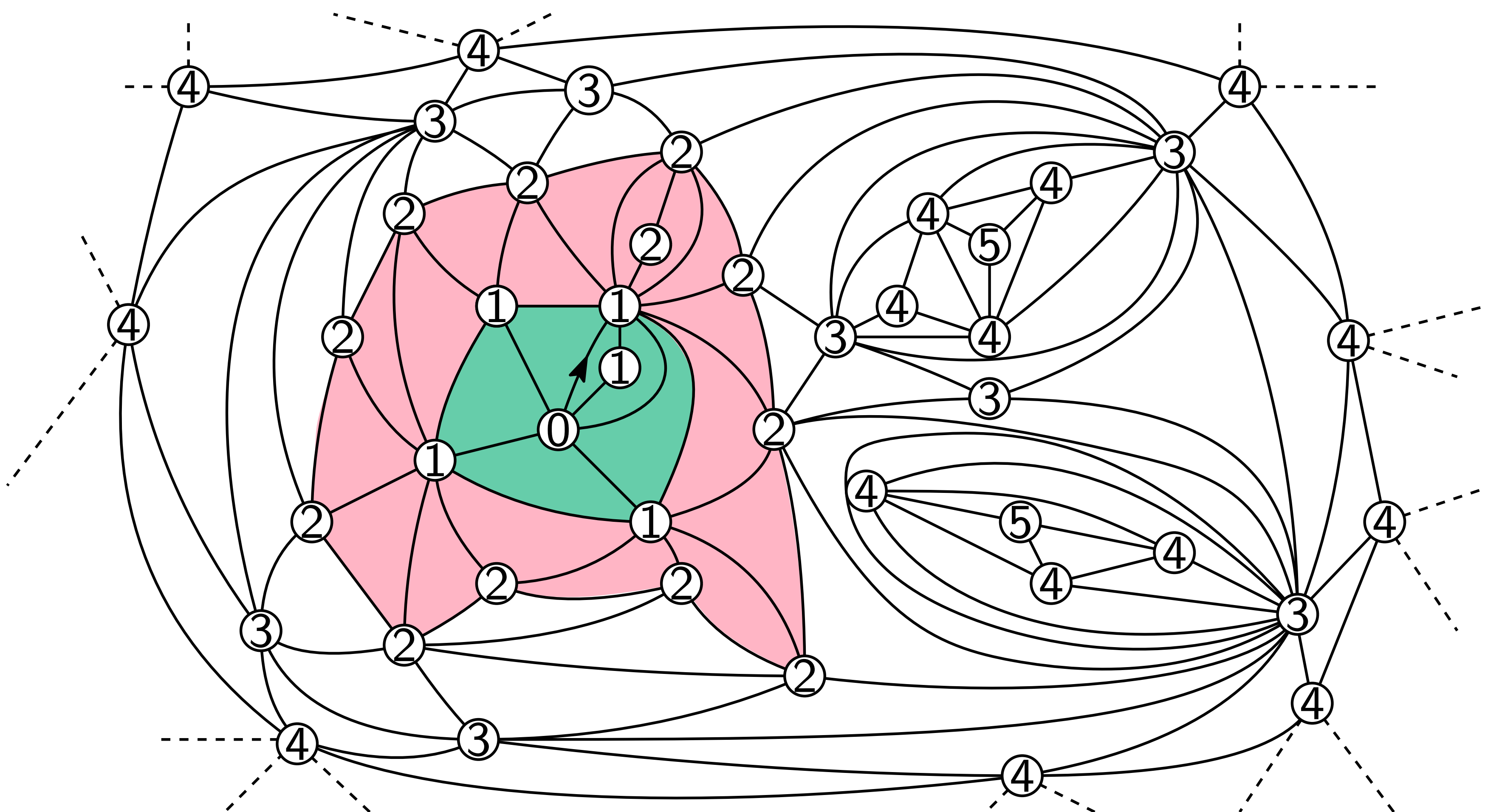

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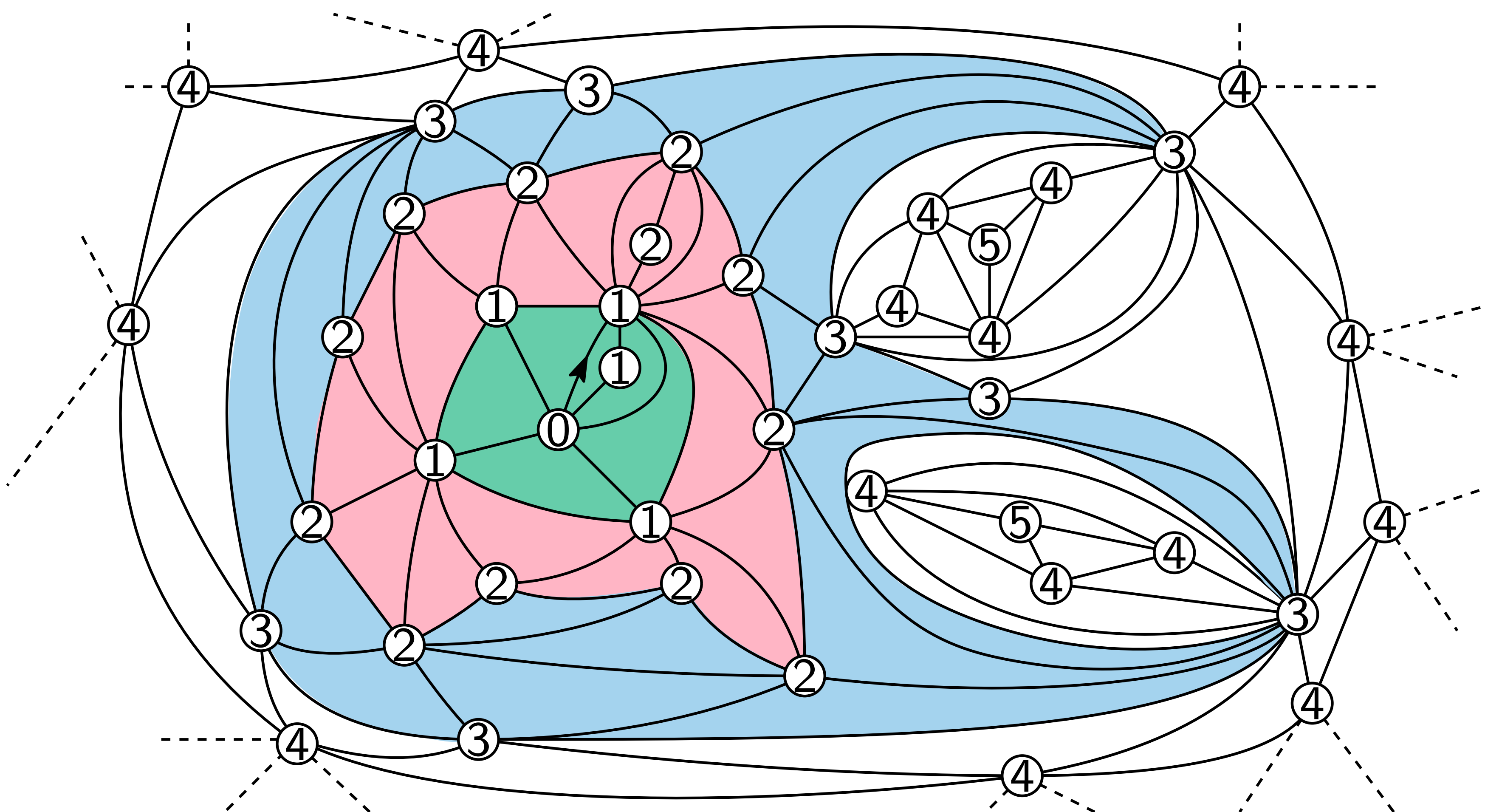

