On the geometry of polar varieties[1]

B. Bank [2], M. Giusti [3], J. Heintz [4],
M. Safey El Din [5], E. Schost [6]

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Abstract

We have developed in the past several algorithms with intrinsic complexity bounds for the problem of point finding in real algebraic varieties. Our aim here is to give a comprehensive presentation of the geometrical tools which are necessary to prove the correctness and complexity estimates of these algorithms. Our results form also the geometrical main ingredients for the computational treatment of singular hypersurfaces.
In particular, we show the non–emptiness of suitable generic dual polar varieties of (possibly singular) real varieties, show that generic polar varieties may become singular at smooth points of the original variety and exhibit a sufficient criterion when this is not the case. Further, we introduce the new concept of meagerly generic polar varieties and give a degree estimate for them in terms of the degrees of generic polar varieties.
The statements are illustrated by examples and a computer experiment.

Keywords: Real polynomial equation solving; singularities; classic polar varieties; dual polar varieties; generic polar varieties; meagerly generic polar varieties

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[2]Humboldt-Universität zu Berlin, Institut für Mathematik, D–10099 Berlin, Germany. bank@mathematik.hu-berlin.de

[3]CNRS, École Polytechnique, Laboratoire LIX, F–91228 Palaiseau Cedex, France. Marc.Giusti@Polytechnique.fr

[4]Departamento de Computación, Universidad de Buenos Aires and CONICET, Ciudad Univ., Pab.I, 1428 Ciudad Autónoma de Buenos Aires, Argentina, and Departamento de Matemáticas, Estadística y Computación, Facultad de Ciencias, Universidad de Cantabria, 39071 Santander, Spain.
joos@dc.uba.ar

[5]INRIA, Paris-Rocquencourt, SALSA Project; UPMC, Univ Paris 06, LIP6; CNRS, UMR 7606, LIP6; UFR Ingénierie 919, LIP6 Passy-Kennedy; Case 169, 4, Place Jussieu, F-75252 Paris Mohab.Safey@lip6.fr

[6]Computer Science Department, Room 415, Middlesex College,
University of Western Ontario, Canada.
eschost@uwo.ca

1 Preliminaries and results

Let $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ be the fields of rational, real and complex numbers, respectively, let $X := (X_1, \ldots, X_n)$ be a vector of indeterminates over $\mathbb{C}$ and let be given a reduced regular sequence $F_1, \ldots, F_p$ of polynomials in $\mathbb{Q}[X]$ such that the ideal $(F_1, \ldots, F_p)$ generated by them is the ideal of definition of a closed, $\mathbb{Q}$–definable subvariety S of the n–dimensional complex affine space $\mathbb{A}^n := \mathbb{C}^n$. Thus S is a non–empty equidimensional affine variety of dimension $n-p$, i.e., each irreducible component of S is of dimension $n-p$. Said otherwise, S is of pure codimension p (in $\mathbb{A}^n$).

We denote by S_{reg} the locus of *regular points* of S, i.e., the points of S, where the Jacobian $J(F_1, \ldots, F_p) := \left[\frac{\partial F_k}{\partial X_l}\right]_{\substack{1\le k\le p\\ 1\le l\le n}}$ has maximal rank p, and by $S_{sing} := S\backslash S_{reg}$ the singular locus of S.

Let $\mathbb{A}^n_{\mathbb{R}} := \mathbb{R}^n$ be the n–dimensional real affine space. We denote by $S_{\mathbb{R}} := S \cap \mathbb{A}^n_{\mathbb{R}}$ the real trace of the complex variety S. Moreover, we denote by $\mathbb{P}^n$ the n–dimensional complex projective space and by $\mathbb{P}^n_{\mathbb{R}}$ its real counterpart. We shall use also the following notations:

$$S := \{F_1 = 0, \ldots, F_p = 0\} \text{ and } S_{\mathbb{R}} := \{F_1 = 0, \ldots, F_p = 0\}_{\mathbb{R}}.$$

We denote the coordinate ring of the affine variety S by $\mathbb{C}[S]$. Thus $\mathbb{C}[S]$ is a finitely generated, reduced, equidimensional $\mathbb{C}$–algebra which is a domain when S is irreducible.
By $\mathbb{C}(S)$ we denote the total quotient ring of $\mathbb{C}[S]$ (or simply of S) which consists of all rational functions of S whose domain has non–empty intersection with every irreducible component of S. When S is irreducible, then $\mathbb{C}(S)$ becomes the usual field of rational functions of S.
The Chinese Remainder Theorem implies that the $\mathbb{C}$–algebra $\mathbb{C}(S)$ is isomomorphic to the direct product of the function fields of the irreducible components of S.

All varieties that occur in this paper are defined set–theoretically, and not scheme–theoretically. Thus the affine ones have always reduced coordinate rings and when we formulate an algebraic property of a given variety like normality or Cohen–Macaulayness we refer always to the (reduced) coordinate ring of the variety.

Let $1 \le i \le n-p$ and let $\alpha := [a_{k,l}]_{\substack{1\le k\le n-p-i+1\\ 0\le l\le n}}$ be a complex $((n-p-i+1)\times(n+1))$–matrix and suppose that $\alpha_* := [a_{k,l}]_{\substack{1\le k\le n-p-i+1\\ 1\le l\le n}}$ has maximal rank $n-p-i+1$. In case $(a_{1,0}, \ldots, a_{n-p-i+1,0}) = 0$ we denote by $\underline{K}(\alpha) := \underline{K}^{n-p-i}(\alpha)$ and in case $(a_{1,0}, \ldots, a_{n-p-i+1,0}) \neq 0$ by $\overline{K}(\alpha) := \overline{K}^{n-p-i}(\alpha)$ the $(n-p-i)$–dimensional linear subvarieties of the projective space $\mathbb{P}^n$ which for $1 \le k \le n-p-i+1$ are spanned by the the points $(a_{k,0} : a_{k,1} : \cdots : a_{k,n})$.

We define the classic and the dual ith polar varieties of S associated with the linear varieties $\underline{K}(\alpha)$ and $\overline{K}(\alpha)$ as the closures of the loci of the regular points of

S where all $(n-i+1)$–minors of the respective polynomial $((n-i+1)\times n)$ matrix

$$\begin{bmatrix} & J(F_1,\ldots,F_p) & \\ a_{1,1} & \cdots & a_{1,n} \\ \vdots & \vdots & \vdots \\ a_{n-p-i+1,1} & \cdots & a_{n-p-i+1,n} \end{bmatrix}$$

and

$$\begin{bmatrix} & J(F_1,\ldots,F_p) & \\ a_{1,1}-a_{1,0}X_1 & \cdots & a_{1,n}-a_{1,0}X_n \\ \vdots & \vdots & \vdots \\ a_{n-p-i+1,1}-a_{n-p-i+1,0}X_1 & \cdots & a_{n-p-i+1,n}-a_{n-p-i+1,0}X_n \end{bmatrix}$$

vanish. We denote these polar varieties by

$$W_{\underline{K}(\alpha)}(S) := W_{\underline{K}^{n-p-i}(\alpha)}(S) \ \text{ and } \ W_{\overline{K}(\alpha)}(S) := W_{\overline{K}^{n-p-i}(\alpha)}(S),$$

respectively. They are of expected pure codimension i in S. Observe also that the polar varieties $W_{\underline{K}^{n-p-i}(\alpha)}(S)$ and $W_{\overline{K}^{n-p-i}(\alpha)}(S)$ are determined by the $((n-p-i+1)\times n)$–matrix $a := \alpha_*$.

If α is a real $((n-p-i+1)\times(n+1))$–matrix, we denote by

$$W_{\underline{K}(\alpha)}(S_{\mathbb{R}}) := W_{\underline{K}^{n-p-i}(\alpha)}(S_{\mathbb{R}}) := W_{\underline{K}(\alpha)}(S) \cap \mathbb{A}^n_{\mathbb{R}}$$

and

$$W_{\overline{K}(\alpha)}(S_{\mathbb{R}}) := W_{\overline{K}^{n-p-i}(\alpha)}(S_{\mathbb{R}}) := W_{\overline{K}(a)}(S) \cap \mathbb{A}^n_{\mathbb{R}}$$

the real traces of $W_{\underline{K}(\alpha)}(S)$ and $W_{\overline{K}(\alpha)}(S)$.

In this paper we shall work with this purely calculatory definition of the classic and dual polar varieties of S. On the other hand, both notions may alternatively be charcterized in terms of intrinsic (i.e., coordinate-free), geometric concepts. For example, the classic polar variety $W_{\underline{K}(\alpha)}(S)$ is the Zariski closure of all regular points x of S such that the tangent space at x is not transversal to the linear variety $\underline{K}(\alpha)$ and the real polar variety $W_{\underline{K}(\alpha)}(S_{\mathbb{R}})$ may be characterized similarly.

In the same vein, fixing an embedding of the affine variety S into the projective space $\mathbb{P}^n$ and fixing a non–degenerate hyperquadric in $\mathbb{P}^n$, we introduced in [5] and [6] the notion of a *generalized polar variety* of S which contains as particular instances the notions of classic and dual polar varieties. From the intrinsic, geometric characterization of these generalized polar varieties we derived then the present calculatory definition of classic and dual polar varieties. For details we refer the reader to [5] and [6].

The argumentation of this paper will substantially depend on this calculatory definition. This leads to a notation which at first glance looks overloaded by diacritic marks as e.g. the underscores and overbars we use to distinguish classic from dual polar varieties. In view of the context, the reader may overlook in most cases these diacritic marks, however omitting them would introduce an inadmissible amount of ambiguities in statements and proofs.

For the rest of this section let us assume that for fixed $1 \leq i \leq n-p$ there is given a generic complex $((n-p-i+1) \times n)$–matrix $a = [a_{k,l}]_{\substack{1\leq k\leq n-p-i+1 \\ 1\leq l\leq n}}$ (the use of the word "generic" will be clarified at the end of this section).
For $1 \leq k \leq n-p-i+1$ and $0 \leq l \leq n$ we introduce the following notations:

$$\underline{a}_{k,l} := 0 \text{ and } \overline{a}_{k,l} := 1 \text{ if } l = 0, \ \underline{a}_{k,l} := \overline{a}_{k,l} := a_{k,l} \text{ if } 1 \leq l \leq n,$$

$$\underline{a} := [\underline{a}_{k,l}]_{\substack{1\leq k\leq n-p-i+1 \\ 0\leq l\leq n}} \text{ and } \overline{a} := [\overline{a}_{k,l}]_{\substack{1\leq k\leq n-p-i+1 \\ 0\leq l\leq n}}$$

(thus we have $\underline{a}_* = \overline{a}_* = [a_{k,l}]_{\substack{1\leq k\leq n-p-i+1 \\ 1\leq l\leq n}}$).

The corresponding polar varieties $W_{\underline{K}(\underline{a})}(S)$ and $W_{\overline{K}(\overline{a})}(S)$ will be called *generic.* In this case we have shown in [5] and [6] that the polar varieties $W_{\underline{K}(\underline{a})}(S)$ and $W_{\overline{K}(\overline{a})}(S)$ are either empty or of pure codimension i in S. Further, we have shown that the local rings of $W_{\underline{K}(\underline{a})}(S)$ and $W_{\overline{K}(\overline{a})}(S)$ at any regular point are Cohen–Macaulay ([5], Theorem 9).

In the case of the classic polar variety $W_{\underline{K}(\underline{a})}(S)$ these results are well known to specialists as a consequence of Kleiman's variant of the Bertini Theorems ([22]) and general properties of determinantal varieties (see [28], proof of Lemma 1.3, or [39], Chapitre IV, proof of Proposition 2). In the case of the dual polar variety $W_{\overline{K}(\overline{a})}(S)$ this kind of general argumentation cannot be applied and more specific tools are necessary like those developed in [5] and [6].

The modern concept of (classic) polar varieties was introduced in the 1930's by F. Severi ([35], [36]) and J. A. Todd ([43], [42]), while the intimately related notion of a reciprocal curve goes back to the work of J.-V. Poncelet in the period of 1813–1829.

As pointed out by Severi and Todd, generic polar varieties have to be understood as being organized in certain equivalence classes which embody relevant geometric properties of the underlying algebraic variety S. This view led to the consideration of rational equivalence classes of the generic polar varieties.

Around 1975 a renewal of the theory of polar varieties took place with essential contributions due R. Piene ([28]) (global theory), B. Teissier, D. T. Lê ([23], [39]), J. P. Henry and M. Merle ([19]), A. Dubson ([13], Chapitre IV) (local theory), J. P. Brasselet and others (the list is not exhaustive, see [40],[28] and [9] for a historical account and references). The idea was to use rational equivalence classes of generic polar varieties as a tool which allows to establish numerical formulas in order to classify singular varieties by their intrinsic geometric character ([28]).

At the same time, classes of generic polar varieties were used in order to formulate a manageable local equisingularity criterion which implies the Whitney conditions in analytic varieties in view of an intended concept of canonical stratifications (see [39]).

On the other hand, classic polar varieties became about ten years ago a fundamental tool for the design of efficient computer procedures with *intrinsic* complexity which find real algebraic sample points for the connected components of $S_{\mathbb{R}}$, if $(F_1, \ldots, F_p)$ is a reduced regular sequence in $\mathbb{Q}[X]$ and $S_{\mathbb{R}}$ is smooth and compact. The sequential time complexity of these procedures, in its essence, turns out to be polynomially bounded by the maximal degree of the generic polar varieties $W_{\underline{K}^{n-p-i}(\underline{a})}(S)$, for $1 \leq i \leq n-p$. As we shall see in Section 4 as a consequence of Theorem 13 , this maximal degree is an invariant of the input system $(F_1, \ldots, F_p)$ and the variety S, and even of the real variety $S_{\mathbb{R}}$, if all irreducible components of S contain a regular real point. In this sense we refer to the complexity of these algorithms as being intrinsic (see [3, 4], [31] and [34]) .

The compactness assumption on $S_{\mathbb{R}}$ was essential in order to guarantee the non–emptiness of the classic polar varieties $W_{\underline{K}^{n-p-i}(\underline{a})}(S_{\mathbb{R}})$, for $1 \leq i \leq n-p$. If $S_{\mathbb{R}}$ is singular or unbounded, the generic classic polar varieties $W_{\underline{K}^{n-p-i}(\underline{a})}(S_{\mathbb{R}})$ may become empty (this becomes also a drawback for the geometric analysis of singular varieties described above).

In order to overcome this difficulty at least in the case of a non–singular real variety $S_{\mathbb{R}}$, the notion of dual polar varieties was introduced in [5] and [6]. The usefulness of dual polar varieties is highlighted by the following statement:

If $S_{\mathbb{R}}$ is smooth, then the dual polar variety $W_{\overline{K}^{n-p-i}(\overline{a})}(S_{\mathbb{R}})$ contains a sample point for each connected component of $S_{\mathbb{R}}$ (see [5, 6], Proposition 2 and [26], Proposition 2.2).

In case that $S_{\mathbb{R}}$ is singular, we have the following, considerably weaker result which will be shown in Section 2 as Theorem 3.

Let $1 \leq i \leq n-p$ and let C be a connected component of the real variety $S_{\mathbb{R}}$ containing a regular point. Then, with respect to the Euclidean topology, there exists a non–empty, open subset $O_C^{(i)}$ of $\mathbb{A}_{\mathbb{R}}^{(n-p-i+1)\times n}$ such that any $((n-p-i+1)\times n)$–matrix a of $O_C^{(i)}$ has maximal rank $n-p-i+1$ and such that the real dual polar variety $W_{\overline{K}(\overline{a})}(S_{\mathbb{R}})$ is generic and contains a regular point of C.

In view of the so–called "lip of Thom" ([41]), a well studied example of a singular curve, this result cannot be improved. Although it is not too expensive to construct algorithmically non–empty open conditions which imply the conclusion of Theorem 3, the search for rational sample points satisfying these conditions seems to be as difficult as the task of finding smooth points on singular real varieties.

Dual polar varieties represent a complex counterpart of the Lagrange multipliers. Therefore their geometric meaning concerns more real than complex algebraic varieties. Maybe this is the reason why, motivated by the search for *real* solutions of polynomial equation systems, they were only recently introduced in (complex) algebraic geometry. In the special case of $p := 1$ and $i := n-p$ the notion of a dual polar variety appears implicitly in [32] (see also [29], [2] and [33]).
The consideration of general $(n-p)$th classic (or dual) polar varieties was introduced in complexity theory by [16] and got the name "critical point method". The emerging of elimination procedures of intrinsic complexity made it necessary to take into account also the higher dimensional ith polar varieties of S (for $1 \leq i < n-p$).

Under the name of "reciprocal polar varieties" generic dual varieties of real singular plane curves are exhaustively studied in [26] and a manageable sufficient condition for their non–emptiness is exhibited.

An alternative procedure of intrinsic complexity to find sample points in not necessarily compact smooth semialgebraic varieties was exhibited in [31]. This procedure is based on the recursive use of *classic* polar varieties.

We are now going to describe the further content of this paper. For the sake of simplicity of exposition let us suppose for the moment that the given variety S is smooth. Refining the tools developed in [5] and [6] we shall show in the first part of Section 3 that the generic classic and dual polar varieties $W_{\underline{K}(\underline{a})}(S)$ and $W_{\overline{K}(\overline{a})}(S)$ are normal (see Theorem 6 and Corollary 8). Hence the generic polar varieties of S are both, normal and Cohen–Macaulay.
Unfortunately, this is the best result we can hope for. In the second part of Section 3 we shall exhibit a general method which allows to obtain smooth varieties S whose *higher* dimensional generic polar varieties are singular. Hence, the assertion Theorem 10 (i) of [6], which claims that *all* generic polar varieties of S are empty or smooth, is wrong in fact.
On the other hand we shall describe a sufficient combinatorial condition in terms of the parameters n, p and $1 \leq i \leq n-p$, that guarantees that the *lower* dimensional generic polar varieties of S are empty or non–singular.

Let us mention here that in case $p := 1$, i.e., if S is a nonsingular hypersurface, the classic polar variety $W_{\underline{K}(\underline{a})}(S)$ is smooth. This is an immediate consequence of the transversality version of Kleiman's theorem (see also [3] for an elementary proof). However, in case $p := 1$, the higher dimensional generic *dual* polar varieties of the smooth hypersurface S may contain singularities.

Finally we explain in Section 3 in a more systematic way how singularities in higher dimensional generic polar varieties of S may arise.
Using for generic $a \in \mathbb{A}^{(n-p-i+1)\times n}$ a natural desingularization of the (open) polar variety $W_{\underline{K}^{n-p-i}(\underline{a})}(S) \cap S_{reg}$ in the spirit of Room–Kempf [30, 21], we shall include the singularities of $W_{\underline{K}^{n-p-i}(\underline{a})}(S) \cap S_{reg}$ in a kind of algebraic geometric "discriminant locus" of this desingularization.

On the other hand the generic complex $((n-p-i+1)\times n)$–matrix a induces an analytic map from S_{reg} to $\mathbb{A}^{n-p-i+1}$. We shall show that $W_{\underline{K}^{n-p-i}(\underline{a})}(S) \cap S_{reg}$ may be decomposed into smooth Thom–Boardman strata of this map.

In the geometric analysis of singular varieties as well as in real polynomial equation solving, generic polar varieties play a fundamental role as providers of geometric invariants which characterize the underlying (complex or real) algebraic variety, in our case S or $S_{\mathbb{R}}$. We have already seen that for $1 \leq i \leq n-p$ the generic polar varieties $W_{\underline{K}^{n-p-i}(\underline{a})}(S)$ and $W_{\overline{K}^{n-p-i}(\overline{a})}(S)$ are empty or of codimension i in S and therefore their dimension becomes an invariant of the algebraic variety S. On the other hand, Theorem 3 and the example of Thom's lip imply that for $a \in \mathbb{Q}^{(n-p-i+1)\times n}$ generic the dimension of the real polar variety $W_{\overline{K}(\overline{a})}(S_{\mathbb{R}})$ is not an invariant of $S_{\mathbb{R}}$, since $W_{\overline{K}(\overline{a})}(S_{\mathbb{R}})$ may be empty or not, according to the choice of a.

It may occur that the degrees of generic polar varieties represent a too coarse measure for the complexity of elimination procedures which solve real polynomial equations. Therefore it is sometimes convenient to replace for $1 \leq i \leq n-p$ the generic polar varieties of S and $S_{\mathbb{R}}$ by more special ones of the form

$W_{\underline{K}^{n-p-i}(\underline{a})}(S)$, $W_{\underline{K}^{n-p-i}(\underline{a})}(S_{\mathbb{R}})$ and $W_{\overline{K}^{n-p-i}(\overline{a})}(S)$, $W_{\overline{K}^{n-p-i}(\overline{a})}(S_{\mathbb{R}})$ (or suitable non–empty Zariski open subsets of them). Here a ranges over a Zariski dense subset of a suitable irreducible subvariety of $\mathbb{A}^{(n-p-i+1)\times n}$ (containing generally a Zariski dense set of rational points). We call these special polar varieties *meagerly generic.* They share important properties with generic polar varieties (e.g. dimension and reducedness) and often they are locally given as transversal intersections of closed form equations and therefore smooth. A particular class of meagerly generic polar varieties with this property was studied in [4].

Another class of meagerly generic polar varieties appears implicitly in [7], where the problem of finding smooth algebraic sample points for the (non–degenerated) connected components of *singular* real hypersurfaces is studied.

It is not hard to see that the (geometric) degrees of the *generic* polar varieties of S constitute invariants of S, i.e., the degrees of $W_{\underline{K}^{n-p-i}(\underline{a})}(S)$ and $W_{\overline{K}^{n-p-i}(\overline{a})}(S)$ are independent of the choice of the (generic) complex $((n-p-i+1)\times n)$–matrix a.

The main result of Section 4 may be paraphrased as follows: for $1 \leq i \leq n-p$ the degrees of the ith meagerly generic classic and dual polar varieties of S are bounded by the degree of the corresponding ith generic polar variety of S.

The rest of Section 4 is devoted to the discussion of the notion of a meagerly generic polar variety.

Before finishing this introductory presentation, we add a clarification about our use of the word *generic.* To this aim we adopt the point of view of René Thom [41].

Definition 1
Let n, $1 \leq p \leq n$ and $1 \leq i \leq n-p$ be as above. The ith polar varieties of S depend on linear subvarieties of $\mathbb{P}^n$ which are given by full–rank matrices of the form $a = [a_{k,l}]_{\substack{1\leq k\leq n-p+1\\ 1\leq l\leq n}}$ with complex, real or rational entries. We say that a given statement is valid for the generic ith classic or dual polar varieties of S if there exists a non–empty Zariski open (and hence residual dense) subset O of full–rank matrices of $\mathbb{A}^{(n-p-i+1)\times n}$ (or $\mathbb{A}_{\mathbb{R}}^{(n-p-i+1)\times n}$) such that for any a in O the statement is verified by $W_{\underline{K}(\underline{a})}$ or $W_{\overline{K}(\overline{a})}$ (or their real traces).

In Section 4 we shall face a more general situation: let be given an irreducible affine subvariety E of $\mathbb{A}^{(n-p-i+1)\times n}$, e.g. an affine linear subspace of $\mathbb{A}^{(n-p-i+1)\times n}$. With reference to E we shall say that a given statement is valid for *meagerly generic* ith polar varieties of S if there exists a non–empty Zariski open subset O of full–rank matrices of E such that for any $a \in O$ the statement is verified by $W_{\underline{K}(\underline{a})}$ or $W_{\overline{K}(\overline{a})}$ (observe that O is residual dense in E).

The aim of this paper is a comprehensive presentation of the geometrical tools which are necessary to prove the correctness of algorithms with intrinsic complexity bounds for real root finding. We developed these algorithms in the past and we think to develop them further. This leads us to frequent references to already published work on applications of geometric reasoning to computer science. Thus, in part, this paper has also survey character.

2 Real dual polar varieties

This section is concerned with the proof of Theorem 3, which was announced in Section 1. We start with the following technical statement.

Lemma 2
Let C be a connected component of the real variety $S_{\mathbb{R}}$ containing an regular point. Then, with respect to the Euclidean topology , there exists a non–empty, open subset U_C of $\mathbb{A}^n_{\mathbb{R}} \setminus S_{\mathbb{R}}$ that satisfies the following condition: Let u be an arbitrary point of U_C and let x be any point of $S_{\mathbb{R}}$ that minimizes the Euclidean distance to u with respect to $S_{\mathbb{R}}$. Then x is a regular point belonging to C.

Proof
For any two points $z_1, z_2 \in \mathbb{A}^n_{\mathbb{R}}$ and any subset Y of $\mathbb{A}^n_{\mathbb{R}}$ we denote by $d(z_1, z_2)$ the Euclidean distance between z_1 and z_2 and by $d(z_1, Y)$ the Euclidean distance from z_1 to Y, i.e., $d(z_1, Y) := \inf\{d(z_1, y) \,|\, y \in Y\}$. Let $C_1, \ldots, C_s$ be the connected components of $S_{\mathbb{R}}$ and suppose without loss of generality that $C = C_1$ holds. By assumption there exists a regular point z of C. Observe that the distance $d(z, S_{sing} \cap \mathbb{A}^n_{\mathbb{R}})$ is positive.

Choose now an open ball B of $\mathbb{A}^n_{\mathbb{R}}$ around the origin which intersects $C_1, \ldots, C_s$ and contains the point z in its interior. Since the non–empty sets $C_1 \cap \bar{B}, \ldots, C_s \cap \bar{B}$ are disjoint and compact, they have well–defined, positive distances. Therefore, there exists a positive real number d stricly smaller than all these distances and $d(z, S_{sing} \cap \mathbb{A}^n_{\mathbb{R}})$ such that the open sphere of radius d and center z is contained in B. Let $U^* := \{u^* \in A^n_{\mathbb{R}} \,|\, d(u^*, z) < \frac{d}{2}\}$ be the open ball of radius $\frac{d}{2}$ and center z and let $U_C := U^* \setminus S_{\mathbb{R}}$.

Since S has positive codimension in $\mathbb{A}^n$, we conclude that U_C is a non–empty open set with respect to the Euclidean topology which is contained in $\mathbb{A}^n_{\mathbb{R}} \setminus S_{\mathbb{R}}$.

Let u be an arbitrary point of U_C. Since u does not belong to the closed set $S_{\mathbb{R}}$ we have $0 < d(u, S_{\mathbb{R}}) \leq d(u, z) < \frac{d}{2}$. Let x be any point of $S_{\mathbb{R}}$ that minimizes the distance to u, thus satisfying the condition $d(u, x) = d(u, S_{\mathbb{R}})$. From $d(u, x) < \frac{d}{2}$, $d(z, u) < \frac{d}{2}$ and the triangle inequality one concludes now that

$$d(z, x) \leq d(z, u) + d(u, x) < d$$

holds. Therefore x cannot be contained in $C_2, \ldots, C_s$ and neither in $S_{sing} \cap \mathbb{A}^n_{\mathbb{R}}$. Thus, necessarily x is a regular point of C. $\square$

Now we are going to formulate and prove the main result of this section.

Theorem 3
Let $1 \leq i \leq n - p$ and let C be a connected component of the real variety $S_{\mathbb{R}}$ containing a regular point. Then, with respect to the Euclidean topology, there exists a non–empty, open subset $O^{(i)}_C$ of $\mathbb{A}^{(n-p-i+1)\times n}_{\mathbb{R}}$ such that any $((n - p - i + 1) \times n)$–matrix a of $O^{(i)}_C$ has maximal rank $n - p - i + 1$ and such that the real dual polar variety $W_{\overline{K(\underline{a})}}(S_{\mathbb{R}})$ is generic and contains a regular point of C.

Proof
Taking into account Lemma 2, we follow the arguments contained in the proof of [5] and [6], Proposition 2.

Let C be a connected component of $S_{\mathbb{R}}$ containing a regular point and let U_C be the non–empty open subset of $\mathbb{A}^n \setminus S_{\mathbb{R}}$ introduced in Lemma 2. Without loss of generality we may assume $0 \notin U_C \setminus S_{\mathbb{R}}$ and that for any point $a \in U_C \setminus S_{\mathbb{R}}$ the dual polar variety $W_{\overline{K}^0(a)}(S)$ is empty or generic. Thus, putting $O_C^{(n-p)} := U_C \setminus S_{\mathbb{R}}$, we show first that the statement of Theorem 3 is true for C and $i := n-p$.

Let $(a_{1,1}, \ldots, a_{1,n})$ be an arbitrary element of $O_C^{(n-p)}$, $a_{1,0} := 1$, $a := (a_{1,0}, \ldots, a_{1,n})$ and let $T_a^{(n-p)}$ be the polynomial $((p+1) \times n)$–matrix

$$T_a^{(n-p)} := T_a^{(n-p)}(X) := \begin{bmatrix} & J(F_1, \ldots, F_p) & \\ a_{1,1} - X_1 & \cdots & a_{1,n} - X_n \end{bmatrix}.$$

For $p+1 \leq k \leq n$ denote by $N_k := N_k(x)$ the $(p+1)$–minor of $T_a^{(n-p)}$ given by the columns $1, \ldots, p, k$. There exists a point x of $S_{\mathbb{R}}$ that minimizes the distance to $(a_{1,1}, \ldots, a_{1,n})$ with respect to $S_{\mathbb{R}}$. From Lemma 2 we deduce that x is a regular point belonging to C. Without loss of generality we may assume that $\det \left[\dfrac{\partial F_j}{\partial X_k} \right]_{1 \leq j,k \leq p}$ does not vanish at x.

There exists therefore a chart Y of the real differentiable manifold $S_{reg} \cap \mathbb{A}^n_{\mathbb{R}}$, passing through x, with local parameters $X_{p+1|_Y}, \ldots, X_{n|_Y}$. Consider now the restriction φ of the polynomial function $(a_{1,1} - X_1)^2 + \cdots + (a_{1,n} - X_n)^2$ to the chart Y and observe that $x \in Y$ minimizes the function φ with respect to Y. From the Lagrange–Multiplier–Theorem [38] we deduce easily that the polynomials $N_{p+1}, \ldots, N_n$ vanish at x. Taking into account that $\det \left[\dfrac{\partial F_j}{\partial X_k} \right]_{1 \leq j,k \leq p}$ does not vanish at x we infer from the Exchange–Lemma of [3] that any $(p+1)$–minor of $T_a^{(n-p)}$ must vanish at x. Therefore, x is a regular point belonging to $W_{\overline{K}^0(a)}(S_{\mathbb{R}}) \cap C$ and $W_{\overline{K}^0(a)}(S)$ is generic.

Now let $1 \leq i \leq n-p$ be arbitrary and let $[a_{k,l}]_{\substack{1 \leq k \leq n-p-i+1 \\ 1 \leq l \leq n}}$ be a real $((n-p-i+1) \times n)$–matrix of maximal rank $n-p-i+1$, $a_{1,0} := \cdots := a_{n-p-i+1,0} := 1$ and $a := [a_{k,l}]_{\substack{1 \leq k \leq n-p-i+1 \\ 0 \leq l \leq n}}$.
Further, let $T_a^{(i)}$ be the polynomial $((n-i+1) \times n)$–matrix

$$T_a^{(i)} := T_a^{(i)}(X) := \begin{bmatrix} & J(F_1, \ldots, F_p) & \\ a_{1,1} - X_1 & \cdots & a_{1,n} - X_n \\ \vdots & \vdots & \vdots \\ a_{n-p-i+1,1} - X_1 & \cdots & a_{n-p-i+1,n} - X_n \end{bmatrix}$$

and recall that the ith (complex) dual polar variety of S associated with the linear variety $\overline{K}(a) := \overline{K}^{n-p-i}(a)$, namely $W_{\overline{K}(a)}(S)$, is defined as the closure of the locus of the regular points of S, where all $(n-p-i)$–minors of $T_a^{(i)}$ vanish.

We choose now a non–empty open subset $O_C^{(i)}$ of $\mathbb{A}_{\mathbb{R}}^{(n-p-i+1) \times n}$ satisfying the following conditions:

(i) Any $((n-p-i+1)\times n)$–matrix $[a_{k,l}]_{\substack{1\le k\le n-p-i+1\\ 1\le l\le n}}$ belonging to $O_C^{(i)}$ has maximal rank $n-p-i+1$, and for $a_{1,0}:=\cdots:=a_{n-p-i+1,0}:=1$ and $a:=[a_{k,l}]_{\substack{1\le k\le n-p-i+1\\ 0\le l\le n}}$ the dual polar variety $W_{\overline{K}^{n-p-i}(a)}(S)$ is empty or generic.

(ii) The point $(a_{1,1},\dots,a_{1,n})$ belongs to $O_C^{(n-p)}$.

Let be given a $((n-p-i+1)\times(n+1))$–matrix a satisfying both conditions above. From the structure of the polynomial matrix $T_a^{(i)}$ one infers easily that $W_{\overline{K}^0(a)}(S)$ is contained in $W_{\overline{K}^{n-p-i}(a)}(S)$. As we have seen above, $W_{\overline{K}^0(a)}(S)\cap C$ contains a regular point and therefore so does $W_{\overline{K}^{n-p-i}(a)}(S)\cap C$. Moreover, by the choice of $O_C^{(n-p)}$, the dual polar variety $W_{\overline{K}^{n-p-i}(a)}(S)$ is generic. $\square$

Corollary 4
Suppose that the real variety $S_{\mathbb{R}}$ contains a regular point and let $1\le i\le n-p$. Then, with respect to the Euclidean distance, there exists a non-empty, open subset $O^{(i)}$ of $\mathbb{A}_{\mathbb{R}}^{(n-p-i+1)\times n}$ such that any $((n-p-i+1)\times n)$–matrix $[a_{k,l}]_{\substack{1\le k\le n-p-i+1\\ 1\le l\le n}}$ of $O^{(i)}$ has maximal rank $n-p-i+1$ and such that for $a_{1,0}:=\cdots:=a_{n-p-i+1,0}:=1$ and $a:=[a_{k,l}]_{\substack{1\le k\le n-p-i+1\\ 0\le l\le n}}$ the dual polar variety $W_{\overline{K}^{n-p-i}(a)}(S)$ is generic and non–empty.

3 On the smoothness of generic polar varieties

Using the ideas and tools developed in [5] and [6], we are going to prove in the first and main part of this section that the generic classic and dual polar varieties of S are normal and Cohen–Macaulay at any point, where $J(F_1,\dots,F_p)$ has maximal rank. Then we show by means of an infinite family of examples that it is not always true that *all* generic polar varieties of S are smooth at any point where $J(F_1,\dots,F_p)$ has maximal rank.

We finish the section with two explanations of this phenomenon of non–smoothness.

For the sake of simplicity of exposition, let us assume for the moment $1\le p<n$ and that any point of S is regular. Further, let $[A_{k,l}]_{\substack{1\le k\le n-p\\ 1\le l\le n}}$ be a $((n-p)\times n)$–matrix of new indeterminates $A_{k,l}$.

Recursively in $1\le i\le n-p$ and relative to $F_1,\dots,F_p$, we are now going to introduce two genericity conditions for complex $((n-p)\times n)$–matrices, namely $U_{classic}^{(i)}$ and $U_{dual}^{(i)}$, such that

$$U_{classic}:=U_{classic}^{(1)}\subset\cdots\subset U_{classic}^{(n-p)}\subset\mathbb{A}^{(n-p)\times n}$$

and

$$U_{dual}:=U_{dual}^{(1)}\subset\cdots\subset U_{dual}^{(n-p)}\subset\mathbb{A}^{(n-p)\times n}$$

form two filtrations of $\mathbb{A}^{(n-p)\times n}$ by suitable constructible, Zariski dense subsets (in fact, we shall choose them as being non–empty and Zariski open). The sets

$U_{classic}$ and U_{dual} will then give an appropriate meaning to the concept of a generic decreasing sequence of classic and dual polar varieties of S or simply to the concept of a generic polar variety. The properties we are going to ensure with this concept of genericity refer to dimension and Cohen–Macaulayness and are used for example in the proofs of Lemma 5 and Theorem 6 (see Definition 1 for the interpretation of the word "generic" in this paper).

Let us begin by introducing for $1 \leq i \leq n-p$ the genericity conditions $U^{(i)}_{classic}$. We start with $i := n-p$.

We fix temporarily a $(p \times p)$–minor m of the Jacobian $J(F_1, \dots, F_p)$. For the sake of conciseness we shall assume $m := \det \left[\frac{\partial F_j}{\partial X_k}\right]_{\substack{1\leq j\leq p\\ 1\leq k\leq p}}$. We suppose first that $1 \leq p < n-1$ holds. In this case we fix for the moment an arbitrary selection of indices $1 \leq k_1 \leq \cdots \leq k_{n-p+1} \leq n-p$ and $p < l_1 \leq \cdots \leq l_{n-p+1} \leq n$, such that $(k_1, l_1), \dots, (k_{n-p+1}, l_{n-p+1})$ are all distinct (observe that the condition $1 \leq p < n-1$ ensures that such selections exist). For $1 \leq j \leq n-p+1$, let

$$M^{(n-p)}_{k_j,l_j} := \begin{bmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_p} & \frac{\partial F_1}{\partial X_{l_j}} \\ \vdots & \cdots & \vdots & \vdots \\ \frac{\partial F_p}{\partial X_1} & \cdots & \frac{\partial F_p}{\partial X_p} & \frac{\partial F_p}{\partial X_{l_j}} \\ A_{k_j,1} & \cdots & A_{k_j,p} & A_{k_j,l_j} \end{bmatrix}.$$

Writing ${\mathbb{A}^n}_m := \{x \in \mathbb{A}^n \mid m(x) \neq 0\}$, we may suppose without loss of generality that $S_m := S \cap {\mathbb{A}^n}_m$ is non–empty.

We consider now two polynomial maps of smooth varieties, namely

$$\Phi : {\mathbb{A}^n}_m \times \mathbb{A}^{(n-p)\times n} \to \mathbb{A}^n \ \text{ and } \ \Psi : {\mathbb{A}^n}_m \times \mathbb{A}^{(n-p)\times n} \to \mathbb{A}^{n+1}$$

which are defined as follows:

For $x \in \mathbb{A}^n_m$ and any complex $((n-p)\times n)$–matrix $a' = [a_{k,l}]_{\substack{1\leq k\leq n-p\\ 1\leq l\leq n}}$, that contains for each $1 \leq j \leq n-p+1$ at the slots indicated by the indices $(k_j, 1), \dots, (k_j, p)$, (k_j, l_j) the entries of the point $a_j := (a_{k_j,1}, \dots, a_{k_j,p}, a_{k_j,l_j})$ of $\mathbb{A}^{p+1}$, the maps Φ and Ψ take the values

$$\Phi(x, a') := \Phi(x, a_1, \dots, a_{n-p}) := \\ (F_1(x), \dots, F_p(x),\ M_{k_1,l_1}(x, a_1), \dots, M_{k_{n-p},l_{n-p}}(x, a_{n-p}))$$

and

$$\Psi(x, a') := \Psi(x, a_1, \dots, a_{n-p+1}) := \\ (F_1(x), \dots, F_p(x), M_{k_1,l_1}(x, a_1), \dots, M_{k_{n-p+1},l_{n-p+1}}(x, a_{n-p+1})).$$

Let (x, a') be an arbitrary point of ${\mathbb{A}^n}_m \times \mathbb{A}^{(n-p)\times n}$ which satisfies the condition $\Phi(x, a') = 0$. Then the Jacobian of Φ at (x, a') contains a complex $(n \times (2n-p))$–matrix of the following form:

$$\begin{bmatrix} J(F_1, \dots, F_p)(x) & 0 & 0 & \cdots & 0 \\ & m(x) & 0 & \cdots & 0 \\ & 0 & m(x) & \cdots & 0 \\ * & \vdots & \vdots & \ddots & \vdots \\ & 0 & 0 & \cdots & m(x) \end{bmatrix}.$$

Taking into account $m(x) \neq 0$ and that $J(F_1, \ldots, F_p)(x)$ has rank p, one sees easily that this matrix has maximal rank n. Therefore Φ is regular at (x, a'). Since this point was chosen arbitrarily in $\Phi^{-1}(0)$, we conclude that $0 \in \mathbb{A}^n$ is a regular value of the polynomial map Φ.

By a similar argument one infers that $0 \in \mathbb{A}^{n+1}$ is a regular value of Ψ.

Therefore there exists by the Weak Transversality Theorem of Thom–Sard (see e.g. [12], Ch.3, Theorem 3.7.4, p.79) a non–empty Zariski open subset U of $\mathbb{A}^{(n-p)\times n}$ satisfiying the following conditions:

For any complex $((n-p) \times n)$–matrix $a' \in U$ such that a' contains at the slots indicated by the indices $(k_j, 1), \ldots, (k_j, p), (k_j, l_j)$, $1 \leq j \leq n-p+1$, the entries of suitable points $a_1, \ldots, a_{n-p+1}$ of $\mathbb{A}^{p+1}$, the equations

$$(1) \quad F_1(X) = \cdots = F_p(X) = 0, \qquad M_{k_1,l_1}(X, a_1) = \cdots = M_{k_{n-p},l_{n-p}}(X, a_{n-p}) = 0$$

intersect transversally at any of their common solutions in $\mathbb{A}^n_m$ and the equations

$$(2) \quad F_1(X) = \cdots = F_p(X) = 0, \qquad M_{k_1,l_1}(X, a_1) = \cdots = M_{k_{n-p+1},l_{n-p+1}}(X, a_{n-p+1}) = 0$$

have no common zero in $\mathbb{A}^n_m$.

Remember now that the construction of U depends on the selection of the minor m and the indices $(k_1, l_1), \ldots, (k_{n-p+1}, l_{n-p+1})$. There are only finitely many of these choices and each of them gives rise to a non–empty Zariski open subset of $\mathbb{A}^{(n-p)\times n}$. Cutting the intersection of all these sets with $(\mathbb{A}^1 \setminus \{0\})^{(n-p)\times n}$ we obtain finally $U^{(n-p)}_{classic}$.

The remaining case $p := n-1$ is treated similarly, considering only the polynomial map Φ.

The general step of our recursive construction of the genericity conditions $U^{(i)}_{classic}$ $1 \leq i \leq n-p$, is based on the same kind of argumentation. For the sake of completeness and though our reasoning may appear repetitive, we shall indicate all essential points that contain modifications with respect to our previous argumentation.

Let $1 \leq i < n-p$ and suppose that the genericity condition $U^{(i+1)}_{classic}$ is already constructed. Consider the $(n \times n)$– matrix

$$N := \begin{bmatrix} & J(F_1, \ldots, F_p) & \\ A_{1,1} & \cdots & A_{1,n} \\ \vdots & \vdots & \vdots \\ A_{n-p,1} & \cdots & A_{n-p,n} \end{bmatrix}$$

and fix for the moment an arbitrary $((n-i) \times (n-i))$– submatrix of N which contains $n-i$ entries from each of the rows number $1, \ldots, p$ of N. Let m denote the corresponding $(n-i)$–minor of N. For the sake of conciseness we shall assume that m is the minor

$$m := \det \begin{bmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_{n-i}} \\ \vdots & \cdots & \vdots \\ \frac{\partial F_p}{\partial X_1} & \cdots & \frac{\partial F_p}{\partial X_{n-i}} \\ A_{1,1} & \cdots & A_{1,n-i} \\ \vdots & \vdots & \vdots \\ A_{n-p-i,1} & \cdots & A_{n-p-i,n-i} \end{bmatrix}.$$

Writing $A := [A_{k,l}]_{\substack{1\le k\le n-p-i \\ 1\le l\le n-i}}$ we denote for a'' in $\mathbb{A}^{(n-p-i)\times n}$ by $m(X,a'')$ the polynomial obtained from $m = m(X,A)$ by specializing A to the complex $((n-p-i)\times n)$–matrix a''.
Without loss of generality we may suppose that $(S \times (\mathbb{A}^{(n-p)\times n})_m$ is non–empty.

Let us first suppose that $i^2 > n-p$ holds.

In this case we fix temporarily an arbitrary selection of indices $n-p-i < k_1 \le \cdots \le k_{n-p+1} \le n-p$ and $n-i < l_1 \le \cdots \le l_{n-p+1} \le n$ such that $(k_1,l_1),\dots,(k_{n-p+1},l_{n-p+1})$ are all distinct (observe that the condition $i^2 > n-p$ ensures that such selections exist).