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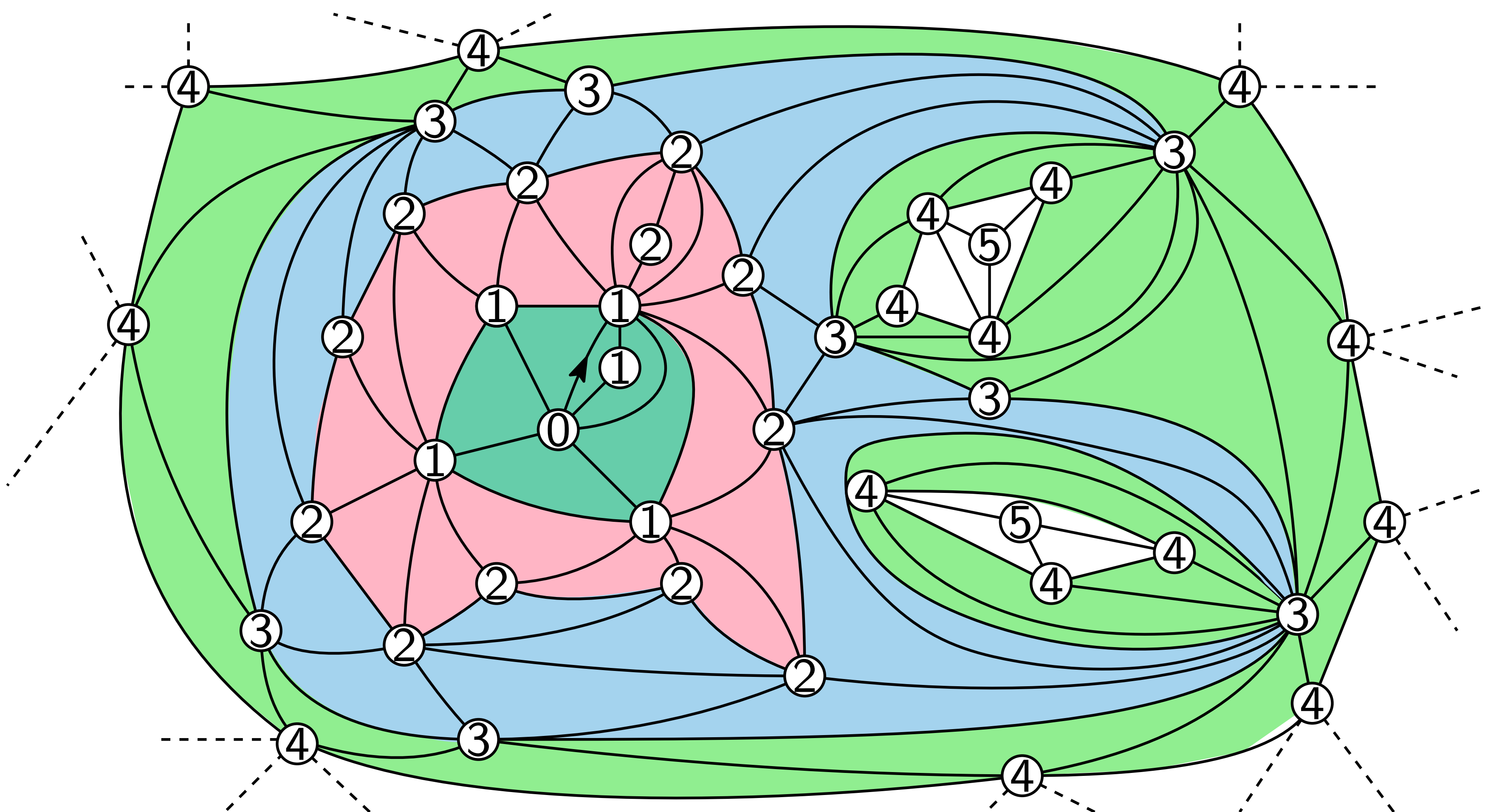


Weak convergence for the local topology

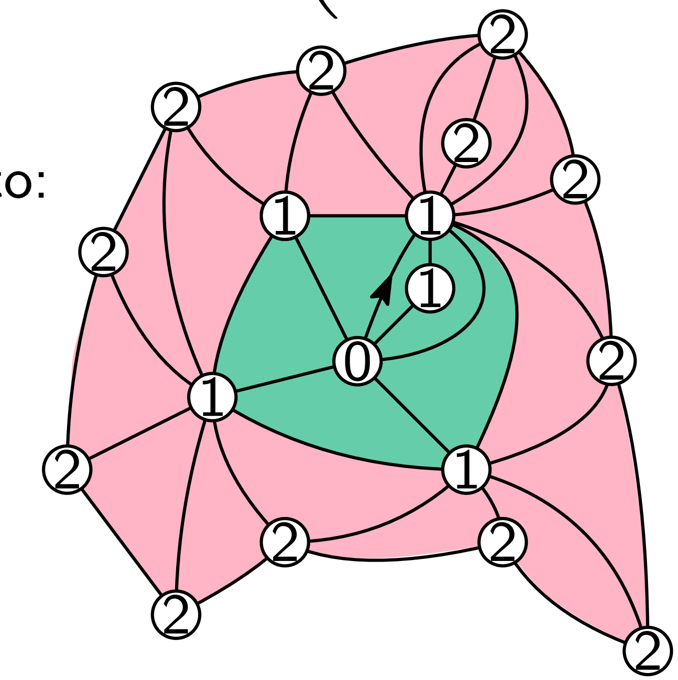
Portemanteau theorem + Levy – Prokhorov metric:

To show that $\mathbb{P}_n^\nu$ converges weakly to $\mathbb{P}^\nu$, prove

1. For every $r > 0$ and every possible ball Δ, show:

$$\mathbb{P}_n^\nu\Big(\{T \in \mathcal{T}_n \,:\, B_r(T) = \Delta\} \Big) \underset{n\to\infty}{\longrightarrow} \mathbb{P}^\nu\Big(\{T \in \mathcal{T}_\infty \,:\, B_r(T) = \Delta\} \Big).$$