For $1 \le j \le n-p+1$ we shall consider in the following the $(n-i+1)$–minor

$$M^{(i)}_{k_j,l_j} := \det \begin{bmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_{n-i}} & \frac{\partial F_1}{\partial X_{l_j}} \\ \vdots & \cdots & \vdots & \vdots \\ \frac{\partial F_p}{\partial X_1} & \cdots & \frac{\partial F_p}{\partial X_{n-i}} & \frac{\partial F_p}{\partial X_{l_j}} \\ A_{1,1} & \cdots & A_{1,n-i} & A_{1,l_j} \\ \vdots & \vdots & \vdots & \vdots \\ A_{n-p-i,1} & \cdots & A_{n-p-i,n-i} & A_{n-p-i,l_j} \\ A_{k_j,1} & \cdots & A_{k_j,n-i} & A_{k_j,l_j} \end{bmatrix}.$$

of N.
In the same spirit as before, we introduce now two polynomial maps of smooth varieties, namely

$$\Phi : (\mathbb{A}^n \times \mathbb{A}^{(n-p-i)\times n})_m \times \mathbb{A}^{i\times n} \longrightarrow \mathbb{A}^n$$

and

$$\Psi : (\mathbb{A}^n \times \mathbb{A}^{(n-p-i)\times n})_m \times \mathbb{A}^{i\times n} \longrightarrow \mathbb{A}^{n+1}$$

which are defined as follows:

For $(x,a'') \in (\mathbb{A}^n \times \mathbb{A}^{(n-p-i)\times n})_m$ and any complex $(i\times n)$–matrix $a''' := [a_{k,l}]_{\substack{n-p-i<k\le n-p \\ 1\le l\le n}}$ that contains for $1 \le j \le n-p+1$ at the slots indicated by the indices $(k_j,1),\dots,(k_j,n-i),(1,l_j),\dots,(n-p-i,l_j),(k_j,l_j)$ the entries of the point $a_j := (a_{k_j,1},\dots,a_{k_j,n-i},a_{1,l_j},\dots,a_{n-p-i,l_j},a_{k_j,l_j})$ of $\mathbb{A}^{2(n-i)-p+1}$, the maps Φ and Ψ take the values

$$\Phi(x,a'',a''') := \Phi(x,a'',a_1,\dots,a_{n-p}) := \\ (F_1(x),\dots,F_p(x),M_{k_1,l_1}(x,a'',a_1),\dots,M_{k_{n-p},l_{n-p}}(x,a'',a_{n-p}))$$

and

$$\Psi(x,a'',a''') := \Psi(x,a'',a_1,\dots,a_{n-p+1}) :=$$
$$:= (F_1(x),\dots,F_p(x), M_{k_1,l_1}(x,a'',a_1),\dots,M_{k_{n-p+1},l_{n-p+1}}(x,a'',a_{n-p+1})).$$

For a fixed full rank matrix $a'' \in \mathbb{A}^{(n-p-i)\times n}$ we denote by

$$\Phi_{a''} : \mathbb{A}^n_{m(X,a'')} \times \mathbb{A}^{i\times n} \to \mathbb{A}^n \quad \text{and} \quad \Psi_{a''} : \mathbb{A}^n_{m(X,a'')} \times \mathbb{A}^{i\times n} \to \mathbb{A}^{n+1}$$

the polynomial maps of smooth varieties induced by Φ and Ψ when we specialize the second argument to the value a''.

Let (x,a'',a''') be an arbitrary point of $(\mathbb{A}^n \times \mathbb{A}^{(n-p-i)\times n})_m \times \mathbb{A}^{i\times n}$ which satisfies the condition $\Phi(x,a'',a''') = 0$. Then the Jacobian of $\Phi_{a''}$ at (x,a''') contains a complex $(n\times(2n-p))$–matrix of the following form

$$\begin{bmatrix} J(F_1,\dots,F_p)(x) & 0 & 0 & \cdots & 0 \\ & m(x,a'') & 0 & \cdots & 0 \\ & 0 & m(x,a'') & \cdots & 0 \\ * & \vdots & \vdots & \ddots & \vdots \\ & 0 & 0 & \cdots & m(x,a'') \end{bmatrix}.$$

Taking into account $m(x,a'') \neq 0$ and that $J(F_1,\dots,F_p)(x)$ has rank p, one sees easily that this matrix and hence the Jacobian of $\Phi_{a''}$ at the point (x,a''') has maximal rank n. Therefore $\Phi_{a''}$ is regular at (x,a'''). Since this point was chosen arbitrarily in $\Phi^{-1}_{a''}(0)$, we conclude that $0 \in \mathbb{A}^n$ is a regular value of the polynomial map $\Phi_{a''}$. Similarly one argues that $0 \in \mathbb{A}^{n+1}$ is a regular value of $\Psi_{a''}$.

We consider now the constructible subset U of $\mathbb{A}^{(n-p)\times n}$ defined by the following conditions:

For any pair consisting of a complex $((n-i)\times n)$–matrix a'' and a complex $(i\times n)$–matrix a''' such that (a'',a''') belongs to U and such that a''' contains at the slots indicated by the indices $(k_j,1),\dots,(k_j,n-i),(1,l_j),\dots,(n-p-i,l_j),(k_j,l_j)$, $1 \le j \le n-p+1$, the entries of suitable points $a_1,\dots,a_{n-p+1}$ of $\mathbb{A}^{2(n-i)-p+1}$, the equations

$$(3) \quad F_1(X) = \cdots = F_p(X) = 0,$$
$$M_{k_1,l_1}(X,a'',a_1) = \cdots = M_{k_{n-p},l_{n-p}}(X,a'',a_{n-p}) = 0$$

intersect transversally at any of their common solutions in $\mathbb{A}^n_{m(X,a'')}$ and the equations

$$(4) \quad F_1(X) = \cdots = F_p(X) = 0,$$
$$M_{k_1,l_1}(X,a'',a_1) = \cdots = M_{k_{n-p+1},l_{n-p+1}}(X,a'',a_{n-p+1}) = 0$$

have no common zero in $\mathbb{A}^n_{m(X,a'')}$.

Applying now for every full rank matrix $a'' \in \mathbb{A}^{(n-p-i)\times n}$ the Weak Transversality Theorem of Thom–Sard to $\Phi_{a''}$ and $\Psi_{a''}$, we conclude from the constructibility of U that it is Zariski dense in $\mathbb{A}^{(n-p)\times n}$. Therefore we may suppose without loss of generality that U is a non–empty Zariski open subset of $\mathbb{A}^{(n-p)\times n}$ (here and in the sequel we use the fact that a residual dense, constructible subset of an affine space contains a non–empty Zariski open subset).

Remember now that the construction of U depends on the selection of the minor m and the indices $(k_1, l_1), \ldots, (k_{n-p+1}, l_{n-p+1})$. There are only finitely many of these choices and each of them gives rise to a non–empty Zariski open subset of $\mathbb{A}^{(n-p)\times n}$. Cutting the intersection of all these sets with $U^{(i+1)}_{classic}$ we obtain finally $U^{(i)}_{classic}$.

The remaining case $i^2 \leq n-p$ is treated similarly. In order to explain the differences with the previous argumentation, let us fix again the $(n-i)$– minor

$$m := \det \begin{bmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_{n-i}} \\ \vdots & \cdots & \vdots \\ \frac{\partial F_p}{\partial X_1} & \cdots & \frac{\partial F_p}{\partial X_{n-i}} \\ A_{1,1} & \cdots & A_{1,n-i} \\ \vdots & \vdots & \vdots \\ A_{n-p-i,1} & \cdots & A_{n-p-i,n-i} \end{bmatrix}$$

of the $n \times n$ matrix N, and an arbitrary selection of indices $n-p-i < k_1 \leq \cdots \leq k_{i^2} \leq n-p$ and $n-i < l_1 \leq \cdots \leq l_{i^2} \leq n$, such that $(k_1, l_1), \ldots, (k_{i^2}, l_{i^2})$ are all distinct (observe that the condition $i^2 \leq n-p$ ensures that such selections exist).

The $(n-i+1)$– minors M_{k_j, l_j}, $1 \leq j \leq i^2$ are the same as before. We consider now instead of Φ and Ψ only the polynomial map

$$\widetilde{\Phi} : (\mathbb{A}^n \times \mathbb{A}^{(n-p-i)\times n})_m \times \mathbb{A}^{i\times n} \longrightarrow \mathbb{A}^{p+i^2}$$

which is defined as follows:

For $(x, a'') \in (\mathbb{A}^n \times \mathbb{A}^{(n-p-i)\times n})_m$ and any complex $(i \times n)$–matrix $a''' := [a_{k,l}]_{\substack{n-p-i<k\leq n-p \\ 1\leq l\leq n}}$ that contains for $1 \leq j \leq i^2$ at the slots indicated by the indices $(k_j, 1), \ldots, (k_j, n-i), (1, l_j), \ldots, (n-p-i, l_j), (k_j, l_j)$ the entries of the point $a_j := (a_{k_j,1}, \ldots, a_{k_j,n-i}, a_{1,l_j}, \ldots, a_{n-p-i,l_j}, a_{k_j,l_j})$ of $\mathbb{A}^{2(n-i)-p+1}$, the map $\widetilde{\Phi}$ takes the value

$$\widetilde{\Phi}(x, a'', a''') := \widetilde{\Phi}(x, a'', a_1, \ldots, a_{i^2}) := \\ (F_1(x), \ldots, F_p(x), M_{k_1,l_1}(x, a'', a_1), \ldots, M_{k_{i^2},l_{i^2}}(x, a'', a_{i^2})).$$

Similarly as before we denote for a fixed full rank matrix $a'' \in \mathbb{A}^{(n-p-i)\times n}$ by

$$\widetilde{\Phi}_{a''} : \mathbb{A}^n_{m(X,a'')} \times \mathbb{A}^{i\times n} \longrightarrow \mathbb{A}^{p+i^2}$$

the polynomial map of smooth varieties induced by $\widetilde{\Phi}$ when we specialize the second argument to the value a''. Then we conclude again that for any $a'' \in \mathbb{A}^{(n-p-i)\times n}$ the point $0 \in \mathbb{A}^{p+i^2}$ is a regular value of the polynomial map $\widetilde{\Phi}_{a''}$ and that there exists a non–empty Zariski open subset $\widetilde{U}$ of $\mathbb{A}^{(n-p)\times n}$ satisfying the following condition:

For any pair consisting of a complex $((n-i)\times n)$–matrix a'' and a complex $(i\times n)$–matrix a''' such that (a'',a''') belongs to $\widetilde{U}$ and such that a''' contains at the slots indicated by the indices $(k_j,1),\ldots,(k_j,n-i),(1,l_j),\ldots,(n-p-i,l_j),(k_j,l_j)$, $1\le j\le i^2$, the entries of suitable points $a_1,\ldots,a_{i^2}$ of $\mathbb{A}^{2(n-i)-p+1}$, the equations

$$
(5)\quad F_1(X)=\cdots=F_p(X)=0,\qquad M_{k_1,l_1}(X,a'',a_1)=\cdots=M_{k_{n-p},l_{n-p}}(X,a'',a_{i^2})=0
$$

intersect transversally at any of their common solutions in $\mathbb{A}^n_{m(X,a'')}$.

Replacing in the previous argumentation the set U by $\widetilde{U}$ we define now $U^{(i)}_{classic}$ in the same way as before.

The construction of a filtration of $\mathbb{A}^{(n-p)\times n}$ by non-empty Zariski open subsets $U^{(i)}_{dual}$, $1\le j\le n-p$, follows the same line of reasoning, where the $n\times n-$ matrix N has to be replaced by

$$
\begin{bmatrix} & J(F_1,\ldots,F_p) & \\ A_{1,1}-X_1 & \cdots & A_{1,n}-X_n \\ \vdots & \vdots & \vdots \\ A_{n-p,1}-X_1 & \cdots & A_{n-p,n}-X_n \end{bmatrix}.
$$

One has only to take care to add in the construction of the sets $U^{(i)}_{dual}$, $1\le i\le n-p$, suitable Zariski open conditions for the minors of the $((n-p)\times n)-$ matrix

$$
[A_{k,l}]_{\substack{1\le k\le n-p\\ 1\le l\le n}}.
$$

For the rest of this section we fix a complex $((n-p)\times n)$–matrix $[a_{k,l}]_{\substack{1\le k\le n-p\\ 1\le l\le n}}$ belonging to the genericity condition $U_{classic}\cap U_{dual}$.

For $1\le i\le n-p,\ \ 1\le k\le n-p-i+1,\ \ 0\le l\le n$ we introduce the following notations:

$$
\underline{a}^{(i)}_{k,l}:=0 \ \text{ and } \ \overline{a}^{(i)}_{k,l}:=1 \ \text{ if } \ l=0,\ \ \underline{a}^{(i)}_{k,l}=\overline{a}^{(i)}_{k,l}:=a_{k,l} \ \text{ if } \ 1\le l\le n,
$$

$$
\underline{a}^{(i)}:=[\underline{a}^{(i)}_{k,l}]_{\substack{1\le k\le n-p-i+1\\ 0\le l\le n}} \ \text{ and } \ \overline{a}^{(i)}:=[\overline{a}^{(i)}_{k,l}]_{\substack{1\le k\le n-p-i+1\\ 0\le l\le n}}
$$

(thus we have in terms of the notation of Section 1 the identity ${\underline{a}^{(i)}}_*={\overline{a}^{(i)}}_*=[a_{k,l}]_{\substack{1\le k\le n-p-i+1\\ 1\le l\le n}}$).

Since $[a_{k,l}]_{\substack{1\le k\le n-p-i+1\\ 1\le l\le n}}$ belongs by assumption to the non–empty Zariski open subset $U_{classic}\cap U_{dual}$ of $\mathbb{A}^{(n-p)\times n}$, we shall consider for $1\le i\le n-p$ the matrices $\underline{a}^{(i)}$ and $\overline{a}^{(i)}$ and the corresponding classic and dual polar varieties

$$
W_{\underline{K}(\underline{a}^{(i)})}(S)=W_{\underline{K}^{n-p-i}(\underline{a}^{(i)})}(S) \ \text{ and } \ W_{\overline{K}(\overline{a}^{(i)})}(S)=W_{\overline{K}^{n-p-i}(\overline{a}^{(i)})}(S)
$$

as "generic". These polar varieties are organized in two nested sequences

$$
W_{\underline{K}(\underline{a}^{(n-p)})}(S)\subset\cdots\subset W_{\underline{K}(\underline{a}^{(1)})}(S)\subset S\subset\mathbb{A}^n
$$

and

$$W_{\overline{K}(\overline{a}^{(n-p)})}(S) \subset \cdots \subset W_{\overline{K}(\overline{a}^{(1)})}(S) \subset S \subset \mathbb{A}^n.$$

Since by construction $U_{classic}$ satisfies the conditions (1), (2), (3), (4), (5) and U_{dual} behaves mutatis mutandis in the same way, we conclude that for $1 \leq i \leq n-p$ the polar varieties $W_{\underline{K}(\underline{a}^{(i)})}(S)$ and $W_{\overline{K}(\overline{a}^{(i)})}(S)$ are also generic in the sense of [5] and [6].

Let $1 \leq i < n-p$ and $1 \leq h \leq n-p-i+1$. We denote by $\underline{a}^{(i,h)}$ and $\overline{a}^{(i,h)}$ the $((n-p-i)\times(n+1))$–matrices obtained by from $\underline{a}^{(i)}$ and $\overline{a}^{(i)}$ by deleting their row number h, namely

$$\underline{a}^{(i,h)} := [\underline{a}^{(i)}_{k,l}]_{\substack{1\leq k\leq n-p-i+1\\ k\neq h\\ 0\leq l\leq n}} \quad \text{and} \quad \overline{a}^{(i,h)} := [\overline{a}^{(i)}_{k,l}]_{\substack{1\leq k\leq n-p-i+1\\ k\neq h\\ 0\leq l\leq n}}.$$

Thus we have, in particular,

$$\underline{a}^{(i+1)} = \underline{a}^{(i,n-p-i+1)} \ \text{ and } \ \overline{a}^{(i+1)} = \overline{a}^{(i,n-p-i+1)}.$$

According to the notations introduced in [5] and [6] we write

$$\underline{\Delta}_i := \bigcap_{1\leq h\leq n-p-i+1} W_{\underline{K}(\underline{a}^{(i,h)})}(S) \ \text{ and } \ \overline{\Delta}_i := \bigcap_{1\leq h\leq n-p-i+1} W_{\overline{K}(\overline{a}^{(i,h)})}(S).$$

From [6] Proposition 6 and the subsequent commentaries we conclude that the singular loci of $W_{\underline{K}(\underline{a}^{(i)})}(S)$ and $W_{\overline{K}(\overline{a}^{(i)})}(S)$ are contained in $\underline{\Delta}_i$ and $\overline{\Delta}_i$, respectively.

The crucial point of the construction of the genericity conditions $U_{classic}$ and U_{dual} may be summarized in the following statement.

Lemma 5
Let $1 \leq i < n-p$. Then $\underline{\Delta}_i$ and $\overline{\Delta}_i$ are empty or closed subvarieties of $W_{\underline{K}(\underline{a}^{(i)})}(S)$ and $W_{\overline{K}(\overline{a}^{(i)})}(S)$ of codimension ≥ 2, respectively.

Proof
For the sake of simplicity of exposition, we restrict our attention to $\underline{\Delta}_i$. The case of $\overline{\Delta}_i$ can be treated in the same way.

Let us first assume $i = n-p-1$. Suppose that $\underline{\Delta}_{n-p-1}$ is non–empty and consider an arbitrary point $x \in \underline{\Delta}_{n-p-1}$. Since S is smooth, there is a p–minor m of $J(F_1,\ldots,F_p)$, say

$$m := \det \begin{bmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_p} \\ \vdots & \cdots & \vdots \\ \frac{\partial F_p}{\partial X_1} & \cdots & \frac{\partial F_p}{\partial X_p} \end{bmatrix}$$

with $m(x) \neq 0$. Since x belongs to $\underline{\Delta}_{n-p-1}$, we have

$$F_1(x) = 0\ ,\ldots, F_p(x) = 0$$

and

$$M_{1,p+1}(x, a^{(n-p-1,2)}) = 0\ ,\ldots, M_{1,n}(x, a^{(n-p-1,2)}) = 0, \quad M_{2,p+1}(x, a^{(n-p-1,1)}) = 0,$$

in contradiction to the conditions (1), (2) satisfied by $U_{classic}$. Therefore $\underline{\Delta}_{n-p-1}$ must be empty. This proves Lemma 5 in case $i := n-p-1$.

Now suppose $1 \leq i < n-p-1$ and that $\underline{\Delta}_{i+1}$ is either empty or of codimension ≥ 2 in $W_{\underline{K}(\underline{a}^{(i+1)})}(S)$. Let

$$N_i := \begin{bmatrix} & J(F_1,\dots,F_p) & \\ a_{1,1} & \cdots & a_{1,n} \\ \vdots & \vdots & \vdots \\ a_{n-p-i,1} & \cdots & a_{n-p-i,n} \end{bmatrix}.$$

If $\underline{\Delta}_i$ is empty we are done. Otherwise consider an arbitrary irreducible component C of $\underline{\Delta}_i$. Assume first that for any $((n-i-1)\times(n-i-1))$– submatrix of N_i, which contains $n-i-1$ entries from each of the rows number $1,\dots,p$ of N_i, the determinant vanishes identically on C. This implies that C is contained in $\underline{\Delta}_{i+1}$. From our assumptions on $\underline{\Delta}_{i+1}$ we deduce that C must be of codimension ≥ 2 in $W_{\underline{K}(\underline{a}^{(i+1)})}(S)$ and hence in $W_{\underline{K}(\underline{a}^{(i)})}(S)$.

Therefore we may suppose without loss of generality that there exists a $((n-i-1)\times(n-i-1))$–submatrix of N_i containing $n-i-1$ entries from each of the rows number $1,\dots,p$ of N_i, whose determinant, say m, does not vanish identically on C.

For the sake of conciseness we shall assume

$$m := \det \begin{bmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_{n-i-1}} \\ \vdots & \cdots & \vdots \\ \frac{\partial F_p}{\partial X_1} & \cdots & \frac{\partial F_p}{\partial X_{n-i-1}} \\ a_{1,1} & \cdots & a_{1,n-i-1} \\ \vdots & \cdots & \vdots \\ a_{n-p-i-1,1} & \cdots & a_{n-p-i-1,n-i-1} \end{bmatrix}.$$

Since C is contained in $\underline{\Delta}_i$, any point of C is a zero of the equation system

$$\begin{aligned} (6)\quad & F_1(X)=0,\dots,F_p(X)=0, \\ & M_{n-p-i,n-i}(X,a^{(i,n-p-i+1)})=0\ ,\dots,M_{n-p-i,n}(X,a^{(i,n-p-i+1)})=0, \\ & \qquad M_{n-p-i+1,n-i}(X,a^{(i,n-p-i)})\ ,\dots,M_{n-p-i+1,n}(X,a^{(i,n-p-i)})=0. \end{aligned}$$

Since $U_{classic}$ satisfies the conditions (3), (4) and (5), we conclude that the equations of (6) intersect transversally at any point of the (non–empty) set C_m. This implies $\dim C \leq n-p-2(i+1)$.

On the other hand we deduce from [5], [6], Proposition 8 that $\dim W_{\underline{K}(\underline{a}^{(i)})}(S) = n-p-i$ holds. Since C is a closed irreducible subvariety of $W_{\underline{K}(\underline{a}^{(i)})}(S)$, we infer that the codimension of C in $W_{\underline{K}(\underline{a}^{(i)})}(S)$ is at least $(n-p-i)-(n-p-2(i+1)) = i+2 \geq 2$. □

From Lemma 5 we draw now the following conclusion.

Theorem 6

Suppose that S is smooth and let $1 \leq i \leq n-p$. Then the generic polar varieties $W_{\underline{K}(\underline{a}^{(i)})}(S)$ and $W_{\overline{K}(\overline{a}^{(i)})}(S)$ are empty or normal Cohen–Macaulay subvarieties of S of pure codimension i.

Proof
We may restrict our attention to $W_{\underline{K}(\underline{a}^{(i)})}(S)$. The case of $W_{\overline{K}(\overline{a}^{(i)})}(S)$ can be treated in a similar way.

From [5], Theorem 9 and Corollary 10 we deduce that $W_{\underline{K}(\underline{a}^{(i)})}(S)$ is empty or a Cohen–Macaulay subvariety of S of codimension i.

We are now going to show that $W_{\underline{K}(\underline{a}^{(i)})}(S)$ is normal.

From [5], [6], Lemma 7 we deduce immediately that $W_{\underline{K}(\underline{a}^{(n-p)})}(S)$ is empty or 0–dimensional and hence normal (recall from Section 1 that $W_{\underline{K}(\underline{a}^{(n-p)})}(S)$ is defined set–theoretically and that its coordinate ring is therefore reduced).

Therefore we may assume without loss of generality $1 \leq i < n-p$ and $W_{\underline{K}(\underline{a}^{(i)})}(S) \neq \emptyset$. As observed above, the singular locus of $W_{\underline{K}(\underline{a}^{(i)})}(S)$ is contained in $\underline{\Delta}_i$ and hence, by Lemma 5, empty or of codimension ≥ 2 in $W_{\underline{K}(\underline{a}^{(i)})}(S)$. Consequently, $W_{\underline{K}(\underline{a}^{(i)})}(S)$ is regular in codimension one.

Since $W_{\underline{K}(\underline{a}^{(i)})}(S)$ is Cohen–Macaulay, we infer from Serre's normality criterion (see e.g. [25], Theorem 23.8) that $W_{\underline{K}(\underline{a}^{(i)})}(S)$ is a normal variety. $\square$

Observation 7
From [5], Theorem 9 one deduces that the defining ideal of $W_{\underline{K}(\underline{a}^{(i)})}(S)$ in $\mathbb{Q}[X]$ is generated by $F_1, \ldots, F_p$ and all $(n-i+1)$–minors of the polynomial matrix $((n-i+1) \times n)$–matrix

$$\begin{bmatrix} & J(F_1, \ldots, F_p) & \\ a_{1,1} & \cdots & a_{1,n} \\ \vdots & \vdots & \vdots \\ a_{n-p-i+1,1} & \cdots & a_{n-p-i+1,n} \end{bmatrix}.$$

For the rest of this section let us drop the assumption that the variety S is smooth. Then Theorem 6 may be reformulated as follows.

Corollary 8
Let $1 \leq i \leq n-p$. Then the generic polar varieties $W_{\underline{K}(\underline{a}^{(i)})}(S)$ and $W_{\overline{K}(\overline{a}^{(i)})}(S)$ are empty or subvarieties of S of pure codimension i that are normal and Cohen–Macaulay at any point where $J(F_1, \ldots, F_p)$ has maximal rank.

Proof
Let m be an arbitrary p–minor of the Jacobian $J(F_1, \ldots, F_p)$. The proof of Theorem 6 relies only on the assumption that S is smooth. Its correctness does not require that S is closed. Therefore the statement of Theorem 6 remains mutatis mutandis correct if we replace S by S_m. Therefore $S_m \neq \emptyset$ implies that

$$(W_{\underline{K}(\underline{a}^{(i)})}(S))_m = W_{\underline{K}(\underline{a}^{(i)})}(S_m) \text{ and } (W_{\overline{K}(\overline{a}^{(i)})}(S))_m = W_{\overline{K}(\overline{a}^{(i)})}(S_m)$$

are empty or normal Cohen–Macaulay subvarieties of S_m of codimension i. Since m was an arbitary p–minor of $J(F_1, \ldots, F_p)$ the assertion of Corollary 8 follows immediately. $\square$

Here the following geometric comment is at order. Schubert varieties are normal and Cohen–Macaulay, whereas classic polar varieties may be interpreted as pre–images of Schubert varieties under suitable polynomial maps (see [28], proof of Lemma 1.3 (ii)). This fact suggests the statements of Theorem 6 and Corollary 8 and an alternative proof of them. In the dual case these statements are new.

We deduced for $a \in \mathbb{A}^{(n-p)\times n}$ generic and $1 \leq i \leq n-p$ the normality of the polar varieties $W_{\underline{K}(\underline{a}^{(i)})}(S)$ and $W_{\overline{K}(\overline{a}^{(i)})}(S)$ of S from the consideration of the Zariski closed subsets $\underline{\Delta}_i$ and $\overline{\Delta}_i$ of $\mathbb{A}^n$ which contain the singular locus of $W_{\underline{K}(\underline{a}^{(i)})}(S)$ and $W_{\overline{K}(\overline{a}^{(i)})}(S)$. The main tool was Lemma 5 which in its turn relied on the elementary, but rather tedious construction of the non–empty Zariski open subsets $\mathcal{U}_{classic}$ and $\mathcal{U}_{dual}$ of $\mathbb{A}^{(n-p)\times n}$. It is necessary to invest some work in proofs because $\underline{\Delta}_i$ is, unlike the polar varieties, not determinantal.

General statements describing the singular locus of a determinantal variety as another determinantal variety (see e.g. [10]) cannot be applied in a straightforward manner to polar varieties. The problem arises from the fact that this property of determinantal varieties is generally not preserved under pre–images. Therefore a proof of the normality of the generic classic polar varieties along these lines would require a similar technical effort as above.

We discuss now under which conditions it may be guaranteed that a generic polar variety of S is smooth at any point where the $J(F_1, \ldots, F_p)$ has maximal rank.

Proposition 9
Let $1 \leq i \leq n-p$ with $2i+2 > n-p$. Then the generic polar varieties $W_{\underline{K}(\underline{a}^{(i)})}(S)$ and $W_{\overline{K}(\overline{a}^{(i)})}(S)$ are smooth at any point where $J(F_1, \ldots, F_p)$ has maximal rank.

Proof
Again we restrict our attention to $W_{\underline{K}(\underline{a}^{(i)})}(S)$. The case of $W_{\overline{K}(\overline{a}^{(i)})}(S)$ is treated similarly.

From [6], Proposition 6 and the subsequent commentaries we conclude that the points at which $W_{\underline{K}(\underline{a}^{(i)})}(S)$ is singular and $J(F_1, \ldots, F_p)$ has maximal rank are contained in $\underline{\Delta}_i$ and from the proof of Lemma 5 we deduce that the codimension of $\underline{\Delta}_i$ in $W_{\underline{K}(\underline{a}^{(i)})}(S)$ is at least $i+2$. This implies that $\underline{\Delta}_i$ has codimension at least $2i+2$ in S. Since S is of codimension p in $\mathbb{A}^n$, we infer from the assumption $2i+2 > n-p$ that $\underline{\Delta}_i$ is empty. Therefore $W_{\underline{K}(\underline{a}^{(i)})}(S)$ is smooth at any point where $J(F_1, \ldots, F_p)$ has maximal rank. $\square$

Proposition 9 says that the lower dimensional generic polar varieties of S (e.g. generic polar curves in case $p < n-1$, generic polar surfaces in case $p < n-2$ etc.) are empty or smooth at any point where $J(F_1, \ldots, F_p)$ has maximal rank.

We are now going to show by an infinite family of examples that this conclusion is not always true for higher dimensional generic polar varieties.

3.1 A family of singular generic polar varieties

Let

$$n \geq 6, \quad c := \begin{bmatrix} c_{1,1} & \cdots & c_{1,n} \\ c_{2,1} & \cdots & c_{2,n} \end{bmatrix} \in \mathbb{A}_{\mathbb{R}}^{2\times n}, \quad a := \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \cdots & \vdots \\ a_{n-2,1} & \cdots & a_{n-2,n} \end{bmatrix} \in \mathbb{A}_{\mathbb{R}}^{(n-2)\times n}$$

such that the composed $(n \times n)$–matrix $\begin{bmatrix} c \\ a \end{bmatrix}$ represents a generic choice in $\mathbb{A}^{n\times n}$ which will be specified later. For the moment it suffices to suppose that $\begin{bmatrix} c \\ a \end{bmatrix}$ and all (2×2)–submatrices of c are regular and that all entries of c are non–zero.

Let

$$E_n := \{(x_1, \ldots, x_n) \in \mathbb{A}_{\mathbb{R}}^n \mid rk \begin{bmatrix} c_{1,1}x_1 & \cdots & c_{1,n}x_n \\ & a & \end{bmatrix} = rk \begin{bmatrix} c_{2,1}x_1 & \cdots & c_{2,n}x_n \\ & a & \end{bmatrix} = n-2\}$$

and observe that E_n is a linear subspace of $\mathbb{A}_{\mathbb{R}}^n$ of dimension at least $2(n-2)-n = n-4 \geq 2$ (here rk denotes the matrix rank).

Therefore we may assume without loss of generality that there exists an element $\xi = (\xi_1, \ldots, \xi_n)$ of E_n with $\xi_1 \neq 0$ and $\xi_2 \neq 0$. Let

$$c_1 := c_{1,1}\xi_1^2 + \cdots + c_{1,n}\xi_n^2 \ \text{ and } \ c_2 := c_{2,1}\xi_1^2 + \cdots + c_{2,n}\xi_n^2,$$

$$F_1^{(n)} := c_{1,1}X_1^2 + \cdots + c_{1,n}X_n^2 - c_1, \quad F_2^{(n)} := c_{2,1}X_1^2 + \cdots + c_{2,n}X_n^2 - c_2$$

and

$$S^{(n)} := \{F_1^{(n)} = F_2^{(n)} = 0\}.$$

For the sake of notational simplicity we shall write $E := E_n$, $F_1 := F_1^{(n)}$, $F_2 := F_2^{(n)}$ and $S := S^{(n)}$.

By construction ξ belongs to $S_{\mathbb{R}}$, which implies that $S_{\mathbb{R}}$, and hence S, is non–empty.

Observe that any irreducible component of the complex variety S has dimension at least $n-2 \geq 4$.

Let D be an arbitrary irreducible component of S. Thus we have $\dim D \geq 4$. We claim that D contains a point $x = (x_1, \ldots, x_n)$ such that there exists two indices $1 \leq u < v \leq n$ with $x_u \neq 0$ and $x_v \neq 0$. Otherwise, for any $x = (x_1, \ldots, x_n)$ of D there exists an index $1 \leq j \leq n$ with $x_1 = \cdots = x_{j-1} = x_{j+1} = \cdots = x_n = 0$. From $0 = F_1(x) = c_{1,j}x_j^2 - c_1$ and $c_{1,j} \neq 0$ we deduce that S, and therefore D, contains only finitely many such points. This implies $\dim D = 0$ in contradiction to $\dim D \geq 4$.

Therefore there exists a point $x = (x_1, \ldots, x_n)$ of D and indices $1 \leq u < v \leq n$ with $x_u \neq 0$ and $x_v \neq 0$. From

$$J(F_1,\ F_2) = 2 \begin{bmatrix} c_{1,1}X_1 & \cdots & c_{1,n}X_n \\ c_{2,1}X_1 & \cdots & c_{2,n}X_n \end{bmatrix}$$

and the "genericity" of c we deduce that

$$\det \begin{bmatrix} \frac{\partial F_1}{\partial X_u}(x) & \frac{\partial F_1}{\partial X_v}(x) \\ \frac{\partial F_2}{\partial X_u}(x) & \frac{\partial F_2}{\partial X_v}(x) \end{bmatrix} = 4 \det \begin{bmatrix} c_{1,u}x_u & c_{1,v}x_v \\ c_{2,u}x_u & c_{2,v}x_v \end{bmatrix} = 4\, x_u x_v \det \begin{bmatrix} c_{1,u} & c_{1,v} \\ c_{2,u} & c_{2,v} \end{bmatrix} \neq 0$$

holds. Thus the Jacobian $J(F_1,\, F_2)$ has maximal rank two at the point x. Since D was an arbitrary irreducible component of S, we conclude that any irreducible component of S contains a point where $J(F_1, F_2)$ has maximal rank. This implies that the defining polynomials $F_1,\, F_2$ of S form a reduced regular sequence in $\mathbb{Q}[X]$. In particular, S is of pure codimension two in $\mathbb{A}^n$. Let

$$N_*(X) := \begin{bmatrix} J(F_1,\, F_2) \\ a \end{bmatrix} = \begin{bmatrix} 2c_{1,1}X_1 & \cdots & 2c_{1,n}X_n \\ 2c_{2,1}X_1 & \cdots & 2c_{2,n}X_n \\ & a & \end{bmatrix}.$$

By construction ξ belongs to E and satisfies the conditions $F_1(\xi) = F_2(\xi) = 0$. Therefore the real $((n-1) \times n)$–matrices

$$\begin{bmatrix} c_{1,1}\xi_1 & \cdots & c_{1,n}\xi_n \\ & a & \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} c_{2,1}\xi_1 & \cdots & c_{2,n}\xi_n \\ & a & \end{bmatrix} \quad \text{have rank } n-2.$$

Moreover, the $((n-2)\times n)$–matrix a has rank $n-2$. This implies that the vectors $(c_{1,1}\xi_1, \ldots, c_{1,n}\xi_n)$ and $(c_{2,1}\xi_1, \ldots, c_{2,n}\xi_n)$ belong to the row span of a. Hence the real $(n \times n)$–matrix $N_*(\xi)$ has rank $n-2$. Thus we have $\det N_*(\xi) = 0$.

Suppose for the moment that $a \in \mathbb{A}^{(n-2)\times n}$ and $\xi \in E$ were chosen in such a way that $W_{\underline{K}^{n-3}(\underline{a})}(S)$ is, with respect to the properties of polar varieties treated here, generic in the sense of Definition 1. From Observation 7 we deduce then that the polynomials $F_1,\, F_2$ and $\det N_*$ generate the ideal of definition of the polar variety $W_{\underline{K}^{n-3}(\underline{a})}(S)$. Hence ξ belongs to the polar variety $W_{\underline{K}^{n-3}(\underline{a})}(S)$ which therefore turns out to be non–empty and of pure codimension three in $\mathbb{A}^n$. By the way, we see that $F_1,\, F_2$ and $\det N_*$ form a reduced regular sequence in $\mathbb{Q}[X]$.

In the same vein as before we deduce from $\xi_1 \neq 0$ and $\xi_2 \neq 0$ that ξ is a regular point of S. For $1 \leq u \leq 2$ and $1 \leq l \leq n$ we denote by $(-1)^{u+l}m_{u,l}$ the $(n-1)$–minor of $\left[\begin{smallmatrix} J(F_1,\, F_2) \\ a \end{smallmatrix}\right]$ obtained by deleting row number u and column number l. From $\xi \in E$ we deduce $m_{u,l}(\xi) = 0$.

Let $1 \leq j \leq n$. From the identity

$$\frac{\partial}{\partial X_j} \det N_* = \sum_{1 \leq l \leq n} (m_{1,l}\, \frac{\partial^2 F_2}{\partial X_l\, \partial X_j} + m_{2,l}\, \frac{\partial^2 F_1}{\partial X_l\, \partial X_j}) = 2(c_{2,j}m_{1,j} + c_{1,j}m_{2,j})$$

we infer $(\frac{\partial}{\partial X_j} \det N_*)(\xi) = 0$. This implies

$$\left((\frac{\partial}{\partial X_1} \det N_*)(\xi), \ldots, (\frac{\partial}{\partial X_n} \det N_*)(\xi) \right) = (0, \ldots, 0).$$

Therefore the rank of the Jacobian $J(F_1,\, F_2, \det N_*)$ is two at the point $\xi \in W_{\underline{K}^{n-3}(\underline{a})}(S)$.

Since the polynomials F_1, F_2 and $\det N_*$ generate the ideal of definition of the polar variety $W_{\underline{K}^{n-3}(\underline{a})}(S)$, we conclude that $W_{\underline{K}^{n-3}(\underline{a})}(S)$ is singular at the point ξ.

On the other hand, we deduce from $\xi_1 \neq 0$ and $\xi_2 \neq 0$ that

$$\det \begin{bmatrix} \frac{\partial F_1}{\partial X_1}(\xi) & \frac{\partial F_1}{\partial X_2}(\xi) \\ \frac{\partial F_2}{\partial X_1}(\xi) & \frac{\partial F_2}{\partial X_2}(\xi) \end{bmatrix} = 4\,\xi_1\,\xi_2 \det \begin{bmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{bmatrix} \neq 0$$

holds. Thus the Jacobian $J(F_1, F_2)$ has maximal rank at ξ. In particular, there exists a point where the polar variety $W_{\underline{K}^{n-3}(\underline{a})}(S)$ is not smooth and where $J(F_1, F_2)$ has maximal rank.

We are now going to show that, for a suitable choice of ξ in E, the polar variety $W_{\underline{K}^{n-3}(\underline{a})}(S)$ is generic.

For this purpose let

$$C = \begin{bmatrix} C_{1,1} & \cdots & C_{1,n} \\ C_{2,1} & \cdots & C_{2,n} \end{bmatrix}$$

and (C_1, C_2) be a $(2 \times n)$–matrix and a pair of new indeterminates. Further, let

$$A := \begin{bmatrix} A_{1,1} & \cdots & A_{1,n} \\ & \cdots & \\ A_{n-2,1} & \cdots & A_{n-2,n} \end{bmatrix}.$$

Observe that S represents not a fixed algebraic variety but rather a constructible *family* of examples that depend on the choice of the $(n \times n)$–matrix $\begin{bmatrix} c \\ a \end{bmatrix}$ and the point $\xi \in E$ (recall that these data determine the values c_1 and c_2). To this family corresponds a genericity condition in the sense of Definition 1 which takes into account the properties of polar varieties we treat in this paper. This genericity condition may be expressed by the non-vanishing of a suitable non–zero polynomial $G \in \mathbb{Q}[A, C, C_1, C_2]$. The "generic choice" of the real $(n \times n)$–matrix $\begin{bmatrix} c \\ a \end{bmatrix}$ means now that the bivariate polynomial $G(a, c, C_1, C_2)$ is non–zero and that all these matrices satisfy the requirements formulated at the beginning of this subsection. For the genericity of the polar variety $W_{\underline{K}^{n-3}(\underline{a})}(S)$ it suffices therefore to prove that the point $\xi = (\xi_1, \ldots, \xi_n)$ may be chosen in E in such a way that

$$c_1 := c_{1,1}\xi_1^2 + \cdots + c_{1,n}\xi_n^2 \quad \text{and} \quad c_2 := c_{2,1}\xi_1^2 + \cdots + c_{2,n}\xi_n^2$$

satisfy the condition $G(a, c, c_1, c_2) \neq 0$ (observe that E depends only on the $(n \times n)$–matrix $\begin{bmatrix} c \\ a \end{bmatrix}$ which we consider now as fixed). To this end recall that $\dim E \geq 2$ holds and consider the polynomial map of affine spaces $\varphi : E \to \mathbb{A}^2$ defined for $\xi = (\xi_1, \ldots, \xi_n)$ in E by

$$\varphi(\xi) := (c_{1,1}\xi_1^2 + \cdots + c_{1,n}\xi_n^2,\ c_{2,1}\xi_1^2 + \cdots + c_{2,n}\xi_n^2).$$

It follows from our previous argumentation that for $\xi_1 \neq 0$ and $\xi_2 \neq 0$ the smooth map φ is a submersion at ξ. Therefore the image of φ is Zariski dense in $\mathbb{A}^2$. This implies that there exists a point $\xi = (\xi_1, \ldots, \xi_n)$ with $\xi_1 \neq 0$ and

$\xi_2 \neq 0$ in E such that the bivariate polynomial $G(a, c, C_1, C_2)$ does not vanish at $(c_1, c_2) := \varphi(\xi)$. For such a choice of ξ in E, the polar variety $W_{\underline{K}^{n-3}(\underline{a})}(S)$ turns out to be generic in the sense stated before.

In particular, we may suppose without loss of generality that

$$c := \begin{bmatrix} c_{1,1} & \cdots & c_{1,n} \\ c_{2,1} & \cdots & c_{2,n} \end{bmatrix} \in \mathbb{A}_{\mathbb{R}}^{2\times n} \text{ and } (c_1, c_2) \in \mathbb{A}_{\mathbb{R}}^2$$

were chosen in such a way that all entries of c and c_1, c_2 are non–zero, that all (2×2)–submatrices of c are regular and that, for any $1 \leq j \leq n$, the values $\frac{c_1}{c_{1,j}}$ and $\frac{c_2}{c_{2,j}}$ are distinct.

Under this restriction the affine variety S is empty or smooth.

Assume that S contains a singular point $x = (x_1, \ldots, x_n)$. From our preceding argumentation, we deduce that there exists an index $1 \leq k \leq n$ such that $x_j = 0$ holds for any $1 \leq j \neq k \leq n$. This implies $c_{1,k}x_k^2 = c_1$ and $c_{2,k}x_k^2 = c_2$ and therefore $\frac{c_1}{c_{1,k}} = \frac{c_2}{c_{2,k}}$ which contradicts our genericity condition.

Therefore we have constructed for each $n \geq 6$ a smooth complete intersection variety $S^{(n)}$ of pure codimension two and a generic polar variety $W_{\underline{K}^{n-3}(\underline{a})}(S^{(n)})$ of $S^{(n)}$ which contains a singular point.

Observe that our construction may easily be modified in order to produce in the case $p := 1$ *singular* generic *dual* polar varieties of regular hypersurfaces. The behaviour of generic *classic* polar varieties is different in this case: they are always empty or smooth at regular points of the given hypersurface.

Our family of examples certifies that the *generic* higher dimensional polar varieties of a smooth complete intersection variety S may become singular. This contradicts the statement Theorem 10 (i) of [6].

On the other hand we shall meet in Section 4 three families of so–called *meagerly generic* polar varieties which are smooth and which therefore satisfy the conclusion of [6] Theorem 10 (i).

Thus we face a quite puzzling situation, where the most general objects under consideration, namely the generic polar varieties, may become singular whereas the more special meagerly generic polar varieties may turn out to be smooth.

3.2 Desingularizing generic polar varieties

For the rest of this section we suppose that S is a smooth variety. We are now going to try to explain in a more systematic way how singularities in higher dimensional generic polar varieties of S may arise.

For expository reasons we limit our attention to the case of the classic polar varieties of S. Fix $1 \leq i \leq n-p$ and consider the locally closed algebraic variety $\mathcal{H}_i$

of $\mathbb{A}^n \times \mathbb{A}^{(n-p-i+1)\times n} \times \mathbb{P}^{n-i}$ defined by

$$\begin{aligned}\mathcal{H}_i := \{(x, a, (\lambda : \vartheta)) \in \mathbb{A}^n \times \mathbb{A}^{(n-p-i+1)\times n} \times \mathbb{P}^{n-i} \mid x \in S_{reg},\, rk\ a = n-p-i+1,\\ \lambda = (\lambda_1, \dots, \lambda_p) \in \mathbb{A}^p,\, \vartheta = (\vartheta_1, \dots, \vartheta_{n-p-i+1}) \in \mathbb{A}^{n-p-i+1},\\ (\lambda, \vartheta) \neq 0,\ J(F_1, \dots, F_p)(x)^T \cdot \lambda^T + a^T \cdot \vartheta^T = 0\},\end{aligned}$$

where $(\lambda : \vartheta) := (\lambda_1 : \dots : \lambda_p : \vartheta_1 : \dots : \vartheta_{n-p-i+1})$ belongs to $\mathbb{P}^{n-i}$, while $(\lambda, \vartheta) \in \mathbb{A}^{n-i+1}$ is its affine counterpart and $J(F_1, \dots, F_p)(x)^T$ denotes the transposed matrix of $J(F_1, \dots, F_p)(x)$, etc.