For instance for $r = 2$, Δ might be equal to:



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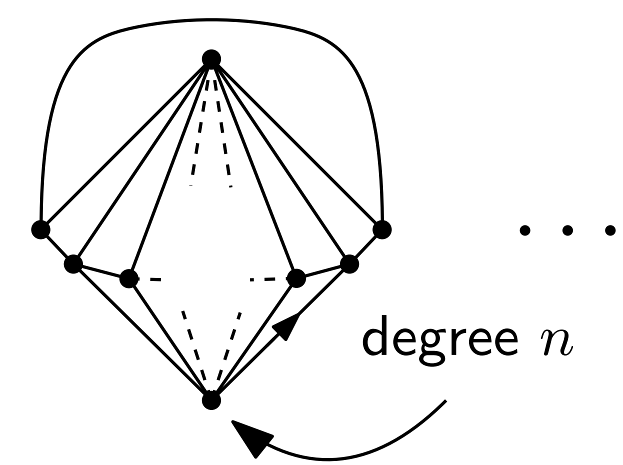

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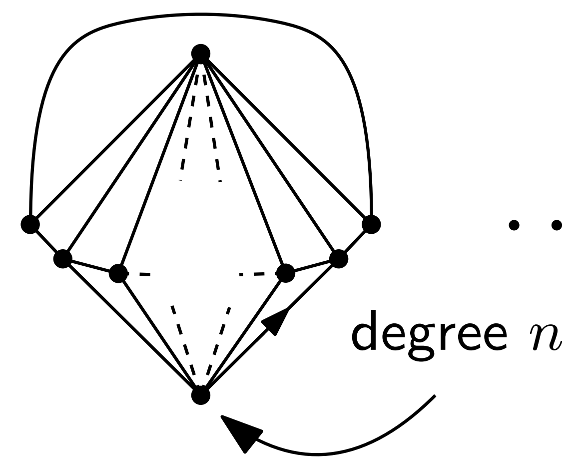

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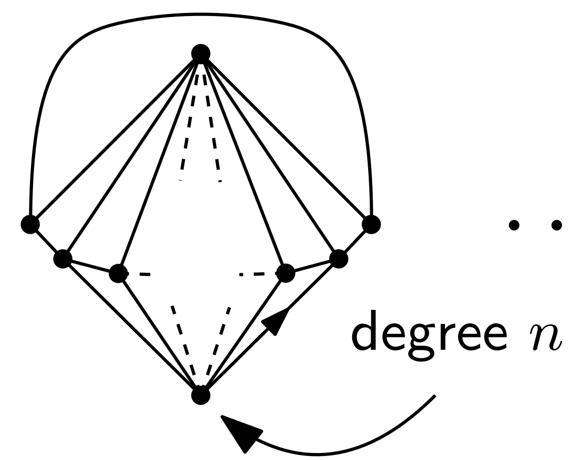

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$$\forall r \geq 0, \qquad \sum_{r-\text{balls}\,\Delta} \mathbb{P}^\nu\Big(\{T \in \mathcal{T}_\infty \,:\, B_r(T) = \Delta\} \Big) = 1.$$

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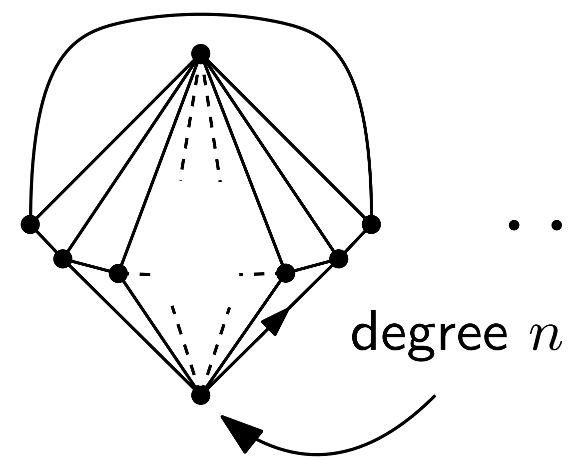

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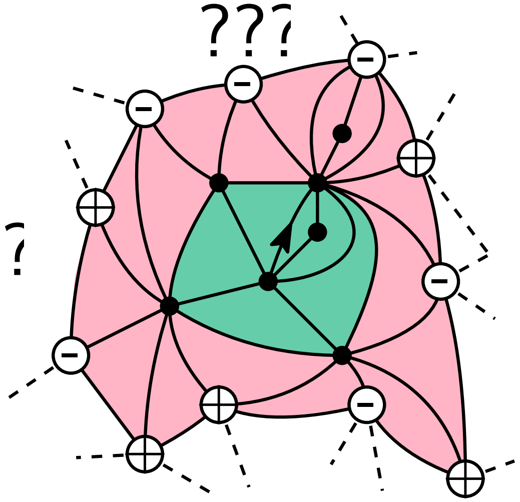
Enough to prove a **tightness** result, which amounts here to say that $\deg(\text{root})$ cannot be too big.

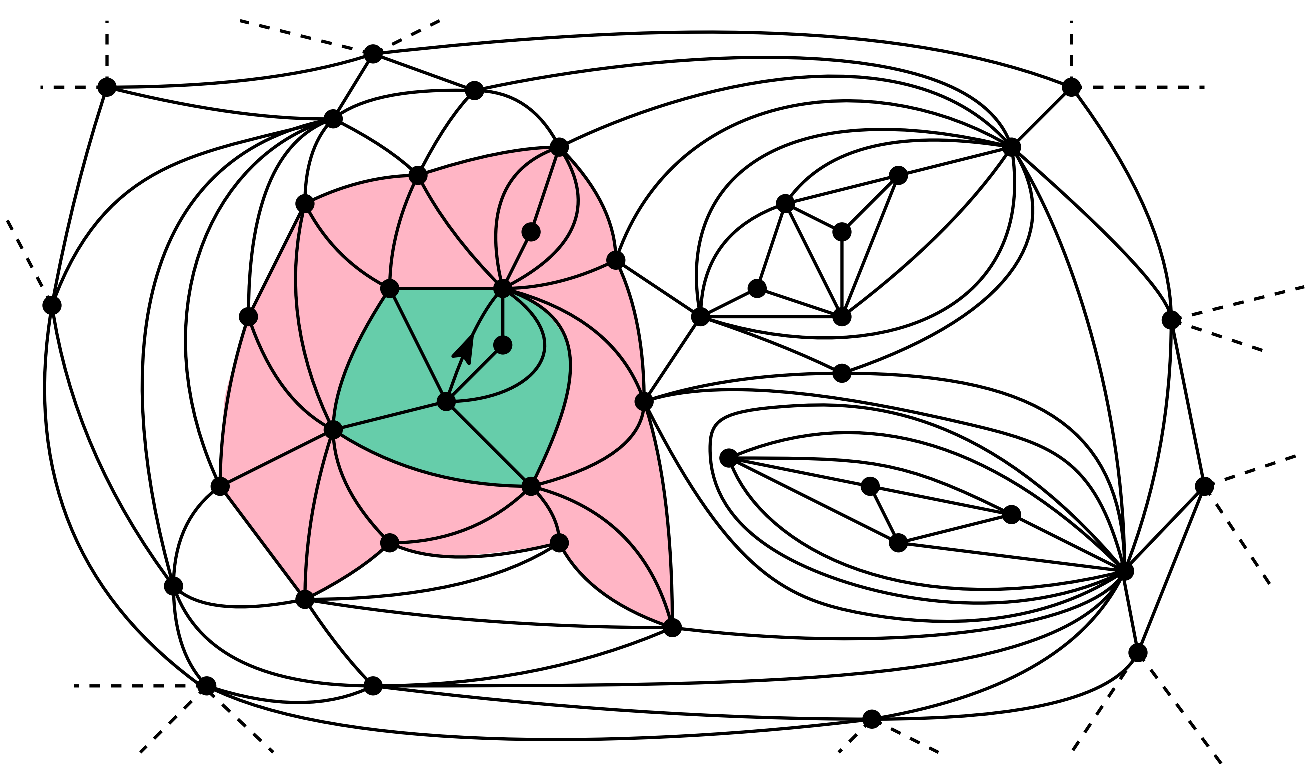
Local convergence and generating series

Need to evaluate, for every possible ball Δ

???

$\mathbb{P}_n\Big($???  ??? $\Big)$

???



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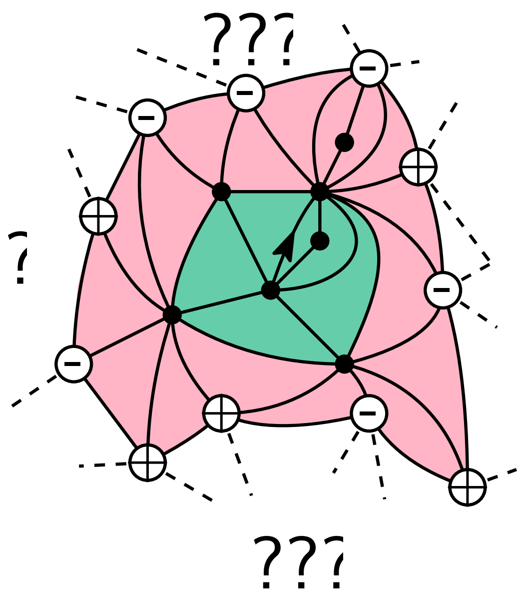

$$\mathbb{P}_n\Bigg(??? \qquad ??? \Bigg) = \frac{\nu^{m(\Delta)-m(\omega)}\,[t^{3n-e(\Delta)+|\omega|}]\mathbf{Z}_\omega(\nu,t)}{[t^{3n}]Q(\nu,t)}$$

Generating series of triangulations with simple boundary and boundary conditions given by ω.
Here $\omega = + - + - - - + - + + -$

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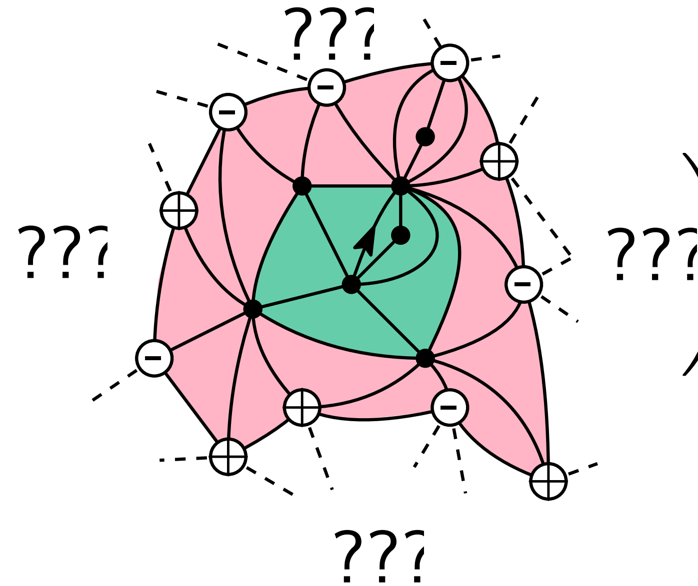
$\mathbb{P}_n\Big($ 

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Generating series of triangulations with simple boundary and boundary conditions given by ω.
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Theorem [AMS]
For every ω, the series $t^{|\omega|}Z_\omega(\nu,t)$ is algebraic, has ρ_ν as unique dominant singularity and satisfies

$$[t^{3n}]t^{|\omega|}Z_\omega(\nu,t) \underset{n\to\infty}{\sim} \begin{cases} \kappa_\omega(\nu_c)\,\rho_{\nu_c}^{-n}\,n^{-7/3} & \text{if } \nu=\nu_c=1+\frac{1}{\sqrt{7}}, \\ \kappa_\omega(\nu)\,\rho_\nu^{-n}\,n^{-5/2} & \text{if } \nu\neq\nu_c. \end{cases}$$

Triangulations with simple boundary

Fix a word ω, with injections from and into triangulations of the sphere:

$$[t^{3n}]t^{|\omega|}Z_\omega = \Theta\left(\rho_\nu^{-n} n^{-\alpha}\right), \text{ with } \alpha = 5/2 \text{ of } 7/3 \text{ depending on } \nu.$$

To get exact asymptotics we need, as series in t^3,

1. algebraicity,
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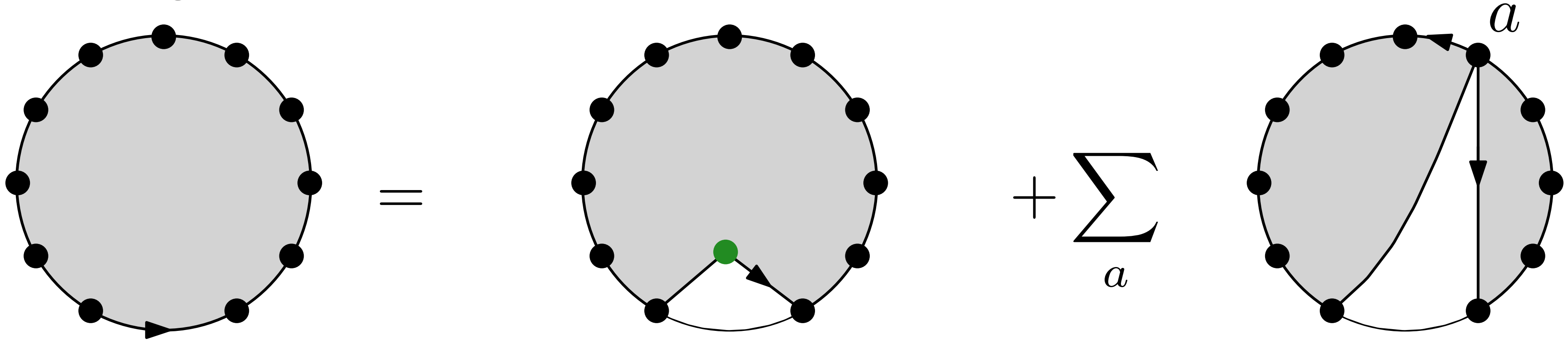
Peeling equation :



$$|\omega| \leq 3, \qquad Z_\omega = \left(Z_{\color{red}{\oplus}\omega} + Z_{\color{blue}{\ominus}\omega} + \sum_{\omega=\omega_1 a \omega_2} Z_{a\omega_1} \cdot Z_{a\omega_2} \right) \times \nu^{\mathbf{1}_{\overleftarrow{\omega}=\overrightarrow{\omega}}}\, t$$