Geometrically $\mathcal{H}_i$ may be interpreted as an incidence variety whose projection to the space $\mathbb{A}^n \times \mathbb{A}^{(n-p-i+1)\times n}$ describes the locally closed variety $\mathcal{W}_i$ of all pairs (x, a) with $x \in S_{reg}$ and a being a full–rank matrix of $\mathbb{A}^{(n-p-i+1)\times n}$ such that x belongs to $W_{\underline{K}(\underline{a})}(S)$. Below we shall see that $\mathcal{H}_i$ is non–empty, smooth and equidimensional. Thus $\mathcal{H}_i$ is a natural desingularisation of the variety $\mathcal{W}_i$ in the sense of Room–Kempf ([30, 21]).

Consider now an arbitrary point $(x, a, (\lambda : \vartheta))$ of $\mathcal{H}_i$ with $a = [a_{k,l}]_{\substack{1\leq k\leq n-p-i+1\\ 1\leq l\leq n}}$, $\lambda = (\lambda_1, \dots, \lambda_p) \in \mathbb{A}^p$ and $\vartheta = (\vartheta_1, \dots, \vartheta_{n-p-i+1}) \in \mathbb{A}^{n-p-i+1}$. Thus we have $(\lambda, \vartheta) \neq 0$. We claim that $\vartheta \neq 0$ holds. Otherwise we would have $\vartheta = 0$ and $J(F_1, \dots, F_p)(x)^T \cdot \lambda^T = 0$. Since x is a regular point, this implies $\lambda = 0$ in contradiction to $(\lambda, \vartheta) \neq 0$.

Therefore we may assume without loss of generality $\vartheta_{n-p-i+1} = 1$. Let $\tilde{a} := [a_{k,l}]_{\substack{1\leq k\leq n-p-i\\ 1\leq l\leq n}}$ and denote for $1 \leq j \leq n$ by θ_j the complex $(n \times (n-p-i+1))$–matrix containing in row number j the entries $1, \vartheta_1, \dots, \vartheta_{n-p-i+1}$, whereas all other entries of θ_j are zero. From $rk\, J(F_1, \dots, F_p)(x) = p$ we deduce now that the complex $((n+p) \times (2n-i+(n-p-i+1)n))$–matrix

$$\begin{bmatrix} J(F_1, \dots, F_p)(x) & O_{p,p} & O_{p,n-p-i} & O_{p,n-p-i+1} & \cdots & O_{p,n-p-i+1} \\ * & J(F_1, \dots, F_p)(x)^T & \tilde{a}^T & \theta_1 & \cdots & \theta_n \end{bmatrix}$$

is of maximal rank $n+p$ (here $O_{p,p}$ denotes the $(p \times p)$–zero matrix, etc.). This implies that $\mathcal{H}_i$ is smooth and of dimension $n-p-i+(n-p-i+1)n$ at the point $(x, a, (\lambda : \vartheta))$.
In other words, the algebraic variety $\mathcal{H}_i$ is non–empty, smooth and equidimensional of dimension $n-p-i+(n-p-i+1)n$.

Let $\mu_i : \mathcal{H}_i \to \mathbb{A}^{(n-p-i+1)\times n}$ be the canonical projection of $\mathcal{H}_i$ in $\mathbb{A}^{(n-p-i+1)\times n}$ and suppose that for a generic choice of $a \in \mathbb{A}^{(n-p-i+1)\times n}$ the polar variety $W_{\underline{K}^{n-p-i}(\underline{a})}(S)$ is non–empty.

Observe that for any $x \in \mathbb{A}^n$ the following statements are equivalent

- $x \in W_{\underline{K}^{n-p-i}(\underline{a})}(S) \cap S_{reg}$;
- there exist $\lambda \in \mathbb{A}^p$ and $\vartheta \in \mathbb{A}^{n-p-i+1}$ with $(\lambda, \vartheta) \neq 0$ such that $x \in S_{reg}$ and $J(F_1, \dots, F_p)(x)^T \cdot \lambda^T + a^T \cdot \vartheta^T = 0$ holds;
- there exist $\lambda \in \mathbb{A}^p$ and $\vartheta \in \mathbb{A}^{n-p-i+1}$ with $(\lambda, \vartheta) \neq 0$ such that $(x, a, (\lambda : \vartheta))$ belongs to $\mu_i^{-1}(a)$.

Since by assumption $W_{\underline{K}^{n-p-i}(\underline{a})}(S)$ is non–empty for any generic choice $a \in \mathbb{A}^{(n-p-i+1)\times n}$, we conclude that $\mu_i^{-1}(a) \neq \emptyset$ holds. This implies that the image of the morphism of equidimensional algebraic varieties $\mu_i : \mathcal{H}_i \longrightarrow \mathbb{A}^{(n-p-i+1)\times n}$ is Zariski dense in $\mathbb{A}^{(n-p-i+1)\times n}$.

Let $a = [a_{k,l}]_{\substack{1\leq k\leq n-p-i+1\\ 1\leq l\leq n}}$ be a fixed generic choice in $\mathbb{A}^{(n-p-i+1)\times n}$ and let

$$H_i := \{(x,(\lambda:\vartheta)) \in \mathbb{A}^n \times \mathbb{P}^{n-i} \mid x \in S_{reg},\, \lambda \in \mathbb{A}^p,\, \vartheta \in \mathbb{A}^{n-p-i+1},$$
$$(\lambda,\vartheta) \neq 0,\; J(F_1,\dots,F_p)(x)^T \cdot \lambda^T + a^T \cdot \vartheta^T = 0\},$$

Since the morphism μ_i has in $\mathbb{A}^{(n-p-i+1)\times n}$ a Zariski dense image, a is chosen generically in $\mathbb{A}^{(n-p-i+1)\times n}$ and $\mathcal{H}_i$ is smooth and equidimensional of dimension $n-p-i+(n-p-i+1)n$, we deduce from the Weak Transversality Theorem of Thom–Sard that H_i is non–empty, smooth and equidimensional of dimension $n-p-i$.
Let $\omega_i : H_i \to \mathbb{A}^n$ be the canonical projection H_i in $\mathbb{A}^n$. From our previous reasoning we conclude that the image of ω_i is $W_{\underline{K}^{n-p-i}(\underline{a})}(S) \cap S_{reg}$. Since this algebraic variety is equidimensional and of the same dimension as H_i, namely $n-p-i$, we infer that there exists a Zariski dense open subset U of $W_{\underline{K}^{n-p-i}(\underline{a})}(S)$ such that the ω_i–fiber of any $x \in U$ is zero–dimensional.

Using the notations of Section 3 let us now consider an arbitrary point $x \in W_{\underline{K}^{n-p-i}(\underline{a})}(S) \cap S_{reg}$ which does not belong to $\underline{\Delta}_i$. Without loss of generality we may assume that

$$m := \det \begin{bmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_{n-i}} \\ \vdots & \cdots & \vdots \\ \frac{\partial F_p}{\partial X_1} & \cdots & \frac{\partial F_p}{\partial X_{n-i}} \\ a_{1,1} & \cdots & a_{1,n-i} \\ \vdots & \cdots & \vdots \\ a_{n-p-i,1} & \cdots & a_{n-p-i,n-i} \end{bmatrix}.$$

does not vanish at x.
Let $\lambda \in A^p$ and $\vartheta \in \mathbb{A}^{n-p-i+1}$ with $\vartheta = (\vartheta_1,\dots,\vartheta_{n-p-i+1})$ and $(\lambda,\vartheta) \neq 0$ such that $(x,(\lambda:\vartheta))$ belongs to $\omega_i^{-1}(x)$. We claim $\vartheta_{n-p-i+1} \neq 0$. Otherwise we would have for $\tilde{a} := [a_{k,l}]_{\substack{1\leq k\leq n-p-i\\ 1\leq l\leq n}}$ and $\tilde{\vartheta} = (\vartheta_1,\dots,\vartheta_{n-p-i})$ that $(\lambda,\tilde{\vartheta}) \neq 0$ and $J(F_1,\dots,F_p)(x)^T \cdot \lambda^T + \tilde{a}^T \cdot \tilde{\vartheta}^T = 0$ holds. This in turn would imply that the rank of the complex $((n-i)\times n)$–matrix $\begin{bmatrix} J(F_1,\dots,F_p)(x) \\ \tilde{a} \end{bmatrix}$ is at most $n-i-1$, whence $m(x) = 0$, a contradiction.

Therefore we may assume without loss of generality $\vartheta_{n-p-i+1} = 1$. Since the algebraic variety H_i is smooth at the point $(x,(\lambda:\vartheta))$, we conclude that the complex $((n+p)\times(2n-i))$–matrix

$$\begin{bmatrix} J(F_1,\dots,F_p)(x) & O_{p,p} & O_{p,n-p-i} \\ * & J(F_1,\dots,F_p)(x)^T & \tilde{a}^T \end{bmatrix}$$

is of maximal rank $n+p$. From $m(x) \neq 0$ we deduce that the submatrix $\begin{bmatrix} J(F_1,\dots,F_p)(x)^T & \tilde{a}^T \end{bmatrix}$ is of maximal rank $n-i$. Taking into account that

$x \in (W_{\underline{K}^{n-p-i}(\underline{a})}(S) \cap S_{reg}) \setminus \underline{\Delta}_i$ is a smooth point of $W_{\underline{K}^{n-p-i}(\underline{a})}(S) \cap S_{reg}$, we infer that there exists in the Euclidean topology a sufficiently small open neighborhood O of x in $W_{\underline{K}^{n-p-i}(\underline{a})}(S) \cap S_{reg}$ and a smooth map $\mu : O \to H_i$ with $\mu(x) = (x, (\lambda : \vartheta))$ and $\omega_i \circ \mu = id_O$, where id_O denotes the identity map on O. In particular, $\omega_i|_{\omega_i^{-1}(O)} : \omega_i^{-1}(O) \to U$ is a submersion at $(x, (\lambda : \vartheta))$. Translating this reasoning to the language of commutative algebra, we conclude that ω_i as morphism of algebraic varieties $H_i \to W_{\underline{K}^{n-p-i}(\underline{a})}(S)$ is unramified at any point of $\omega_i^{-1}(x)$. In particular, $\omega_i^{-1}(x)$ is a zero–dimensional algebraic variety.

Let us now consider an arbitrary point $x \in \underline{\Delta}_i \cap S_{reg}$. Then the complex $((n - i + 1) \times n)$–matrix $\begin{bmatrix} J(F_1, \dots, F_p)(x) \\ a \end{bmatrix}$ has rank at most $n - i - 1$. Hence the linear variety

$$\Gamma := \{(\lambda : \vartheta) \in \mathbb{P}^{n-i} \mid \lambda \in \mathbb{A}^p,\ \vartheta \in \mathbb{A}^{n-p-i+1},\ (\lambda, \vartheta) \neq 0,$$
$$J(F_1, \dots, F_p)(x)^T \cdot \lambda^T + a^T \cdot \vartheta^T = 0\},$$

which is isomorphic to $\omega_i^{-1}(x)$, has dimension at least one. Thus $\omega_i^{-1}(x)$ is not a zero–dimensional algebraic variety. In particular, if $W_{\underline{K}^{n-p-i}(\underline{a})}(S)$ is smooth at x, then for any sufficiently small open neighborhood O of x in $W_{\underline{K}^{n-p-i}(\underline{a})}(S)$ and any $\lambda \in \mathbb{A}^p$, $\vartheta \in \mathbb{A}^{n-p-i+1}$ with $(\lambda, \vartheta) \neq 0$ and $(x, (\lambda : \vartheta)) \in \omega_i^{-1}(x)$ the smooth map $\omega_i|_{\omega_i^{-1}(O)} : \omega_i^{-1}(O) \to O$ is not a submersion at $(x, (\lambda : \vartheta))$.

We may summarize this argumentation by the following statement.

Proposition 10
Let notations be as above and suppose that for $a \in \mathbb{A}^{(n-p-i+1)\times n}$ generic the polar variety $W_{\underline{K}^{n-p-i}(\underline{a})}(S)$ is not empty. Then we have for any regular point x of $W_{\underline{K}^{n-p-i}(\underline{a})}(S)$: x belongs to $\underline{\Delta}_i$ if and only if $\omega_i^{-1}(x)$ is a zero-dimensional variety. If this is the case, the morphism of algebraic varieties $\omega_i : H_i \to W_{\underline{K}^{n-p-i}(\underline{a})}(S)$ is unramified at any point of $\omega_i^{-1}(x)$.

The morphism of algebraic varieties $\omega_i : H_i \to W_{\underline{K}^{n-p-i}(\underline{a})}(S)$ represents a natural desingularization of the open generic polar variety $W_{\underline{K}^{n-p-i}(\underline{a})}(S) \cap S_{reg}$ in the spirit of Room–Kempf [30, 21]. Proposition 10 characterizes $\underline{\Delta}_i$ as a kind of an algebraic–geometric "discriminant locus" of the morphism ω_i which maps the smooth variety H_i to the possibly non–smooth generic polar variety $W_{\underline{K}^{n-p-i}(\underline{a})}(S)$.

We finish this section by a decomposition of the locus of the regular points of the generic classic polar varieties of S into smooth Thom–Boardman strata of suitable polynomial maps defined on S_{reg}. Our main tool will be Mather's theorem on generic projections [24] (see also [1]). Then we will discuss this decomposition in the light of Proposition 9.

Let us fix an index $1 \leq i \leq n - p$ and a complex $(n \times (n - p - i + 1))$–matrix $a \in \mathbb{A}^{(n-p-i+1)\times n}$ of rank $n - p - i + 1$. Then a induces a linear map $\Pi_a : \mathbb{A}^n \to \mathbb{A}^{n-p-i+1}$ which is for $x \in \mathbb{A}^n$ defined by $x\,a$. Let π_a be the restriction of this linear map to the variety S.

Let x be an arbitrary point of S. We denote by $T_x := T_x(S)$ the Zariski tangent space of S at x, by $d_x(\pi_a) : T_x \to \mathbb{A}^{n-p-i+1}$ the corresponding linear map of tangent spaces at x and $\pi_a(x)$, by $ker\, d_x(\pi_a)$ its kernel and by $\dim_{\mathbb{C}}(ker\, d_x(\pi_a))$ the dimension of the $\mathbb{C}$–vector space $ker\, d_x(\pi_a)$.

Furthermore, for $i \le j \le n-p$ we denote by

$$\Sigma^{(j)}(\pi_a) := \{x \in S_{reg} \mid \dim_{\mathbb{C}}(ker\, d_x(\pi_a)) = j\}$$

the corresponding Thom–Boardman stratum of π_a. Observe that $\Sigma^{(j)}(\pi_a)$ is a constructible subset of $\mathbb{A}^n$. With these notations we have the following result.

Lemma 11

(i) $W_{\underline{K(a)}}(S) \cap S_{reg} = \bigcup_{i \le j \le n-p} \Sigma^{(j)}(\pi_a)$.

(ii) $\underline{\Delta_i} \cap S_{reg} = \bigcup_{i+1 \le j \le n-p} \Sigma^{(j)}(\pi_a)$.

Proof

Let x be an arbitrary point of S_{reg} and let N_x and E_a be the $\mathbb{C}$–vector spaces generated by the rows of the matrices $J(F_1, \ldots, F_p)(x)$ and a, respectively. Thus we have $\dim_{\mathbb{C}}\, T_x = n-p$, $\dim_{\mathbb{C}}\, N_x = p$ and $\dim_{\mathbb{C}}\, E_a = n-p-i+1$.

Let $Y_1, \ldots, Y_n$ be new indeterminates. With respect to the bilinear form $X_1Y_1 + \cdots + X_nY_n$ we may consider T_x and N_x as orthogonal subspaces of $\mathbb{A}^n$.

From the definition of the polar variety $W_{\underline{K(a)}}(S)$ we deduce that the point x belongs to it if and only if $\dim_{\mathbb{C}}(N_x + E_a) \le n-i$ holds. Taking into account the identity

$$\dim_{\mathbb{C}}(N_x + E_a) = \dim_{\mathbb{C}}\, N_x + \dim_{\mathbb{C}}\, E_a - \dim_{\mathbb{C}}(N_x \cap E_a)$$

we conclude that the condition $\dim_{\mathbb{C}}(N_x + E_a) \le n-i$ is equivalent to

$$\dim_{\mathbb{C}}(N_x \cap E_a) \ge \dim_{\mathbb{C}}\, N_x + \dim_{\mathbb{C}}\, E_a - n + i = p + (n-p-i+1) - n + i = 1.$$

Since the $\mathbb{C}$–linear spaces $N_x \cap E_a$ and T_x are orthogonal subspaces of $\mathbb{A}^n$ with respect to the bilinear form $X_1Y_1 + \cdots + X_nY_n$, we infer

$$\dim_{\mathbb{C}}\, \Pi_a(T_x) \le \dim_{\mathbb{C}}\, E_a - \dim_{\mathbb{C}}(N_x \cap E_a) = n-p-i+1 - \dim_{\mathbb{C}}(N_x \cap E_a).$$

Thus the conditions

$$\dim_{\mathbb{C}}(ker\, d_x(\pi_a)) \ge i, \quad \dim_{\mathbb{C}}\, \Pi_a(T_x) \le n-p-i \quad \text{and} \quad x \in W_{\underline{K(a)}}(S) \cap S_{reg}$$

are equivalent. This proves the statement (i) of the lemma.

Statement (ii) can be shown similarly, observing that the condition $x \in \underline{\Delta_i} \cap S_{reg}$ is equivalent to $\dim_{\mathbb{C}}(N_x + E_a) \le n-i-1$. $\square$

The following deep result represents the main ingredient of our analysis.

Lemma 12

For an integer $j \ge i$, let $v_i(j) := (j-i+1)j$. There exists a non–empty Zariski open subset U_i of complex full–rank $((n-p-i+1)\times n)$–matrices of $\mathbb{A}^{(n-p-i+1)\times n}$ such that for any any $a \in U_i$ and any index $i \le j \le n-p$ the constructible set $\Sigma^{(j)}(\pi_a)$ is empty or a smooth subvariety of S of codimension $v_i(j)$.

The proof of Lemma 12 is based on two technical statements which represent particular instances of [1], Theorem 2.31 on one side and Theorems 3.10 and 3.11 on the other. We will not reproduce the details here. These statements go back to Mather's original work [24].

Combining Lemma 11 and Lemma 12 we obtain for generic $a \in \mathbb{A}^{(n-p-i)\times n}$ decompositions of $W_{\underline{K}(\underline{a})}(S) \cap S_{reg}$ and $\underline{\Delta_i} \cap S_{reg}$ in pieces which are empty or smooth and of predetermined codimension in S.

In order to illustrate the power of this analysis we are now giving an alternative proof of Proposition 9 using the following argumentation.

Suppose $2i+2 > n-p$ and let a be a complex full–rank $((n-p-i+1)\times n)$–matrix contained in the set U_i introduced in Lemma 12.

Then we have for $i < j \leq n-p$

$$n - p - v_i(j) \leq n - p - v_i(i+1) = n - p - (2i+2) < 0$$

and therefore

$$\Sigma^{(j)}(\pi_a) = \emptyset.$$

Lemma 11 implies now

$$W_{\underline{K}(\underline{a})}(S) \cap S_{reg} = \Sigma^{(i)}(\pi_a)$$

and from Lemma 12 we deduce finally that $W_{\underline{K}(\underline{a})}(S) \cap S_{reg}$ is empty or a smooth subvariety of S of codimension $v_i(i) = i$.

4 Meagerly generic polar varieties

In terms of the notions and notations introduced in Section 1 and Section 2, for $a := [a_{k,l}]_{\substack{1\leq k\leq n-p \\ 1\leq l\leq n}}$ being a complex full rank matrix, the (geometric) degrees of the classic and dual polar varieties $W_{\underline{K}(\underline{a}^{(i)})}(S)$ and $W_{\overline{K}(\overline{a}^{(i)})}(S)$, $1 \leq i \leq n-p$, are constructible functions of $a \in \mathbb{A}^{(n-p)\times n}$. In view of Definition 1, this implies that, for $a \in \mathbb{A}^{(n-p)\times n}$ generic, these degrees are independent of a and therefore invariants of S. We denote by $\delta_{classic}$ the maximal geometric degree of the classic polar varieties $W_{\underline{K}(\underline{a}^{(i)})}(S)$, $1 \leq i \leq n-p$, when $a \in \mathbb{A}^{(n-p)\times n}$ is generic. In a similar way we define δ_{dual} as the maximal degree of the $W_{\overline{K}(\overline{a}^{(i)})}(S)$, $1 \leq i \leq n-p$, for $a \in \mathbb{A}^{(n-p)\times n}$ generic. Thus $\delta_{classic}$ and δ_{dual} are also well–defined invariants of S. The main outcome of Theorem 13 below is the observation that $\delta_{classic}$ and δ_{dual} dominate (in a suitable sense) the degrees of *all* classic and dual polar varieties of S.

The statement of Theorem 13 is motivated by our aim to apply the geometry of polar varieties to algorithmic problem of real root finding. This can be explained as follows:

Suppose that there is given a division–free arithmetic circuit β in $\mathbb{Q}[X]$ evaluating the reduced regular sequence $F_1, \ldots, F_p \in \mathbb{Q}[X]$ that defines the complex variety S. Let L be the size of β (which measures the number of arithmetic operations required by β for the evaluation of $F_1, \ldots, F_p$) and let d and D be the maximal degrees of the polynomials $F_1, \ldots, F_p$ and the intermediate varieties $\{F_1 = 0, \ldots, F_k = 0\}, 1 \leq k \leq p$, respectively. Suppose furthermore that the real trace $S_{\mathbb{R}}$ of S is non–empty, smooth and of pure codimension p in $\mathbb{A}^n_{\mathbb{R}}$.

Then a set of at most δ_{dual} real algebraic sample points for the connected components of $S_{\mathbb{R}}$ can be found using at most $O(\binom{n}{p} L n^4 p^2 d^2 (\max\{D, \delta_{dual}\})^2)$ arithmetic operations and comparisons in $\mathbb{Q}$ (see [5, 6]).

If additionally $S_{\mathbb{R}}$ is compact, the same conclusion holds true with $\delta_{classic}$ instead of δ_{dual}.
The underlying algorithm can be implemented in the *non–uniform deterministic* or alternatively in the *uniform probabilistic* complexity model.