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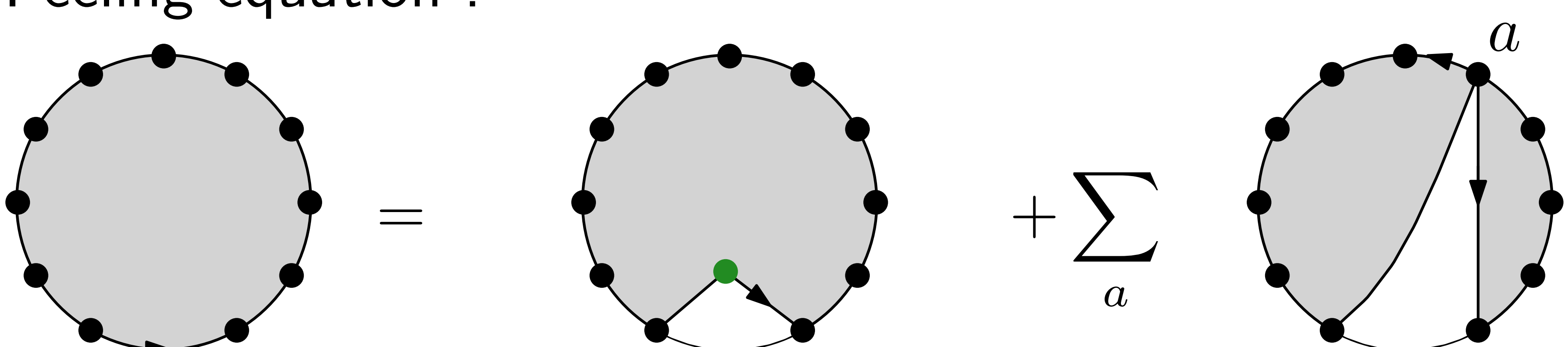

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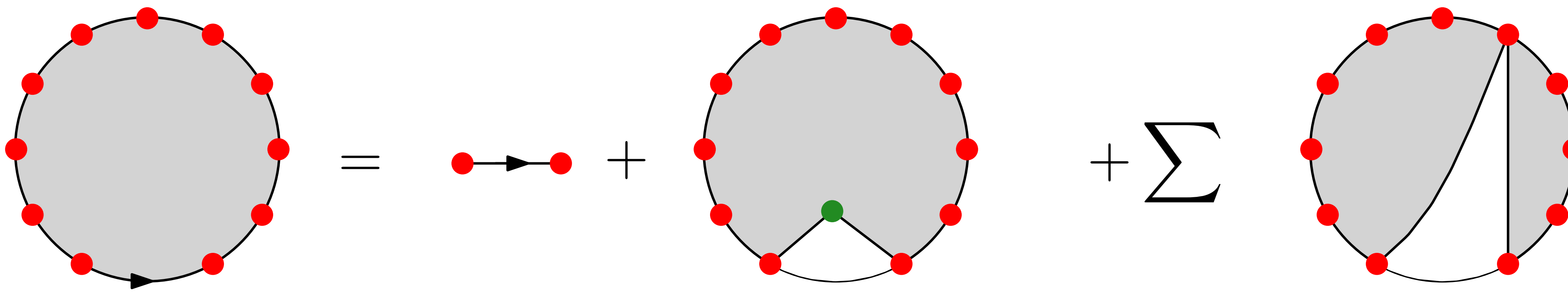
Double recursion on $|\omega|$ and number of $\ominus$'s :
enough to prove 1. and 2. for the $t^p Z_{\oplus^p}$'s

Positive boundary conditions : two catalytic variables



$$A(x) := \sum_{p\geq 1} Z_{\oplus^p} x^p = \nu t x^2 + \qquad + \frac{\nu t}{x}(A(x))^2$$

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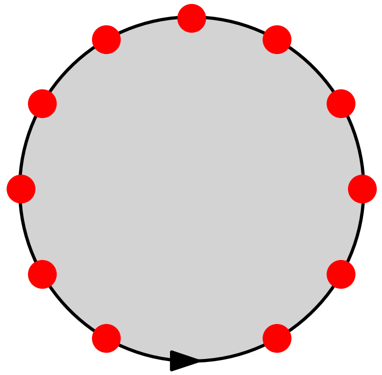
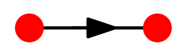
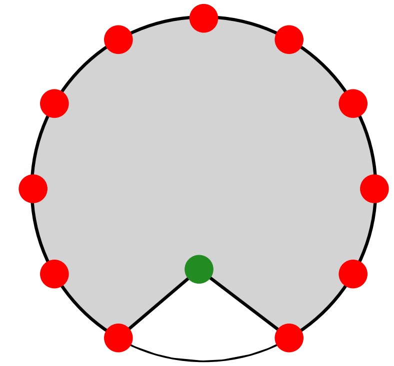
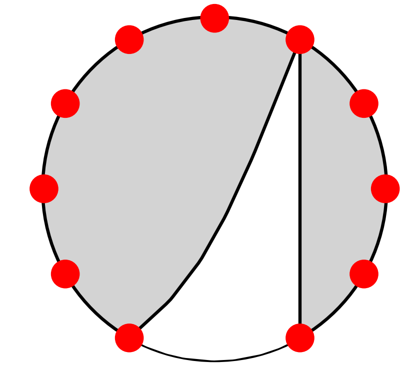
 $=$  $+$  $+\sum$ 

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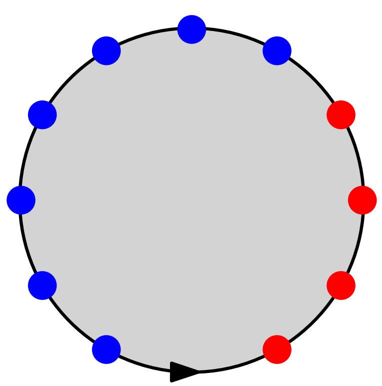
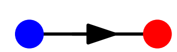
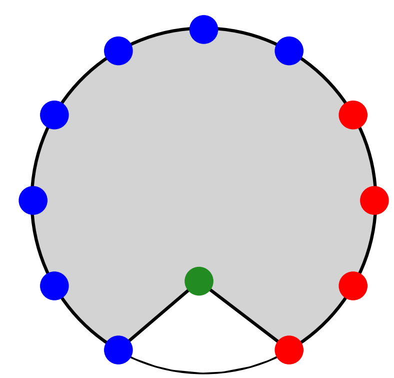
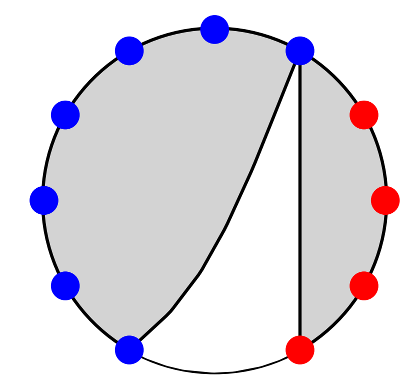
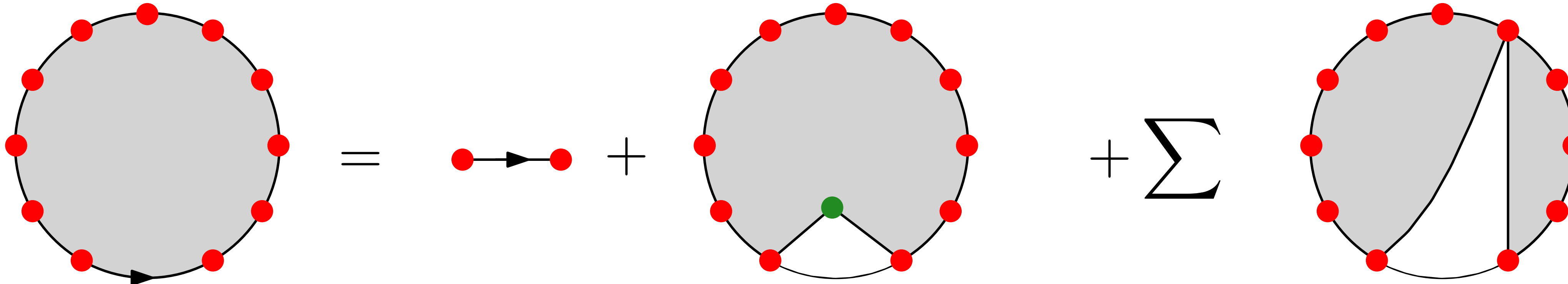
Peeling equation **at interface** $\ominus$–$\oplus$:

 $=$  $+$  $+\sum$ 

$$S(x,y) := \sum_{p,q\geq 1} Z_{\oplus^p \ominus^q} x^p y^q$$

Positive boundary conditions : two catalytic variables



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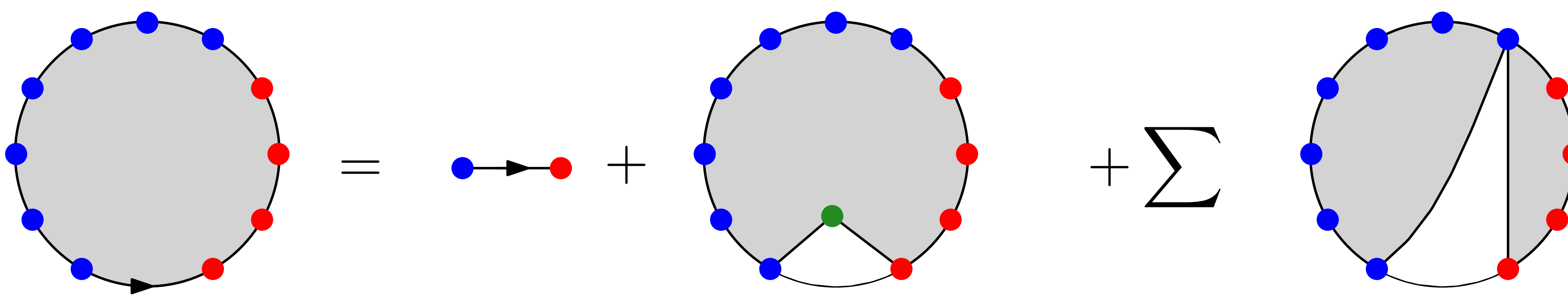

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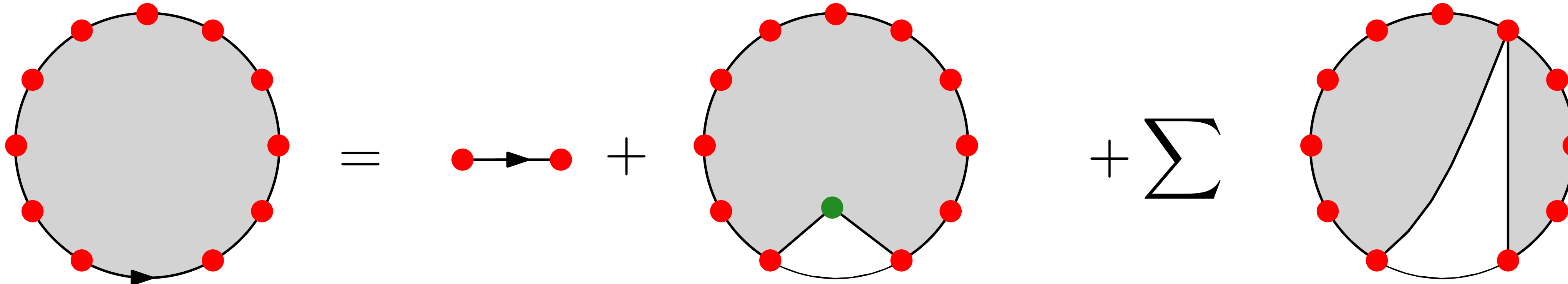

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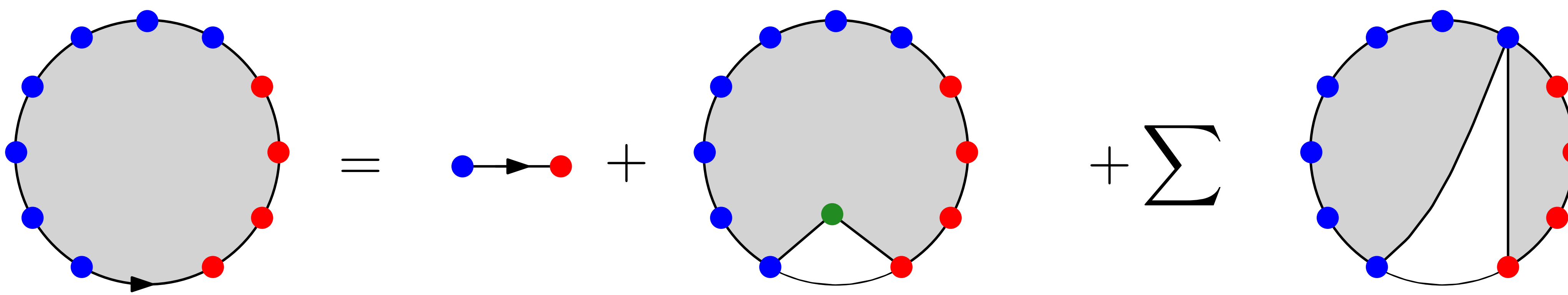

$$S(x,y) := \sum_{p,q\geq 1} Z_{\oplus^p \ominus^q} x^p y^q$$
$$= txy + \frac{t}{x}\left(S(x,y) - x[x]S(x,y)\right) + \frac{t}{y}\left(S(x,y) - y[y]S(x,y)\right)$$
$$+ \frac{t}{x}S(x,y)A(x) + \frac{t}{y}S(x,y)A(y)$$