From the Bézout Inequality [17] one deduces easily that $\delta_{classic}$ and δ_{dual} are bounded by $d^n p^{n-p}$ (see [5] and [6] for proofs). This implies that from the geometrically unstructured and extrinsic point of view, the quantities $\delta_{classic}$ and δ_{dual} and hence the complexity of the procedures of [3], [4], [5], [6], [31] and [34] are in worst case of order $(nd)^{O(n)}$. This meets all previously known algorithmic bounds (see e.g. [16], [18], [8], [29], [32]). From [15] and [11] one deduces a *worst case* lower bound of order $d^{\Omega(n)}$ for the complexity of the real root finding problem under consideration. Thus complexity improvements are only possible if we distinguish between well– and ill–posed input systems $F_1 = 0, \ldots, F_p = 0$ or varieties S. We realize this distinction by means of the invariants $\delta_{classic}$ and δ_{dual}.

For details on the notion of the (geometric) degree of algebraic varieties we refer to [17] (or [14] and [44]). For expository reasons we define the degree of the empty set as zero. For algorithmic aspects we refer to [4], [5], [6], [31] and [34]. The special case of $p := 1$ and $S_{\mathbb{R}}$ compact is treated in [3].

We mention here that the algorithms in the papers [3] and [4], which treat the special case $p := 1$ and the general case $1 \leq p \leq n$ when $S_{\mathbb{R}}$ is compact, use instead of the *generic* polar varieties $W_{\underline{K}(\underline{a}^{(i)})}(S)$, $1 \leq i \leq n-p$, with $a := [a_{k,l}]_{\substack{1 \leq k \leq n-p \\ 1 \leq l \leq n}}$ generic, more special polar varieties, where $a := [a_{k,l}]_{\substack{1 \leq k \leq n-p \\ 1 \leq l \leq n}}$ ranges over a (Zariski) dense set of rational points of a suitable irreducible subvariety of $\mathbb{R}^{(n-p) \times n}$. In the sequel we shall call these special polar varieties *meagerly generic* (see Definition 16 below).

It is not difficult to show that the argumentation of [3] in the case $p = 1$ covers also the instance of generic polar varieties. However, in the case $1 < p \leq n$, the polar varieties $W_{\underline{K}(\underline{a}^{(i)})}(S)$, $1 < i \leq n-p$ appearing in [4] are not generic, though they turn out to be non–empty, smooth and of pure codimension i in S (see Example 1).

Although it was not essential for the final outcome of [4], namely the correctness and complexity bound of the proposed point finding algorithm, we believed at the moment of publishing this paper that the smoothness of these meagerly generic polar varietes implies the smoothness of their generic counterparts. As we have seen in Section 3, this conclusion is wrong.

In order to estimate the complexity of root finding algorithms based on meagerly generic polar varieties (as in [4]) in terms of not too artificial invariants of the underlying variety S (like $\delta_{classic}$ or δ_{dual}), we need to know that the degrees of the generic polar varieties dominate the degrees of their more special meagerly generic counterparts. This is the aim of the next result.

Theorem 13
Let $a := [a_{k,l}]_{\substack{1\le k\le n-p\\ 1\le l\le n}}$ be a complex $((n-p)\times n)$–matrix. Suppose that for $1\le i\le n-p$ the $((n-p-i+1)\times n)$–matrix $[a_{k,l}]_{\substack{1\le k\le n-p-i+1\\ 1\le l\le n}}$ is of maximal rank $n-p-i+1$ and that the polar varieties $W_{\underline{K}(\underline{a}^{(i)})}(S)$ and $W_{\overline{K}(\overline{a}^{(i)})}(S)$ are empty or of pure codimension i in S.

Furthermore, let be given a non–empty Zariski open set U of complex $((n-p)\times n)$–matrices $b := [b_{k,l}]_{\substack{1\le k\le n-p\\ 1\le l\le n}}$ such that for each $b\in U$ and each $1\le i\le n-p$ the $((n-p-i+1)\times n)$–matrix $[b_{k,l}]_{\substack{1\le k\le n-p-i+1\\ 1\le l\le n}}$ is of maximal rank $n-p-i+1$ and such that the polar varieties $W_{\underline{K}(\underline{b}^{(i)})}(S)$ and $W_{\overline{K}(\overline{b}^{(i)})}(S)$ are uniformly empty or of codimension i in S and have fixed geometric degrees, say ${\delta_{classic}}^{(i)}$ and ${\delta_{dual}}^{(i)}$.

Suppose finally that, for $1\le i\le n-p$, the non–emptiness of $W_{\underline{K}(\underline{a}^{(i)})}(S)$ (or $W_{\overline{K}(\overline{a}^{(i)})}(S)$) implies that of $W_{\underline{K}(\underline{b}^{(i)})}(S)$ (or $W_{\overline{K}(\overline{b}^{(i)})}(S)$).

Then we have for $1\le i\le n-p$ the estimates

$$\deg\, W_{\underline{K}(\underline{a}^{(i)})}(S)\le {\delta_{classic}}^{(i)}$$

and

$$\deg\, W_{\overline{K}(\overline{a}^{(i)})}(S)\le {\delta_{dual}}^{(i)}.$$

The proof of Theorem 13 is based on Lemma 14 below. In order to state this lemma, suppose that there is given a morphism of affine varieties $\varphi : V\to\mathbb{A}^r$, where V is equidimensional of dimension r and the restriction map $\varphi_{|C} : C\to\mathbb{A}^r$ is dominating for each irreducible component C of V. Then φ induces an embedding of the rational function field $\mathbb{C}(\mathbb{A}^r)$ of $\mathbb{A}^r$ into the total quotient ring $\mathbb{C}(V)$ of V. Observe that $\mathbb{C}[V]$ is a subring of $\mathbb{C}(V)$ and that $\mathbb{C}(V)$ is isomorphic to the direct product of the function fields of all irreducible components of V. Hence $\mathbb{C}(V)$ is a finite dimensional $\mathbb{C}(\mathbb{A}^r)$–vector space and the degree of the morphism φ, namely $\deg\varphi := \dim_{\mathbb{C}(\mathbb{A}^r)}\mathbb{C}(V)$, is a well–defined quantity. Finally, we denote for $y\in\mathbb{A}^r$ by $\#\,\varphi^{-1}(y)$ the cardinality of the φ–fiber $\varphi^{-1}(y)$ at y and by $\#_{isolated}\,\varphi^{-1}(y)$ the (finite) number of its isolated points.

With these assumptions and notations we are now going to show the following elementary geometric fact which represents a straightforward generalization of [17] Proposition 1 and its proof.

Lemma 14
(i) Any point $y\in\mathbb{A}^r$ satisfies the condition

$$\#_{isolated}\,\varphi^{-1}(y)\le\deg\,\varphi.$$

In particular, if the fiber of φ is finite, we have $\#\,\varphi^{-1}(y)\le\deg\,\varphi$.

(ii) There exists a non–empty Zariski open subset $U \subset \mathbb{A}^r$ such that any point $y \in U$ satisfies the condition

$$\# \varphi^{-1}(y) = \deg \varphi.$$

Proof

Let us show the first statement of Lemma 14 (i), the second one is then obvious. We proceed by induction on r.

The case $r = 0$ is evident. Let us therefore suppose $r > 0$ and that the lemma is true for any morphism of affine varieties $\varphi' : V' \to \mathbb{A}^{r-1}$ which satisfies the previous requirements.

Consider an arbitrary point $y = (y_1, \ldots, y_r)$ of $\mathbb{A}^r$. If $\varphi^{-1}(y)$ does not contain isolated points we are done. Therefore we may consider an arbitrary isolated point z of $\varphi^{-1}(y)$. Denoting the coordinate functions of $\mathbb{A}^r$ by $Y_1, \ldots, Y_r$, let us identify the hyperplane $\{Y_1 - y_1 = 0\}$ of $\mathbb{A}^r$ with $\mathbb{A}^{r-1}$. Thus we have $y \in \mathbb{A}^{r-1}$. From $\varphi^{-1}(y) \neq \emptyset$ and the Dimension Theorem for algebraic varieties (see e.g. [37], Vol. I, Ch. I, Corollary 1 of Theorem 6.2.5) we deduce that $\varphi^{-1}(\mathbb{A}^{r-1})$ is a non–empty subvariety of V of pure codimension one. Since y belongs to $\mathbb{A}^{r-1}$ and $\varphi^{-1}(y)$ contains an isolated point we deduce from the Theorem on Fibers (see e.g. [37], Vol. I, Ch. I, Theorem 6.3.7) that there exists an irreducible component C' of $\varphi^{-1}(\mathbb{A}^{r-1})$ such that the induced morphism of irreducible affine varieties $\varphi_{|_{\mathbb{C}'}} : C' \to \mathbb{A}^{r-1}$ is dominating.

Let C'' be an irreducible component of $\varphi^{-1}(\mathbb{A}^{r-1})$ such that $\varphi(C'')$ is not Zariski dense in $\mathbb{A}^{r-1}$ and denote the Zariski closure of $\varphi(C'')$ in $\mathbb{A}^{r-1}$ by D. Then D is an irreducible subvariety of $\mathbb{A}^{r-1}$ of dimension at most $r-2$.

We claim now that z does not belong to C''. Otherwise, by the Theorem on Fibers, all components of the $\varphi_{|_{\mathbb{C}''}}$–fiber of y, namely $\varphi^{-1}(y) \cap C''$, have dimension at least $\dim C'' - \dim D > 0$ and z belongs to $\varphi^{-1}(y) \cap C''$. This contradicts our assumption that z is an isolated point of $\varphi^{-1}(y)$.

Therefore the isolated points of $\varphi^{-1}(y)$ are contained in $V' := \bigcup_{C' \in \mathcal{Z}} C'$, where

$$\mathcal{Z} := \{C' \,|\, C' \text{ irreducible component of } \varphi^{-1}(\mathbb{A}^{r-1}),\ \varphi_{|_{\mathbb{C}'}} : C' \to \mathbb{A}^{r-1} \text{ dominating}\}.$$

Observe that V' is a (non–empty) equidimensional affine variety of dimension $r-1$ and that φ induces a morphism of affine varieties $\varphi' : V' \to \mathbb{A}^{r-1}$ such that $\varphi'_{|_{C'}} : C' \to \mathbb{A}^{r-1}$ is dominating for any irreducible component C' of V'. Moreover, y belongs to the image of φ' and $(\varphi')^{-1}(y)$ and $\varphi^{-1}(y)$ have the same isolated points.

Hence, by induction hypothesis, the statement (i) of the lemma is valid for φ' and y. Consequently we have

$$\#_{isolated}\, \varphi^{-1}(y) = \#_{isolated}\, (\varphi')^{-1}(y) \leq \deg \varphi'.$$

We finish now the proof of lemma 14, (i) by showing that $\deg \varphi' \leq \deg \varphi$ holds.

For this purpose let us abbreviate $Y := (Y_1, \ldots, Y_r)$ and let us write $\mathbb{C}[Y]$ and $\mathbb{C}(Y)$ instead of $\mathbb{C}[\mathbb{A}^r]$ and $\mathbb{C}(\mathbb{A}^r)$, respectively. Thus we consider $\mathbb{C}[Y]$ as a $\mathbb{C}$–subalgebra of $\mathbb{C}[V]$ and $\mathbb{C}(Y)$ as a $\mathbb{C}$–subfield of the total quotient ring $\mathbb{C}(V)$.

For any irreducible component C of V the morphism of affine varieties $\varphi_{|C} : C \to \mathbb{A}^r$ is by assumption dominating. This implies that $Y_1 - y_1$ is not a zero divisor of $\mathbb{C}[V]$ or $\mathbb{C}(V)$.

The canonical embedding of V' in V induces a surjective $\mathbb{C}$–algebra homomorphism $\mathbb{C}[V] \to \mathbb{C}[V']$ which associates to each $g \in \mathbb{C}[V]$ an image in $\mathbb{C}[V']$ denoted by g'. Therefore there exists for $\delta := \deg \varphi'$ elements $g_1, \ldots, g_\delta$ of $\mathbb{C}[V]$ such that $g_1', \ldots, g_\delta'$ form a vector space basis of $\mathbb{C}(V')$ over $\mathbb{C}(\mathbb{A}^{r-1}) = \mathbb{C}(Y_2, \ldots, Y_r)$. In particular, $g_1', \ldots, g_\delta'$ are $\mathbb{C}(Y_2, \ldots, Y_r)$–linearly independent. We claim that $g_1, \ldots, g_\delta$, considered as elements of $\mathbb{C}(V)$, are also linearly independent over $\mathbb{C}(\mathbb{A}^r) = \mathbb{C}(Y)$.
Otherwise there exist polynomials $b_1, \ldots, b_\delta \in \mathbb{C}[Y]$ not all zero, satisfying the condition

$$b_1\, g_1 + \cdots + b_\delta\, g_\delta = 0 \tag{7}$$

in $\mathbb{C}[V]$.

For $1 \le j \le \delta$ let $b_j = (Y_1 - y_1)^{u_j} c_j$, where u_j is a non–negative integer and c_j is polynomial of $\mathbb{C}[Y]$ which is not divisible by $Y_1 - y_1$. Thus (7) can be written as

$$(Y_1 - y_1)^{u_1} c_1\, g_1 + \cdots + (Y_1 - y_1)^{u_\delta} c_\delta\, g_\delta = 0.$$

Therefore, since $Y_1 - y_1$ is not a zero divisor in $\mathbb{C}[V]$, we may assume without loss of generality that there exists $1 \le j_0 \le \delta$ with $u_{j_0} = 0$. This means that we may suppose b_{j_0} is not divisible by $Y_1 - y_1$ in $\mathbb{C}[Y]$.

Condition (7) implies now that

$$b_1'\, g_1' + \cdots + b_\delta'\, g_\delta' = 0 \tag{8}$$

holds in $\mathbb{C}[V']$ and therefore also in $\mathbb{C}(V')$. Observe that b_j' is the polynomial of $\mathbb{C}[Y_2, \ldots, Y_r]$ obtained by substituting in b_j the indeterminate Y_1 by $y_1 \in \mathbb{C}$, namely $b_j' = b_j(y_1, Y_2, \ldots, Y_r)$, where $1 \le j \le \delta$. Since by assumption b_{j_0} is not divisible by $Y_1 - y_1$ in $\mathbb{C}[Y]$, we have $b_{j_0}' = b_{j_0}(y_1, Y_2, \ldots, Y_r) \neq 0$. Thus (8) implies that $g_1', \ldots, g_\delta' \in \mathbb{C}(V')$ are linearly dependent over $\mathbb{C}(Y_2, \ldots, Y_r) = \mathbb{C}(\mathbb{A}^{r-1})$ which contradicts the hypothesis that $g_1', \ldots, g_\delta'$ is a vector space basis of $\mathbb{C}(V')$.

Thus $g_1, \ldots, g_\delta \in \mathbb{C}(V)$ are linearly independent over $\mathbb{C}(\mathbb{A}^r)$. Hence we may conclude

$$\deg \varphi' = \dim_{\mathbb{C}(\mathbb{A}^{r-1})} \mathbb{C}(V') = \delta \le \dim_{\mathbb{C}(\mathbb{A}^r)} \mathbb{C}(V) = \deg \varphi$$

and we are done.

We are now going to prove statement (ii) of Lemma 14. Let $C_1, \ldots, C_s$ be the irreducible components of V. For $1 \le j \le s$ we denote the dominating morphism of affine varieties $\varphi_{|C_j} : C_j \to \mathbb{A}^r$ by φ_j. Observe that $\deg \varphi = \sum_{1 \le j \le s} \varphi_j$ holds. Since our ground field $\mathbb{C}$ is of characteristic zero, [17], Proposition 1 (ii) implies that we may choose a non–empty Zariski open subset U of $\mathbb{A}^r$ satisfying the following condition:
For any $y \in U$, the fibers $\varphi_j^{-1}(y)$, $1 \le j \le s$ are of cardinality $\deg \varphi_j$ and form

a (disjoint) partition of $\varphi^{-1}(y)$.
This implies that for any $y \in U$

$$\#\,\varphi^{-1}(y) = \sum_{1\leq j\leq s} \#\,\varphi_j^{-1}(y) = \sum_{1\leq j\leq s} \deg\,\varphi_j = \deg\,\varphi.$$

holds. $\square$

Observation 15
Statement and proof of Lemma 14 (i) do not depend on the characteristic of the ground field (in our case $\mathbb{C}$).

Now we are going to prove Theorem 13.

Proof
We limit our attention to the classic polar varieties. The case of the dual polar varieties can be treated similarly (but not identically). In order to simplify the exposition we assume that S is smooth.

Let us fix $1 \leq i \leq n-p$. Without loss of generality we may suppose that $W_{\underline{K}^{n-p-i}(\underline{a}^{(i)})}(S)$ is non–empty. We consider the closed subvariety W_i of $\mathbb{A}^n \times \mathbb{A}^{(n-p-i+1)\times n}$ defined by the vanishing of $F_1,\dots,F_p$ and of all $(n-i+1)$–minors of the $((n-i+1)\times n)$–matrix

$$P_i := \begin{bmatrix} & J(F_1,\dots,F_p) & \\ A_{1,1} & \cdots & A_{1,n} \\ \vdots & \vdots & \vdots \\ A_{n-p-i+1,1} & \cdots & A_{n-p-i+1,n} \end{bmatrix}$$

and the canonical projection $\pi_i := W_i \to \mathbb{A}^{(n-p-i+1)\times n}$ of W_i to $\mathbb{A}^{(n-p-i+1)\times n}$. Let D be an irreducible component of $W_{\underline{K}^{n-p-i}(\underline{a}^{(i)})}(S)$. Then there exists an irreducible component C of W_i which contains $D \times \{a^{(i)}\}$. We claim that there exists a $((n-i)\times(n-i))$– submatrix of P_i containing $(n-i)$ entries of each of the rows number $1,\dots,p$ of P_i, such that the corresponding $(n-i)$–minor of P_i, say

$$m := \det \begin{bmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_{n-i}} \\ \vdots & \cdots & \vdots \\ \frac{\partial F_p}{\partial X_1} & \cdots & \frac{\partial F_p}{\partial X_{n-i}} \\ A_{1,1} & \cdots & A_{1,n-i} \\ \vdots & \vdots & \vdots \\ A_{n-p-i,1} & \cdots & A_{n-p-i,n-i} \end{bmatrix},$$

does not vanish identically on C.
Otherwise C would be contained in the locus defined by the vanishing of $F_1,\dots,F_p$ and the determinants of all $((n-i)\times(n-i))$–submatrices of P_i that contain $n-i$ entries of each of the rows number $1,\dots,p$ of P_i. This would imply that D is contained in $W_{\underline{K}^{n-p-i-1}(\underline{a}^{(i+1)})}(S)$ and hence

$$\dim\, D \leq \dim\, W_{\underline{K}^{n-p-i-1}(\underline{a}^{(i+1)})}(S) = n-p-i-1.$$

Since by assumption the codimension of D in S is i, we have $\dim D = n-p-i$ and therefore a contradiction.
Thus we may assume without loss of generality that the $(n-i)$–minor m of the $((n-i+1)\times n)$–matrix P_i does not vanish identically on C.

For $n-i < j \leq n$ let us denote by M_j the $(n-i+1)$–minor of the $((n-i+1)\times n)$–matrix P_i which corresponds to the first $n-i$ columns and the column number j of P_i.
Let $\widetilde{W}_i$ be the closed subvariety of $\mathbb{A}^n \times \mathbb{A}^{(n-p-i+1)\times n}$ where $F_1,\dots,F_p$ and the $(n-i+1)$–minors $M_{n-i+1},\dots,M_n$ vanish. Observe that any irreducible component of $\widetilde{W}_i$ has dimension at least $n+(n-p-i+1)n-p-i = (n-p-i+2)n-p-i \geq 0$. Obviously $\widetilde{W}_i$ contains W_i and therefore C. Thus there exists an irreducible component $\widetilde{C}$ of $\widetilde{W}_i$ that contains C. Observe that m does not vanish identically on $\widetilde{C}$ and that $\widetilde{C}_m \supset C_m \neq \emptyset$ holds. From the Exchange Lemma in [4] we deduce now that all $(n-i+1)$–minors of the $((n-i+1)\times n)$–matrix P_i vanish identically on $\widetilde{C}_m$ and hence on $\widetilde{C}$. Thus $\widetilde{C}$ is an irreducible closed subset of W_i that contains the component C. This implies $\widetilde{C} = C$. Therefore C is an irreducible component of the variety $\widetilde{W}_i$, whence $\dim C \geq (n-p-i+2)n-p-i$.

We claim now that the morphism of affine varieties $\pi_{i|_C} : C \to \mathbb{A}^{(n-i+1)\times n}$ is dominating. Otherwise, by the Theorem on Fibers, any irreducible component of any (non–empty) fiber of $\pi_{i|_C}$ would have a dimension strictly larger than

$$\dim C - (n-p-i+1)n \geq ((n-p-i+2)n-p-i) - (n-p-i+1)n = n-p-i.$$

Recall that D is an irreducible component of $W_{\underline{K}^{n-p-i}(\underline{a}^{(i)})}(S)$ such that $D \times \{a^{(i)}\}$ is contained in C. Moreover we have $\pi_i^{(-1)}(a^{(i)}) \cong W_{\underline{K}^{n-p-i}(\underline{a}^{(i)})}(S)$. Thus $D \times \{a^{(i)}\}$ is an irreducible component of $\pi_i^{(-1)}(a^{(i)})$ which is contained in $\pi_i^{(-1)}(a^{(i)}) \cap C$. Since by assumption $W_{\underline{K}^{n-p-i}(\underline{a}^{(i)})}(S)$ is of pure codimension i in S, we have $\dim D = n-p-i$ and therefore the $\pi_{i|_C}$–fiber of $a^{(i)} \in \mathbb{A}^{(n-p-i+1)\times n}$, namely $\pi_i^{-1}(a^{(i)}) \cap C$, contains an irreducible component of dimension exactly $n-p-i$, contradicting our previous conclusion that the irreducible components of any $\pi_{i|_C}$–fiber are all of dimension strictly larger than $n-p-i$.

Therefore the the morphism of affine varieties $\pi_{i|_C} : C \to \mathbb{A}^{(n-i+1)\times n}$ is dominating. Taking into account the estimate $\dim C \leq (n-p-i+2)n-p-i$ and that the $\pi_{i|_C}$–fiber of $a^{(i)}$ contains an irreducible component of dimension $n-p-i$, we deduce now from the Theorem on Fibers that $\dim C = (n-p-i+2)n-p-i$ holds.

Let V_i be the union of all $((n-i+2)n-p-i)$–dimensional components C of W_i such that the morphism of affine varieties $\pi_{i|_C} : C \to \mathbb{A}^{(n-p-i+1)\times n}$ is dominating. Let us denote the restriction map $\pi_{i|_{V_i}} : V_i \to \mathbb{A}^{(n-p-i+1)\times n}$ by $\psi_i : V_i \to \mathbb{A}^{(n-p-i+1)\times n}$.

Then V_i is an equidimensional affine variety of dimension $(n-p-i+2)n-p-i$ and ψ_i is an morphism of affine varieties such that the restriction of ψ_i to any irreducible component of V_i is dominating. Observe that our previous argumentation implies

$$\psi_i^{-1}(a^{(i)}) \cong W_{\underline{K}^{n-p-i}(\underline{a}^{(i)})}(S) \text{ and } \psi_i^{-1}(b^{(i)}) \cong W_{\underline{K}^{n-p-i}(\underline{b}^{(i)})}(S) \text{ for any } b \in U.$$

Observe now that there exists a finite set $\mathcal{M}$ of rational $((n-p-i)\times n)$–matrices of maximal rank $n-p-i$ satisfying the following conditions:

- Each $M \in \mathcal{M}$ represents a generic Noether position (see [17], Lemma 1) of the polar variety $W_{\underline{K}^{n-p-i}(\underline{a}^{(i)})}(S)$.
- For each $b \in U$ there exists a $((n-p-i)\times n)$ matrix $M \in \mathcal{M}$ representing a generic Noether position of the affine polar variety $W_{\underline{K}^{n-p-i}(\underline{b}^{(i)})}(S)$ (here we use the assumption that $W_{\underline{K}^{n-p-i}(\underline{a}^{(i)})}(S) \neq \emptyset$ implies $W_{\underline{K}^{n-p-i}(\underline{b}^{(i)})}(S) \neq \emptyset$).

Therefore, restricting, if necessary, the non–empty Zariski open set U, we may suppose without loss of generality that there exists a full–rank matrix $M \in \mathbb{Q}^{(n-p-i)\times n}$ such that M represents a generic Noether position of $W_{\underline{K}^{n-p-i}(\underline{a}^{(i)})}(S)$ and of any $W_{\underline{K}^{n-p-i}(\underline{b}^{(i)})}(S)$, where b belongs to U. The $((n-p-i)\times n)$–matrix M and ψ_i induce now a morphism of $((n-p-i+2)n-p-i)$–dimensional affine varieties $\varphi_i : V_i \to \mathbb{A}^{n-p-i} \times \mathbb{A}^{(n-p-i+1)\times n}$ which associates to any point $(x,c) \in V_i \subset \mathbb{A}^n \times \mathbb{A}^{(n-p-i+1)\times n}$ the value $\varphi_i(x,c) := (Mx, c)$.

Let C be an arbitrary irreducible component of V_i. Recall that we have dim $C = (n-p-i+2)n-p-i$ and the morphism of affine varieties $\psi_{i|_C} : C \to \mathbb{A}^{(n-p-i+1)\times n}$ is dominating. Therefore, by the Theorem on Fibers, there exists $b \in U$ such that $b^{(i)}$ belongs to $\psi_i(C)$ and that the $\psi_{i|_C}$–fiber of $b^{(i)}$, namely $\psi_i^{-1}(b^{(i)} \cap C$, is non–empty and equidimensional of dimension $(n-p-i+2)n-p-i-(n-p-i+1)n = n-p-i$. Observe that $b \in U$ implies that $W_{\underline{K}^{n-p-i}(\underline{b}^{(i)})}$ is non–empty and of pure codimension i in S. Therefore any irreducible component $W_{\underline{K}^{n-p-i}(\underline{b}^{(i)})}$ is of dimension $n-p-i$. Hence any irreducible component of $\psi_i^{-1}(b^{(i)}) \cap C$ is isomorphic to an irreducible component of $W_{\underline{K}^{n-p-i}(\underline{b}^{(i)})}(S)$. This implies that the rational $((n-p-i)\times n)$–matrix M represents also a generic Noether position of the $\psi_{i|_C}$–fiber of $b^{(i)}$, namely $\psi_i^{-1}(b^{(i)}) \cap C$.

Fix now any point $y \in \mathbb{A}^{n-p-i}$ and observe that the $\varphi_{i|_C}$–fiber $(y, b^{(i)})$ is isomorphic to a non–empty subset of the zero–dimensional variety

$$\{x \in W_{\underline{K}^{n-p-i}(\underline{b}^{(i)})}(S) \mid Mx = y\}.$$

Therefore we may conclude that the morphism of affine varieties

$$\varphi_{i|_C} : C \to \mathbb{A}^{n-p-i} \times \mathbb{A}^{(n-p-i+1)\times n}$$

is dominating. Since C was chosen as an arbitrary irreducible component of V_i, we infer that $\varphi_i : V_i \to \mathbb{A}^{n-p-i} \times \mathbb{A}^{(n-p-i+1)\times n}$ satisfies the requirements of Lemma 14.

Let us now fix $b \in U$ and $y \in \mathbb{A}^{n-p-i}$ such that $(y, b^{(i)})$ represents a generic choice in the ambient space $\mathbb{A}^{n-p-i} \times \mathbb{A}^{(n-p-i+1)\times n}$. Observe that ${\varphi_i}^{-1}(y, a^{(i)})$ is isomorphic to

$$\{x \in W_{\underline{K}^{n-p-i}(\underline{a}^{(i)})}(S) \mid Mx = y\}.$$

Since M represents a generic Noether position of the affine variety $W_{\underline{K}^{n-p-i}(\underline{a}^{(i)})}(S)$, we conclude that $\#\,\varphi_i^{-1}(y, a^{(i)}) = \deg\, W_{\underline{K}^{n-p-i}(\underline{a}^{(i)})}(S)$ holds. In particular, $\#\,\varphi_i^{-1}(y, \underline{a}^{(i)})$ is finite. Similarly, one deduces $\#\,\varphi_i^{-1}(y, b^{(i)}) = \deg\, W_{\underline{K}^{n-p-i}(\underline{b}^{(i)})}(S)$.

Lemma 14 implies now

$$\deg\, W_{\underline{K}^{n-p-i}(\underline{b}^{(i)})}(S) = \#\, \varphi_i^{-1}(y, b^{(i)}) = \deg\, \varphi_i$$

and

$$\deg\, W_{\underline{K}^{n-p-i}(\underline{a}^{(i)})}(S) = \#\, \varphi_i^{-1}(y, a^{(i)}) \leq \deg\, \varphi_i = \deg\, W_{\underline{K}^{n-p-i}(\underline{b}^{(i)})}(S).$$

This proves Theorem 13. $\square$

The following remark is now at order. Theorem 13 is not aimed to replace a general intersection theory formulated in terms of rational equivalence classes of cycles of the projective space $\mathbb{P}^n$, it is rather complementary to such a theory. As said before, its main purpose is to allow, in terms of the degrees of generic polar varieties, complexity estimates for real point finding procedures which are based on the consideration of meagerly generic polar varieties. The alluded interplay between a general intersection theory and the statement of Theorem 13 may become more clear by the following considerations.

Under the assumption that the homogenizations of the polynomials $F_1, \ldots, F_p$ define a smooth codimension p subvariety $\widetilde{S}$ of $\mathbb{P}^n$, the projective closures of the affine polar varieties $W_{\underline{K}(\underline{a}^{(i)})}(S)$, $1 \leq i \leq n-p$, form the corresponding polar varieties of $\widetilde{S}$, when $a \in \mathbb{A}^{(n-p)\times n}$ is generic. Moreover, $\widetilde{S}$ is the projective closure of S. The degrees of the generic (classic) polar varieties of $\widetilde{S}$ may be expressed in terms of the degrees of the Chern classes of $\widetilde{S}$ (see [14], Example 14.3.3). Further, since $\widetilde{S}$ is a projective smooth complete intersection variety, the total Chern class of $\widetilde{S}$ may be characterized in terms of the degrees of $F_1, \ldots, F_p$ and the first Chern class c_1 of the normal bundle of $\widetilde{S}$ in $\mathbb{P}^n$ (see [20], Theorem 4.8.1, Section 22.1 and [27], Theorem 1). This implies an upper bound for the geometric degrees of the *affine* polar varieties $W_{\underline{K}(\underline{a}^{(i)})}(S)$ in terms of the degrees of the polynomials $F_1, \ldots, F_p$ and the class c_1.

Of course our previous assumption on $F_1, \ldots, F_p$ is very restrictive. Nevertheless, in this situation we are able to illustrate how results like Theorem 13 (or [17], Proposition 1) interact with facts from a general intersection theory.

Let us now turn back to the discussion of the, up to now informal, concept of meagerly generic polar varieties. We are going to give to this concept a precise mathematical shape and to discuss it by concrete examples.

Definition 16
Let $1 \leq i \leq n-p$ and let m be a non–zero polynomial of $\mathbb{C}[X]$, let E be an irreducible closed subvariety of $\mathbb{A}^{(n-p-i+1)\times n}$ and let O be a non–empty Zariski open subset of E. Suppose that the following conditions are satisfied:

(i) each $b \in O$ is a complex $((n-p)\times n)$–matrix of maximal rank $n-p-i+1$,

(ii) for each $b \in O$ the affine variety $W_{\underline{K}(\underline{b})}(S)_m$ (respectively $W_{\overline{K}(\overline{b})}(S)_m$) is empty or of pure codimension i in S.

Then we call the algebraic family $\{W_{\underline{K}(\underline{b})}(S)_m\}_{b\in O}$ *(respectively* $\{W_{\overline{K}(\overline{b})}(S)_m\}_{b\in O}$ *) of Zariski open subsets of polar varieties of* S meagerly generic.

In the same vein, we call for $b \in O$ *the affine variety* $W_{\underline{K}(\underline{b})}(S)_m$ *(respectively* $W_{\overline{K}(\overline{b})}(S)_m$ *) meagerly generic.*

Typically, the irreducible affine variety E contains a Zariski dense set of rational points and for any $b \in O$ the non–emptiness of $W_{\underline{K}(\underline{b})}(S)_m$ (respectively $W_{\overline{K}(\overline{b})}(S)_m$) implies the non–emptiness of the open locus of the corresponding generic polar variety defined by the non-vanishing of m. From Theorem 13 and its proof we deduce that in this case the sum of the degrees of the irreducible components of $W_{\underline{K}(\underline{b})}(S)$ (respectively $W_{\overline{K}(\overline{b})}(S)$), where m does not vanish identically, is bounded by the sum of the degrees of the components of the corresponding generic polar variety which satisfy the same condition.

We paraphrase this consideration as follows.

Observation 17
For full rank matrices $\mathbb{A}^{(n-p)\times n}$ *and* $1 \leq i \leq n-p$ *the degrees of (suitable open loci) of meagerly generic polar varieties* $W_{\underline{K}(\underline{a}^{(i)})}(S)$ *and* $W_{\overline{K}(\overline{a}^{(i)})}(S)$ *attain their maximum when* a *is generic.*

We are now going to discuss two examples of geometrically relevant algebraic families of Zariski open subsets of polar varieties of S. It will turn out that these sets are empty or smooth affine subvarieties of S.

Example 1

We are going to adapt the argumentation used in [4], Section 2.3 to the terminology of meagerly generic polar varieties.

Let $m \in \mathbb{Q}[X_1, \ldots, X_n]$ denote the $(p-1)$–minor of the Jacobian $J(F_1, \ldots, F_p)$ given by the first $(p-1)$ rows and columns, i.e., let

$$m := \det \left[\frac{\partial F_k}{\partial X_l}\right]_{\substack{1\leq k\leq p-1\\ 1\leq l\leq p-1}}.$$

Moreover, for $p \leq r \leq n, p \leq t < n$ let $Z_{r,t}$ be a new indeterminate. Using the following regular $((n-p+1)\times(n-p+1))$–parameter matrix

$$Z := \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & & \cdots & 0 \\ Z_{p+1,p} & 1 & & & & & & \\ \vdots & \vdots & \ddots & & \text{O} & & & \vdots \\ Z_{p+i-1,p} & Z_{p+i-1,p+1} & \cdots & 1 & & & & \\ Z_{p+i,p} & Z_{p+i,p+1} & \cdots & Z_{p+i,p+i-1} & 1 & & & \\ \vdots & \vdots & & \vdots & \vdots & \ddots & & 0 \\ Z_{n,p} & Z_{n,p+1} & \cdots & Z_{n,p+i-1} & Z_{n,p+i} & Z_{n,p+i+1} & \cdots & 1 \end{bmatrix},$$

we are going to introduce an $(n\times n)$–coordinate transformation matrix $A := A(Z)$ which will represent the key for our ongoing construction of a meagerly generic family of Zariski open subsets of classic polar varieties of S.

Let us fix an index $1 \leq i \leq n-p$. According to our choice of i, the matrix Z may be subdivided into submatrices as follows:

$$Z = \begin{bmatrix} Z_1^{(i)} & O_{i,n-p-i+1} \\ Z^{(i)} & Z_2^{(i)} \end{bmatrix}.$$

Here the matrix $Z^{(i)}$ is defined as

$$Z^{(i)} := \begin{bmatrix} Z_{p+i,p} & \cdots & Z_{p+i,p+i-1} \\ \vdots & \cdots & \vdots \\ Z_{n,p} & \cdots & Z_{n,p+i-1} \end{bmatrix},$$

and $Z_1^{(i)}$ and $Z_2^{(i)}$ denote the quadratic lower triangular matrices bordering $Z^{(i)}$ in Z. Let

$$A := \begin{bmatrix} I_{p-1} & O_{p-1,i} & O_{p-1,n-p-i+1} \\ O_{i,p-1} & Z_1^{(i)} & O_{i,n-p-i+1} \\ O_{n-p-i+1,p-1} & Z^{(i)} & Z_2^{(i)} \end{bmatrix}.$$

Here the submatrix I_{p-1} is the $((p-1) \times (p-1))$–identity matrix and $Z^{(i)}, Z_1^{(i)}$, and $Z_2^{(i)}$ are the submatrices of the parameter matrix Z introduced before. Thus, A is a regular, parameter dependent $(n \times n)$–coordinate transformation matrix.

Like the matrix Z, the matrix $A(Z)$ contains

$$s := \frac{(n-p)(n-p+1)}{2}$$

parameters $Z_{r,t}$ which we may specialize into any point z of the affine space $\mathbb{A}^s$. For such a point $z \in \mathbb{A}^s$ we denote the corresponding specialized matrices by $Z_1^{(i)}(z), Z_2^{(i)}(z), Z^{(i)}(z)$ and $A(z)$.

Let $B_i := B_i(Z)$ be the $((n-p-i+1) \times n)$–matrix consisting of the rows number $p+i, \ldots, n$ of the inverse matrix $A(Z)^{-1}$ of $A(Z)$. From the particular triangular form of the matrix $A(Z)$ we deduce that the entries of $B_i(Z)$ are polynomials in the indeterminates $Z_{r,t}$, $p \leq r \leq n, p \leq t < r$. Moreover, we have the matrix identity

$$B_i A = \begin{bmatrix} O_{n-p-i+1,p+i-1} & I_{n-p-i+1} \end{bmatrix}.$$

Finally, let E be the Zariski closure of $B_i(\mathbb{A}^s)$ in $\mathbb{A}^{(n-p-i+1)\times n}$. Then E is an irreducible closed subvariety of $\mathbb{A}^{(n-p-i+1)\times n}$ consisting of complex $((n-p-i+1) \times n)$–matrices of maximal rank $n-p-i+1$.

According to the structure of the coordinate transformation matrix $A = A(Z)$ we subdivide the Jacobian $J(F_1, \ldots, F_p)$ into three submatrices

$$J(F_1, \ldots, F_p) = \begin{bmatrix} U & V^{(i)} & W^{(i)} \end{bmatrix},$$

with

$$U := \left[\frac{\partial F_k}{\partial X_l}\right]_{\substack{1\leq k\leq p \\ 1\leq l\leq p-1}}, \quad V^{(i)} := \left[\frac{\partial F_k}{\partial X_l}\right]_{\substack{1\leq k\leq p \\ p\leq l\leq p+i-1}}, \quad W^{(i)} := \left[\frac{\partial F_k}{\partial X_l}\right]_{\substack{1\leq k\leq p \\ p+i\leq l\leq n}}.$$

We have then the following matrix identity:

$$J(F_1,\dots,F_p)\; A(Z) = \begin{bmatrix} U & V^{(i)}Z_1^{(i)} + W^{(i)}Z^{(i)} & W^{(i)}Z_2^{(i)} \end{bmatrix}.$$

For $p \leq j \leq p+i-1$ let us denote by $\widetilde{M}_j := \widetilde{M}_j(X,Z)$ the p–minor of the polynomial $(p \times n)$–matrix $J(F_1,\dots,F_p)\; A(Z)$ that corresponds to the columns number $1,\dots,p-1,j$.

From [4], Section 2.3 and Lemma 1 we deduce that there exists a non–empty Zariski open subset $\widetilde{O}$ of $\mathbb{A}^s$ such that any point $z \in \widetilde{O}$ satisfies the following two conditions:

(i) the polynomial equations

$$F_1(X) = 0, \dots, F_p(X) = 0, \widetilde{M}_p(X,z) = 0, \dots, \widetilde{M}_{p+i-1}(X,z) = 0$$

intersect transversally at any of their common solutions in $\mathbb{A}^n_m$,

(ii) for any point $x \in \mathbb{A}^n_m$ with

$$F_1(x) = 0, \dots, F_p(x) = 0, \widetilde{M}_p(x,z) = 0, \dots, \widetilde{M}_{p+i-1}(x,z) = 0$$

all p–minors of the complex $(p \times (p+i-1))$–matrix

$$\begin{bmatrix} U(x) & V^{(i)}(x)Z_1^{(i)}(z) + W^{(i)}(x)Z^{(i)}(z) \end{bmatrix}$$

vanish, i.e., this matrix has rank at most $p-1$.

Let $O := B_i(\widetilde{O})$ and observe that O is a non–empty Zariski open subset of the irreducible affine variety E.

Let us fix for the moment an arbitrary point $z \in \widetilde{O}$. Then the complex $((n-p-i+1) \times n)$–matrix $B_i(z)$ is of maximal rank $n-p-i+1$.

We consider now the Zariski open subset of $W_{\underline{K}(B_i(z))}(S)_m$ of the classic polar variety
$W_{\underline{K}(B_i(z))}(S)$. By definition the affine variety $W_{\underline{K}(B_i(z))}(S)_m$ consists of the points of S_m where all $(n-i+1)$–minors of the polynomial $((n-i+1) \times n)$–matrix

$$N_i := \begin{bmatrix} J(F_1,\dots,F_p) \\ B_i(z) \end{bmatrix}$$

vanish.

Let x be an arbitrary element of S_m. Then all $(n-i+1)$–minors of N_i vanish at x if and only if N_i has rank at most $n-i$ at x. This is equivalent to the condition that

$$N_i\, A(z) = \begin{bmatrix} J(F_1,\dots,F_p)\, A(z) \\ B_i(z)\, A(z) \end{bmatrix} = \begin{bmatrix} U & V^{(i)}Z_1^{(i)}(z) + W^{(i)}Z^{(i)}(z) & W^{(i)}Z_2^{(i)}(z) \\ & O_{n-p-i+1,p+i-1} & I_{n-p-i+1} \end{bmatrix}$$

has rank at most $n-i$ at x. Hence $W_{\underline{K}(B_i(z))}(S)_m$ consists of the points of S_m where the $(p \times (p+i-1))$–matrix

$$\begin{bmatrix} U & V^{(i)}Z_1^{(i)}(z) + W^{(i)}Z^{(i)}(z) \end{bmatrix}$$

has rank at most $p-1$.

From conditions (i) and (ii) above we deduce that the affine variety $W_{\underline{K}(\underline{B_i(z)})}(S)_m$ is either empty or a smooth subvariety of S of pure codimension i.

Since $z \in \widetilde{O}$ was chosen arbitrarily, we have shown the following restatement of [4], Theorem 1:

Proposition 18
For each point $b \in O$ the Zariski open subset $W_{\underline{K}(\underline{b})}(S)_m$ of the polar variety $W_{\underline{K}(\underline{b})}(S)$ is either empty or a smooth affine subvariety of S of pure codimension i. Thus $\{W_{\underline{K}(\underline{b})}(S)_m\}_{b\in O}$ forms a meagerly generic algebraic family of empty or smooth Zariski open subsets of polar varieties of S.

Observe that the matrix $A(Z)$ (and hence the matrix $A(Z)^{-1}$) does not depend on the index $1 \leq i \leq n-p$. Moreover, $A(Z)^{-1}$ has the same triangular shape as $A(Z)$. Here the entries $Z_{r,t}$, $p \leq r \leq n$, $p \leq t < r$ have to be replaced by suitable polynomials in Z which are algebraically independent over $\mathbb{Q}$.

Consequently
$$B_{n-p}(Z), \ldots, B_1(Z)$$
form a nested sequence of matrices, each contained in the other. Therefore we obtain for $z \in \widetilde{O}$ a descending chain of Zariski open susets of polar varieties

$$W_{\underline{K}(\underline{B_1}(z))}(S)_m \supset \cdots \supset W_{\underline{K}(\underline{B_{n-p}}(z))}(S)_m$$

which are empty or smooth codimension one subvarieties one of the other.

We discuss now Example 1 in view of algorithmic applications. In order to do this let us start with the following considerations.

Let $Y^* = (Y_1, \ldots, Y_{p-1})$ be a $(p-1)$–tupel of new indeterminates . We fix for the moment indices $1 \leq h_1 < \cdots < h_p \leq n$ and make a sub–selection of $p-1$ of them, say $h'_1 < \cdots < h'_{p-1}$. For the sake of notational simplicity suppose $h_1 = 1, \ldots, h_p = p$ and $h'_1 = 1, \ldots, h'_{p-1} = p-1$.

Let
$$U^* := \left[\frac{\partial F_k}{\partial X_l}\right]_{1\leq k,l\leq p}, \quad V^* := \left[\frac{\partial F_k}{\partial X_l}\right]_{\substack{1\leq k\leq p\\ p+1\leq l\leq n}}, M^* := \det U^*$$
and let $N^* := N^*(X, Y^*)$ be the polynomial $((p+1)\times p)$–matrix
$$N^* := \begin{bmatrix} U^* \\ Y_1 \cdots Y_{p-1} \; 1 \end{bmatrix}.$$

There exists a polynomial $(p\times p)$–matrix $C^* := C^*(Y^*)$ with $\det C^* = 1$ satisfying the condition
$$N^* C^* = \begin{bmatrix} U^* C^* \\ 0 \cdots 0 \; 1 \end{bmatrix}.$$
In particular we have $M^* = det(U^* C^*)$.