From two catalytic variables to one: Tutte's invariants

Kernel method: equation for S reads

$$K(x,y)\cdot S(x,y)=R(x,y)$$

where $$K(x,y)=1-\frac{t}{x}-\frac{t}{y}-\frac{t}{x}A(x)-\frac{t}{y}A(y).$$

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2. Work a bit with the help of $R(x,Y_i/t)=0$ to get a **second invariant** $J(y)$ depending only on $t,\nu,Z_{\oplus}(t),y$ and $A(y/t)$.

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3. Prove that $J(y) = C_0(t) + C_1(t)I(y) + C_2(t)I^2(y)$ with C_i's explicit polynomials in $t, Z_{\oplus}(t)$ and $Z_{\oplus^2}(t)$.

$\boxed{\textbf{Equation with one catalytic variable} \text{ for } A(y) \text{ with } Z_{\oplus} \text{ and } Z_{\oplus^2}\ !}$

A simple tightness argument

A **"double counting"** argument to study the degree of the root vertex δ:

Mark a uniform edge conditionally on the triangulation

$$\overline{\mathbb{P}_n}\left(\delta \in e\right) = \sum_{k=1}^{3n} \overline{\mathbb{P}}\left(\delta \in e \middle| \deg(\delta) = k\right) \cdot \overline{\mathbb{P}_n}\left(\deg(\delta) = k\right)$$

$$\geq \sum_{k=1}^{3n} \frac{k}{2 \cdot 3n} \overline{\mathbb{P}_n}\left(\deg(\delta) = k\right) = \frac{1}{6n} \mathbb{E}_n\left[\deg(\delta)\right]$$

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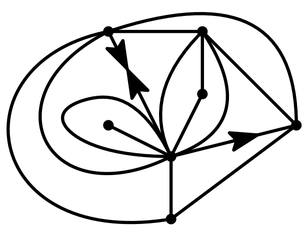

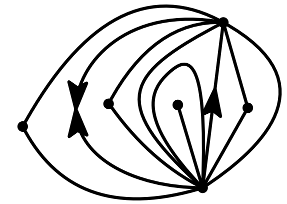

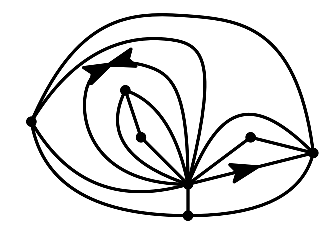

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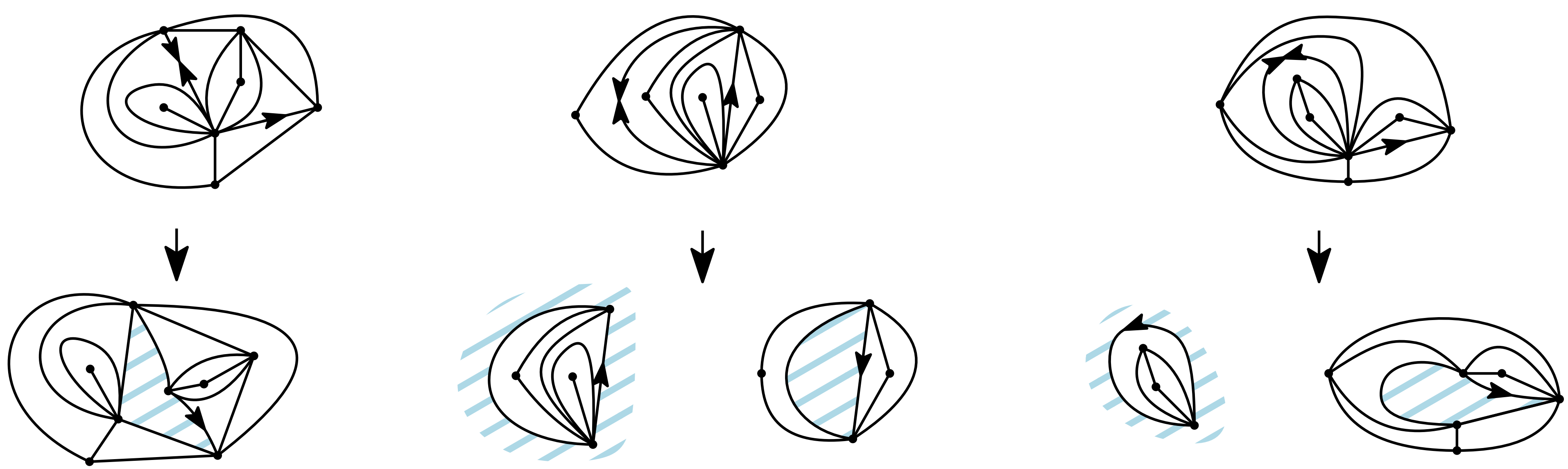

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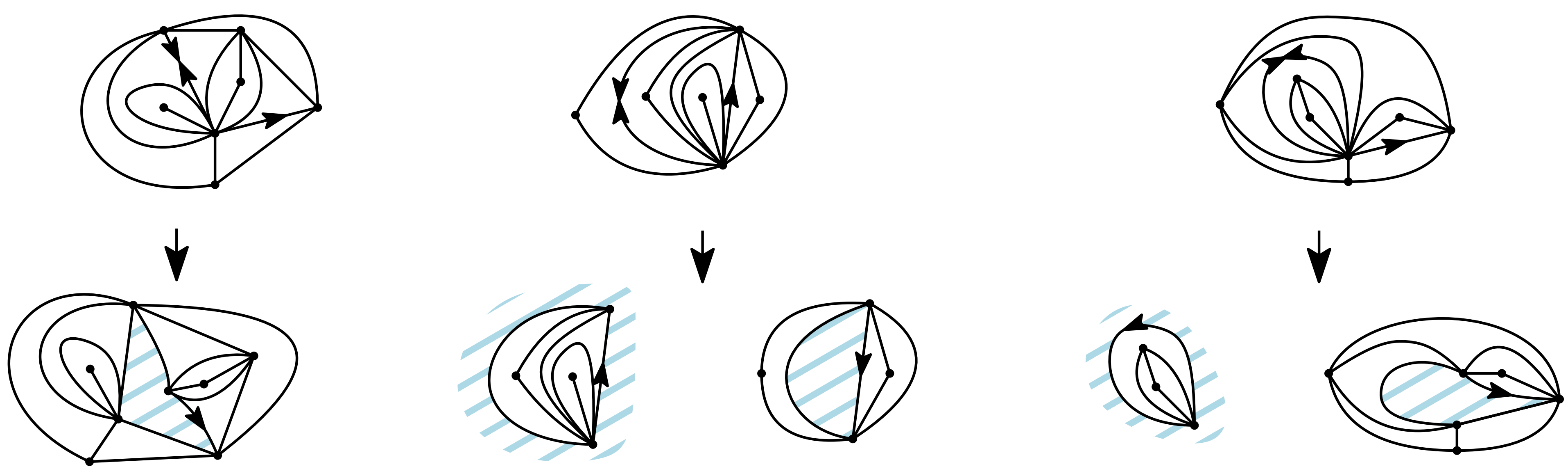