To our sub–selection of indices corresponds the $(p-1)$–minor $m^* := m^*(X, Y^*)$ determined by the rows and columns number $1, \ldots, p-1$ of $U^* C^*$. For the moment let us fix an index $1 \leq i \leq n-p$. According to our choice of indices and the sub–selection made from them, let us denote for $p \leq j \leq p+i-1$ by $M_j^* = M_j^*(X, Y^*, Z)$ the p–minor of the polynomial $(p \times n)$–matrix $[U^* \, C^* \;\; V^*] \; A(Z)$ that corresponds to the columns number $1, \ldots, p-1, j$.

We consider now the polynomial map

$$\Phi^* : \mathbb{A}^n_{M^*} \times \mathbb{A}^{p-1} \times \mathbb{A}^s \to \mathbb{A}^p \times \mathbb{A}^i$$

defined for $(x, y^*, z) \in \mathbb{A}^n_{M^*} \times \mathbb{A}^{p-1} \times \mathbb{A}^s$ by

$$\Phi^*(x, y^*, z) := (F_1(x), \ldots, F_p(x), M_p^*(x, y^*, z), \ldots, M_{p+i-1}^*(x, y^*, z)).$$

As in [4], Section 2.3 we may now argue that $0 \in \mathbb{A}^{p-1} \times \mathbb{A}^i$ is a regular value of the polynomial map Φ^*. From the Weak Transversality Theorem of Thom–Sard we then deduce that there exists a non–empty Zariski open subset O^* of $\mathbb{A}^{p-1} \times \mathbb{A}^i$ such that for any point (y^*, z) of O^* the polynomial equations

$$F_1(X) = 0, \ldots, F_p(X) = 0, M_p^*(X, y^*, z) = 0, \ldots, M_{p+i-1}^*(X, y^*, z) = 0$$

intersect transversally at any of their common solutions in $\mathbb{A}^n_{M^*}$.

Let (y^*, z) be an arbitrary point of O^*.

Notice that for any point $x \in S_{M^*}$ there exists a $(p-1)$–minor of $U^* \, C^*(y^*)$ that does not vanish at x. Let us suppose $m^*(x, y^*) \neq 0$. Now we may conclude as in the proof of Proposition 18 that the assumption

$$M_p^*(x, y^*, z) = 0, \ldots, M_{p+i-1}^*(x, y^*, z) = 0$$

implies that the polynomial $((n-i+1) \times n)$–matrix

$$\begin{bmatrix} U^* \, C^*(y^*) \;\; V^* \\ B_i(z) \end{bmatrix}$$

has rank at most $n-i$ at x.

Observe that the row number $n-p-i+1$ of $B_i(Z)$ has the form $(0, \ldots, 0, B_{n,p}(Z), \ldots, B_{n,n-1}(Z), 1)$, where $B_{n,p}(Z), \ldots, B_{n,n-1}(Z)$ are suitable polynomials in Z which are algebraically independent over $\mathbb{Q}$. Without loss of generality we may suppose that any point (y^*, z) of O^* satisfies the condition $B_{n,p}(z) \neq 0$. Thus there exists a non–empty Zariski open subset Q^* of $\mathbb{A}^n$ such that any point $b = (b_1, \ldots, b_n)$ of Q^* has non–zero entries and satisfies the following condition:

There exists a point (y^*, z) of O^* with $y^* = \frac{1}{b_p}(b_1, \ldots, b_{p-1})$ and

$$\frac{1}{B_{n,p}(z)}(B_{n,p+1}(z), \ldots, B_{n,n-1}(z), 1) = \frac{1}{b_p}(b_{p+1}, \ldots, b_n).$$

All these constructions and arguments depend on a fixed choice of indices $1 \leq h_1 < \cdots < h_p \leq n$, namely $h_1 := 1, \ldots, h_p := p$ and a sub–selection of them, namely $h'_1 := 1, \ldots, h'_{p-1} := p-1$.

Intersecting now for all possible choices of indices and sub–selections of them the resulting subsets of $\mathbb{A}^n$ corresponding to Q^* in case $h_1 = 1, \dots, h_p = p$, $h'_1 = 1, \dots, h'_{p-1} = p-1$, we obtain a non–empty Zariski open subset Q of $\mathbb{A}^n$ which has the same properties as Q^* with respect to any choice of indices $1 \le h_1 < \cdots < h_p \le n$ and any sub–selection $h'_1 < \cdots < h'_{p-1}$ of them.

Let us now suppose that $S_{\mathbb{R}}$ is smooth and compact and let $b = (b_1, \dots, b_n)$ be a fixed point of $Q \cap \mathbb{Q}^n$. Of course, such a point b exists and can be obtained by a random choice.

From [4], Theorem 2 we deduce that $W_{\underline{K}(\underline{b})}(S_{\mathbb{R}})$ is not empty. Since b belongs to Q there exists a point $(y^*, z) \in Q^*$ with

$$y^* = \frac{1}{b_p}(b_1, \dots, b_{p-1}) \text{ and } \frac{1}{B_{n,p}(z)}(B_{n,p+1}(z), \dots, B_{n,n-1}(z), 1) = \frac{1}{b_p}(b_{p+1}, \dots, b_n).$$

We may assume $z \in \mathbb{Q}^s$. Since $S_{\mathbb{R}}$ is smooth and $W_{\underline{K}(\underline{b})}(S_{\mathbb{R}})$ is not empty, we may suppose without loss of generality

$$W_{\underline{K}(\underline{b})}(S)_{M^*(X) m^*(X,y^*)} \neq \emptyset.$$

For $1 \le i \le n-p$ let W_i^* be the affine subvariety of it defined by the vanishing of all $(n-i+1)$–minors of the polynomial $((n-i+1) \times n)$–matrix $\begin{bmatrix} U^* \, C^*(y^*) \;\; V^* \\ B_i(z) \end{bmatrix}$. As before we obtain a descending chain of affine varieties, namely $W_1^* \supset \cdots \supset W_{n-p}^*$ which are empty or smooth codimension one subvarieties one of the other. In order to rule out the possibility of emptiness we are going to show $W_{n-p}^* \neq \emptyset$.

For this purpose we consider an arbitrary element $x \in S_{M^*(X) m^*(X,y^*)}$. Then all $(p+1)$–minors of the polynomial $((p+1) \times n)$–matrix $\begin{bmatrix} J(F_1, \dots, F_p) \\ b \end{bmatrix}$ vanish at x if and only if the same is true for $\begin{bmatrix} J(F_1, \dots, F_p) \\ \frac{1}{b_p} b \end{bmatrix}$ and this is equivalent to the condition that the polynomial $((p+1) \times n)$–matrix

$$\begin{bmatrix} U^* \, C^*(y^*) \;\; V^* \\ 0 \cdots 0 \; 1 \; \frac{1}{b_p} b_{p+1} \cdots \frac{1}{b_p} b_n \end{bmatrix},$$

and hence also

$$\begin{bmatrix} U^* \, C^*(y^*) \;\; V^* \\ 0 \cdots 0 \; b_p \; b_{p+1} \cdots b_n \end{bmatrix} = \begin{bmatrix} U^* \, C^*(y^*) \;\; V^* \\ 0 \cdots 0 \; B_{n,p}(z) \cdots B_{n,n-1}(z), \; 1 \end{bmatrix},$$

have rank at most p at x. This implies

$$W_{\underline{K}(\underline{b})}(S)_{M^*(X) m^*(X,y^*)} = W_{n-p}^*, \quad \text{whence } W_{n-p}^* \neq \emptyset.$$

In order to retrieve finitely many real algebraic sample points for the connected components of the real algebraic variety $S_{\mathbb{R}}$ we may now proceed by applying the general algorithm described in [4, 5] or [6] as follows: For each choice of indices $1 \le h_1 < \cdots < h_p \le n$ and any sub–selection $h'_1 < \cdots < h_{p-1}$ of them we generate the equations and the inequation that define the descending chain of affine varieties corresponding to $W_1^* \supset \cdots \supset W_{n-p}^*$ in order to find the (finitely many) real points contained in the last one.

The set of real points obtained in this way is by virtue of [4], Theorem 2 a set of sample points for the connected components of $S_{\mathbb{R}}$. For details of the algorithm we refer to [4].

We are now going to explain how the affine varieties W_i^*, $1 \le i \le n-p$ may be interpreted as Zariski open subsets of polar varieties of suitable complete intersection varieties.

Suppose we have already fixed indices $1 \le h_1 < \cdots < h_p \le n$ and have made a sub–selection $h_1' < \cdots < h_{p-1}$. For the sake of notational simplicity we suppose again $h_1 = 1, \ldots, h_p = p$, $h_1' = 1, \ldots, h_{p-1}' = p-1$. Furthermore, we suppose that we have chosen a point (y^*, z) of O^*. Let $\Omega = (\Omega_1, \ldots, \Omega_n)$ be an n–tupel of new indeterminates and let us consider the coordinate transformation matrix

$$D := \begin{bmatrix} C^*(y^*) & O_{p,n-p} \\ O_{n-p,p} & I_{n-p} \end{bmatrix}$$

and the polynomials

$$G_1 := G_1(\Omega) := F_1(\Omega\, D^T), \ldots, G_p := G_p(\Omega) := F_p(\Omega\, D^T),$$

where D^T denotes the transposed matrix of D. Let $G := (G_1, \ldots, G_p)$, $S_G := \{G_1 = 0, \ldots, G_p = 0\}$ and M_G the $(p \times p)$–minor of $J(G_1, \ldots, G_p)$ which corresponds to the columns number $1, \ldots, p$.

From the identity

$$J(G)(X(D^{-1})^T) = J(F)D = [U^*\, C^*(y^*)\ \ V^*]$$

we deduce now that for any point $x \in \mathbb{A}^n$ and any index $1 \le i \le n-p$ the conditions

$$x(D^{-1})^T \in W_{\underline{K}(\underline{B_i(z)})}(S_G)_{M_G} \quad \text{and} \quad x \in W_i^*$$

are equivalent. Therefore W_i^* is isomorphic to the polar variety $W_{\underline{K}(\underline{B_i(z)})}(S_G)$.

Example 2

We are going to consider simultaneously two meagerly generic algebraic families of Zariski open subsets of polar varieties of S, one in the classic and the other in the dual case.

Let m be the $(p-1)$–minor of $J(F_1, \ldots, F_p)$ introduced in Example 1, namely $m := \det \left[\frac{\partial F_k}{\partial X_l}\right]_{1 \le k,l \le p-1}$. We fix now an index $1 \le i \le n-p$ and an $(n-i)$–tupel of rational numbers $\gamma = (\gamma_1, \ldots, \gamma_{n-i})$ with $\gamma_{n-i} \ne 0$.

Let $Z = (Z_{n-i+1}, \ldots, Z_n)$ be an i–tupel of new indeterminates and let $B = B_{(i,\gamma)}$ be the polynomial $((n-p-i+1) \times n)$–matrix defined by

$$B := \begin{bmatrix} O_{n-p-i,\,p-1} & I_{n-p-i} & & O_{n-p-i,\,i+1} \\ \gamma_1 \ldots \gamma_{p-1} & \gamma_p \ldots \gamma_{n-i-1} & \gamma_{n-i} & Z_{n-i+1} \ldots Z_n \end{bmatrix}.$$

By $\underline{B} := \underline{B}_{(i,\gamma)}$ and $\overline{B} := \overline{B}_{(i,\gamma)}$ we denote the polynomial $((n-p-i+1)\times(n+1))$–matrices obtained by adding to B as column number zero the transposed $(n-p-i+1)$–tupel $(0,\ldots,0,0)^T$ and $(0,\ldots,0,1)^T$, respectively. So we have $\underline{B}_* = \overline{B}_* = B$.

Let $E := B(\mathbb{A}^i)$. Then E is a closed irreducible subvariety of $\mathbb{A}^{(n-p-i+1)\times n}$ which is isomorphic to $\mathbb{A}^i$.

From $\gamma_{n-i} \neq 0$ we deduce that for any instance $z \in \mathbb{A}^i$ with $z = (z_{n-i+1},\ldots,z_n)$ the complex $((n-p-i+1)\times n)$–matrix $B_i(z) = \underline{B}(z)_* = \overline{B}(z)_*$ has maximal rank $n-p-i+1$ and this gives rise to two polar varieties of S, one classic and the other dual, namely $W_{\underline{K}(\underline{B(z)})}(S)$ and $W_{\overline{K}(\overline{B(z)})}(S)$.

We shall limit our attention to the algebraic family of Zariski open subsets of dual polar varieties $\{W_{\overline{K}(\overline{B(z)})}(S)_m\}_{z\in\mathbb{A}^i}$. The case of the algebraic family $\{W_{\underline{K}(\underline{B(z)})}(S)_m\}_{z\in\mathbb{A}^i}$ is treated similarly.

We consider now the polynomial $((n-i+1)\times n)$–matrix $N := N(i,\gamma)$ defined by

$$N := \begin{bmatrix} & J(F_1,\ldots,F_p) & \\ O_{n-p-i,\,p-1} & I_{n-p-i} & O_{n-p-i,\,i+1} \\ \gamma_1 - X_1 \ldots \gamma_{p-1} - X_{p-1} & \gamma_p - X_p \ldots \gamma_{n-i-1} - X_{n-i+1} \quad \gamma_{n-i} - X_{n-i} & Z_{n-i+1} - X_{n-i+1} \ldots Z_n - X_n \end{bmatrix}$$

Observe that the $(n-i)$–minor of N, which corresponds to the rows and columns $1,\ldots,n-i$, has value m. For $n-i+1 \le j \le n$ let us denote by $\widetilde{M}_j := \widetilde{M}_j(X,Z)$ the $(n-i+1)$–minor of N that corresponds to the columns number $1,\ldots,n-i$ and j. One verifies immediately for $n-i+1 \le j' \le n$ the identities $\frac{\partial \widetilde{M}_j}{\partial Z_{j'}} = 0$ in case $j' \neq j$ and $\frac{\partial \widetilde{M}_j}{\partial Z_{j'}} = m$ in case $j' = j$.

Let us now consider the polynomial map $\Phi : \mathbb{A}^n_m \times \mathbb{A}^i \to \mathbb{A}^{p+i}$ defined for $(x,z) \in \mathbb{A}^n_m \times \mathbb{A}^i$ by $\Phi(x,z) := (F_1(x),\ldots,F_p(x),\widetilde{M}_{n-i+1}(x,z),\ldots,\widetilde{M}_n(x,z))$. The Jacobian $J(\Phi)(x,z)$ of Φ at (x,z) has the following form:

$$J(\Phi)(x,z) = \begin{bmatrix} J(F_1,\ldots,F_p)(x) & & O_{p,i} & \\ & \frac{\partial \widetilde{M}_{n-i+1}}{\partial Z_{n-i+1}}(x,z) & \cdots & \frac{\partial \widetilde{M}_{n-i+1}}{\partial Z_n}(x,z) \\ * & \vdots & \cdots & \vdots \\ & \frac{\partial \widetilde{M}_n}{\partial Z_{n-i+1}}(x,z) & \cdots & \frac{\partial \widetilde{M}_n}{\partial Z_n}(x,z) \end{bmatrix} =$$

$$= \begin{bmatrix} J(F_1,\ldots,F_p)(x) & & O_{p,i} & & \\ & m(x) & 0 & \cdots & 0 \\ & 0 & m(x) & \cdots & 0 \\ * & \vdots & \vdots & \ddots & \vdots \\ & 0 & 0 & \cdots & m(x) \end{bmatrix}$$

and is of maximal rank $p+i$, since x belongs to $\mathbb{A}^n_m$. In particular $0 \in \mathbb{A}^{p+i}$ is a regular value of Φ. From the Weak Transversality Theorem of Thom–Sard we deduce now that there exists a non–empty Zariski open subset $\widetilde{O}$ of $\mathbb{A}^i$ such that for any point $z \in \widetilde{O}$ the equations

$$F_1(X) = 0, \ldots, F_p(X) = 0,\ \widetilde{M}_{n-i+1}(X, z) = 0, \ldots, \widetilde{M}_n(X, z) = 0 \tag{9}$$

intersect transversally at any of their common solutions in $\mathbb{A}^n_m$.

Let O be the image of $\widetilde{O}$ under the given isomorphism which maps $\mathbb{A}^i$ onto E. Then O is a non–empty Zariski open subset of E.

Fix for the moment a point $z \in \widetilde{O}$. From our previous argumentation and [4], Lemma 1 we conclude that at any point $x \in \mathbb{A}^n_m$ with

$$F_1(x) = 0, \ldots, F_p(x) = 0,\ \widetilde{M}_{n-i+1}(x, z) = 0, \ldots, \widetilde{M}_n(x, z) = 0$$

all $(n-i+1)$–minors of $N(x, z)$ are vanishing.

This means that the equations (9) define the intersection of the dual polar variety $W_{\overline{K}(\overline{B(z)})}(S)$ with $\mathbb{A}^n_m$. Therefore $W_{\overline{K}(\overline{B(z)})}(S)_m$ is either empty or a smooth affine subvariety of S of pure codimension i.

Re–parametrizing the algebraic family of affine varieties $\{W_{\overline{K}(\overline{B(z)})}(S)_m\}_{z\in\widetilde{O}}$ by the non–empty Zariski open subset O of E we obtain finally a meagerly generic algebraic family of empty or smooth Zariski open subsets of polar varieties of S.

Let $\gamma = (\gamma_1, \ldots, \gamma_n)$ be a generically chosen point of $\mathbb{Q}^n$ and assume that the real variety $S_\mathbb{R}$ is smooth. Then we deduce from [5] and [6], Proposition 2 that the generic polar variety $W_{\overline{K}(\overline{\gamma})}(S_\mathbb{R})$ is not empty. Hence, without loss of generality, we may assume $W_{\overline{K}(\overline{\gamma})}(S)_m \neq \emptyset$.

For $1 \leq i \leq n-p$ let $z_i := (\gamma_{n-i+1}, \ldots, \gamma_n)$. Then, similarly as in Example 1, we may argue that

$$W_{\overline{K}(\overline{B(z_1)})}(S)_m \supset \cdots \supset W_{\overline{K}(\overline{B(z_{n-p})})}(S)_m = W_{\overline{K}(\gamma)}(S)_m \tag{10}$$

is a descending chain of smooth affine varieties which are codimension one subvarieties one of the other.

Again, in order to retrieve finitely many real algebraic sample points for the connected components of the real variety $S_\mathbb{R}$, we may proceed like in Example 1, applying the general algorithm of [4, 5, 6] as follows: For any choice of indices $1 \leq h_1 < \cdots < h_p \leq n$ we generate the equations and the inequations that define the descending chain of affine varieties corresponding to (10) in order to find the (finitely many) real points contained in the last one. The set of points obtained in this way is again a set of sample points for the connected components of $S_\mathbb{R}$.

A variant of Example 2 is used in [7] in order to find efficiently smooth algebraic sample points for the (non–degenerated) connected components of singular real hypersurfaces.

5 A computational test

In this short section we try to get some feeling about the sharpness of the bound contained in Proposition 9. To this end we performed a series of computer experiments with generic classic polar varieties of smooth complete intersection manifolds.

More precisely, for a given a triple (n, p, i) of integer parameters with $1 \leq p \leq n-1$ and $1 \leq i \leq n-p$, we did the following:
we

- chose random polynomials $F_1, \ldots, F_p$ of degree two in $\mathbb{Z}[X]$,
- verified that $F_1, \ldots, F_p$ form a regular sequence in $\mathbb{Q}[X]$ whose set of common zeros S contains no singular point,
- chose a random integer $((n-p) \times n)$–matrix $a := [a_{k,l}]_{\substack{1 \leq k \leq n-p-i+1 \\ 1 \leq l \leq n}}$, and
- determined the dimension of the singular locus of the classic polar variety $W_{\underline{K}(\underline{a}^{(i)})}(S)$.

The results of such experiments should be interpreted with caution, since we cannot guarantee that the matrix a satisfies the necessary genericity condition. However, running the procedure for several random choices of a may increase our confidence into the experimental results.

Moreover, the aim was not to test the efficiency of the root finding procedures proposed in [4, 5, 6], but to check experimentally a mathematical thesis (the sharpness of the bound contained in Proposition 9). Therefore we chose a rather modest sample of equation systems and relied on the most comfortable software, disregarding its efficiency.

More specifically, we chose equations of degree two in order to avoid a complexity explosion of our computations. In the same vein, we had to control the bit–size of the coefficients of the polynomials created during our procedure. To get rid of this situation, we performed our computations modulo the prime number $q := 10000000019$, taking into account that for any system $F_1, \ldots, F_p$ of $\mathbb{Z}[X]$ and any integer $((n-p) \times n)$–matrix a there are only finitely many primes r for which the dimension of the singular locus of $W_{\underline{K}(\underline{a}^{(i)})}(S)$ differs from that of its counterpart obtained over the finite field $\mathbb{Z}_r$ by reducing $F_1, \ldots, F_p$ and a modulo r. Hence we guess that the experimental results given here reflect the expected generic behavior of the system $F_1, \ldots, F_p$ over $\mathbb{Q}$.

The computations were performed using the MAGMA package, on a Core2 DUO processor with 4 Gb of RAM. The search space consisted of all values of the triple (n, p, i) with $2 \leq n \leq 11$, $1 \leq p \leq n-1$ and $1 \leq i \leq n-p$. Nevertheless, we had to discard all triples of the form $(10, p, i)$ with $p = 6, \ldots, 9$ and $(11, p, i)$ with $p = 5, \ldots, 10$, since computations ran out of memory.
The experimentation showed that in the hypersurface case, i.e., $p = 1$, the resulting polar varieties were always smooth, as expected. For $p > 1$ we found that the dimension of the singular locus of the polar varieties was always $\max\{-1, n-p-(2i+2)\}$, where the dimension of the empty set is defined by -1. In other words, in case $(2i+2) > n-p$, the singular locus was empty, as predicted in Proposition 9, and in case $(2i+2) < n-p$, the experimentation returned exactly the value $n-p-(2i+2)$ as the dimension of the singular locus.

It turned out that in case $2i+2 \leq n-p$ the singular locus of the polar variety under consideration contained always a real subvariety of the codimension $2i+2$.

Observe that by Lemma 12 the value $2i+2$ coincides with the expected codimension of the stratum $\Sigma^{(i+1)}(\pi_{a^{(i)}})$, in case $2i+2 \leq n-p$. This finding suggests that in this case $\bigcup_{i<j\leq n-p} \Sigma^{(j)}(\pi_{a^{(i)}}))$ contains the singular locus of $W_{\underline{K}(\underline{a}^{(i)})} \cap S_{reg}$.

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