$$\overline{\mathbb{P}_n}\left(\delta \in e\right) \leq \max\left\{\frac{1}{\nu}, 1\right\}^2 \frac{[t^{3n+2}](Z_4 + Z_2^2 + Z_1^2 + Z_1^2 Z_2 + Z_1 Z_3)}{3n\left[t^{3n}\right]\mathcal{Z}}$$

$$= \mathcal{O}(1/n)$$

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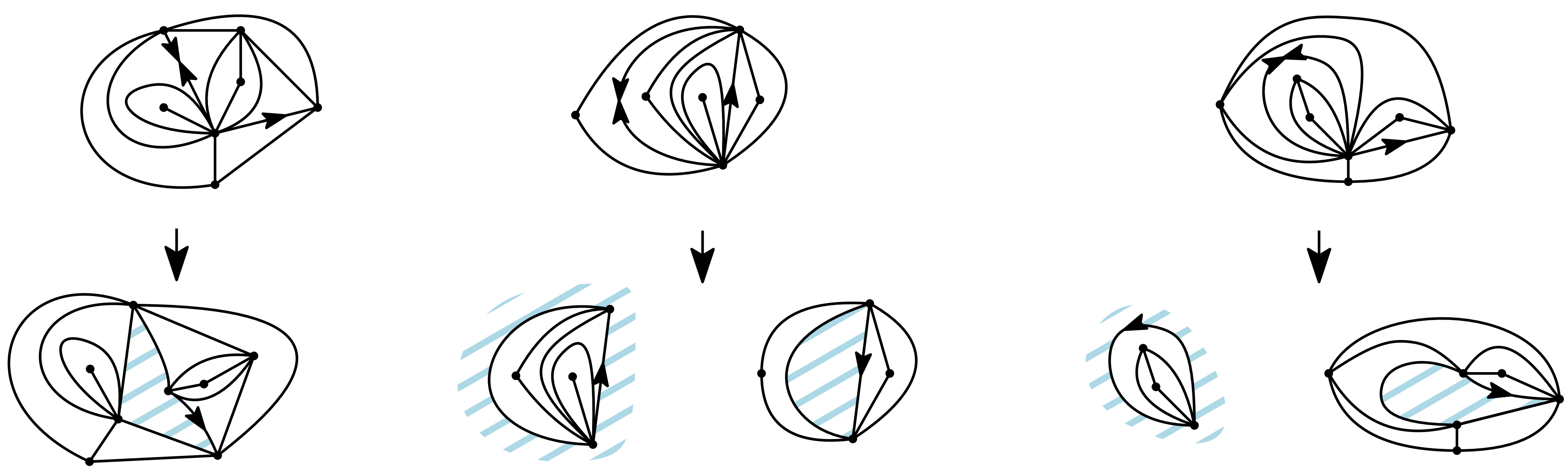

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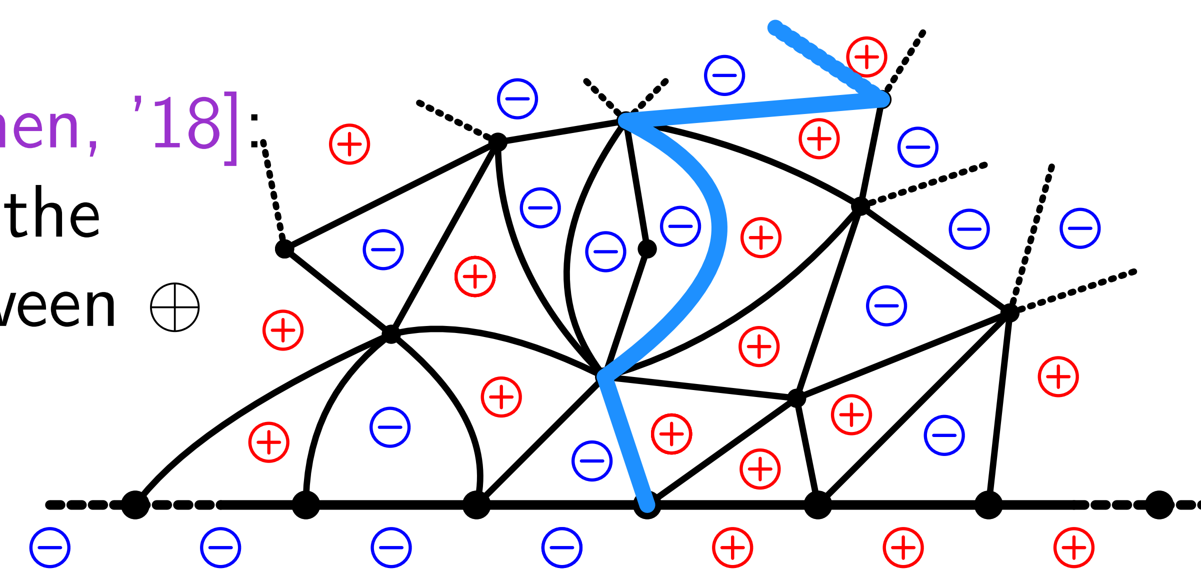
Local convergence of triangulations with spins

Probability measure on triangulations of $\mathcal{T}_n$ with a spin configuration:

$$\mathbb{P}_n^\nu\Big(\{(T,\sigma)\}\Big) = \frac{\nu^{m(T,\sigma)}}{[t^{3n}]Q(\nu,t)}.$$

Theorem [AMS]
As $n \to \infty$, the sequence $\mathbb{P}_n^\nu$ converges weakly to a probability measure $\mathbb{P}^\nu$ for the **local topology**.
The measure $\mathbb{P}^\nu$ is supported on infinite triangulations with one end.

Recent related result by [Chen, Turunen, '18]:
Local convergence for triangulations of the halfplane by studying the interface between $\oplus$ and $\ominus$.



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Thank you for your attention!

Summer school **Random trees and graphs**
July 1 to 5, 2019 in Marseille France
Org. M. Albenque, J. Bettinelli, J. Rué and L.Menard

Summer school **Random walks and models of complex networks**
July 8 to 19, 2019 in Nice
Org. B. Reed and D. Mitsche

Thank you for your attention!