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"Introduction to Deep Inference and Proof Nets"

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These are the notes for the third lecture (written by Lutz Straßburger).

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1 Sequent Calculus for MLL

In a previous lecture you have seen the sequent calculus GS1p for classical logic. In this lecture we remove the rules for weakening and contraction. The result is called *unit-free multiplicative linear logic*. Since this is a different logic, there is also a different notation. Conjunction is written as $\otimes$, disjunction as $\parr$, and negation as $(-)^{\perp}$. What we get is the following system

$$\mathsf{id}\,\frac{}{\vdash a, a^{\perp}} \qquad \otimes\,\frac{\vdash A, \Gamma \quad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta} \qquad \parr\,\frac{\vdash A, B, \Gamma}{\vdash A \parr B, \Gamma} \tag{1}$$

We consider sequents as multisets, i.e., order does not matter. The system in (1) is called MLL^{-}, where the $-$ indicates the fact that the system is unit-free. For adding the units $\perp$ and $\mathbf{1}$ of linear logic, which correspond to *false* and *true* in classical logic, we need to add the rules

$$\mathbf{1}\,\frac{}{\vdash \mathbf{1}} \qquad \text{and} \qquad \perp\,\frac{\vdash \Gamma}{\vdash \perp, \Gamma} \tag{2}$$

The system consisting of the rules in (1) and (2) is denoted by MLL. The logic is called *multiplicative linear logic*.

Note that in MLL-formulas negation is only allowed at the atomic level, but we can define it inductively for all formulas via the deMorgan laws:

$$a^{\perp\perp} = a \qquad \mathbf{1}^{\perp} = \perp \qquad \perp^{\perp} = \mathbf{1} \qquad (A \otimes B)^{\perp} = A^{\perp} \parr B^{\perp} \qquad [A \parr B]^{\perp} = A^{\perp} \otimes B^{\perp} \tag{3}$$

This allows us to write the cut rule as

$$\mathsf{cut}\,\frac{\vdash A, \Gamma \quad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}$$

As for classical logic, we have that the id-rule can be reduced to atoms, but the cut-rule cannot.

1.1 Proposition *The the general rule*

$$\mathsf{id}\,\frac{}{\vdash A, A^{\perp}}$$

is derivable in MLL.

Proof: We proceed by structural induction on A. If A is an atom, then we are done. If A is a unit, then we replace

$$\mathsf{id}\,\frac{}{\vdash \bot, \mathbf{1}} \qquad \text{by} \qquad \bot\,\frac{\mathbf{1}\,\dfrac{}{\vdash \mathbf{1}}}{\vdash \bot, \mathbf{1}}$$

If A is a compound formula, say $A = B \otimes C$, then we replace

$$\mathsf{id}\,\frac{}{\vdash B \otimes C, B^{\perp} \mathbin{⅋} C^{\perp}} \qquad \text{by} \qquad ⅋\,\frac{\otimes\,\dfrac{\mathsf{id}\,\dfrac{}{\vdash B, B^{\perp}} \quad \mathsf{id}\,\dfrac{}{\vdash C, C^{\perp}}}{\vdash B \otimes C, B^{\perp}, C^{\perp}}}{\vdash B \otimes C, B^{\perp} \mathbin{⅋} C^{\perp}}$$

and apply the induction hypothesis. If $A = B \mathbin{⅋} C$ we proceed similarly. □

As before, we have the cut-elimination theorem.

1.2 Theorem *If a sequent* $\vdash \Gamma$ *is provable in* $\mathsf{MLL} + \mathsf{cut}$*, then it is provable in* MLL *without cut.*

The proof of this theorem is for linear logic much simpler than for classical logic. For this reason we can show it here in full. We define the *size* of a proof Π, denoted by $\mathsf{size}(\Pi)$ to be the number of rule applications in Π. Now we begin by showing the following lemma:

1.3 Lemma *A proof of the shape*

$$\mathsf{cut}\,\frac{\overset{\Pi_1}{\vdash A, \Gamma} \qquad \overset{\Pi_2}{\vdash A^{\perp}, \Delta}}{\vdash \Gamma, \Delta} \tag{4}$$

where Π_1 *and* Π_2 *are both cut-free, can be transformed into a cut-free proof*

$$\overset{\Pi_3}{\vdash \Gamma, \Delta} \tag{5}$$

such that $\mathsf{size}(\Pi_3) < \mathsf{size}(\Pi_1) + \mathsf{size}(\Pi_2) + 1$.

Proof: We do this by induction on the size of the proof in (4), i.e., $\mathsf{size}(\Pi_1) + \mathsf{size}(\Pi_2) + 1$. We now proceed by a case analysis on the bottommost rules appearing in Π_1 and Π_2. If these rules do not interfere with the cut, we can permute them down, as in the following cases:

$$\mathsf{cut}\,\dfrac{\bot\,\dfrac{\begin{array}{c}\Pi_1'\\ \vdash A,\Gamma'\end{array}}{\vdash A,\bot,\Gamma'} \qquad \begin{array}{c}\Pi_2\\ \vdash A^\bot,\Delta\end{array}}{\vdash \bot,\Gamma',\Delta} \quad\rightarrow\quad \bot\,\dfrac{\mathsf{cut}\,\dfrac{\begin{array}{c}\Pi_1'\\ \vdash A,\Gamma'\end{array} \qquad \begin{array}{c}\Pi_2\\ \vdash A^\bot,\Delta\end{array}}{\vdash \Gamma',\Delta}}{\vdash \bot,\Gamma',\Delta} \tag{6}$$

$$\mathsf{cut}\,\dfrac{\parr\,\dfrac{\begin{array}{c}\Pi_1'\\ \vdash A,C,D,\Gamma'\end{array}}{\vdash A,C\parr D,\Gamma'} \qquad \begin{array}{c}\Pi_2\\ \vdash A^\bot,\Delta\end{array}}{\vdash C\parr D,\Gamma',\Delta} \quad\rightarrow\quad \parr\,\dfrac{\mathsf{cut}\,\dfrac{\begin{array}{c}\Pi_1'\\ \vdash A,C,D,\Gamma'\end{array} \qquad \begin{array}{c}\Pi_2\\ \vdash A^\bot,\Delta\end{array}}{\vdash C,D,\Gamma',\Delta}}{\vdash C\parr D,\Gamma',\Delta} \tag{7}$$

$$\mathsf{cut}\,\dfrac{\otimes\,\dfrac{\begin{array}{c}\Pi_1'\\ \vdash C,\Gamma'\end{array} \qquad \begin{array}{c}\Pi_1''\\ \vdash A,D,\Gamma''\end{array}}{\vdash A,C\otimes D,\Gamma',\Gamma''} \qquad \begin{array}{c}\Pi_2\\ \vdash A^\bot,\Delta\end{array}}{\vdash C\otimes D,\Gamma',\Gamma'',\Delta} \quad\rightarrow\quad \otimes\,\dfrac{\begin{array}{c}\Pi_1'\\ \vdash C,\Gamma'\end{array} \qquad \mathsf{cut}\,\dfrac{\begin{array}{c}\Pi_1''\\ \vdash A,D,\Gamma''\end{array} \qquad \begin{array}{c}\Pi_2\\ \vdash A^\bot,\Delta\end{array}}{\vdash D,\Gamma'',\Delta}}{\vdash C\otimes D,\Gamma',\Gamma'',\Delta} \tag{8}$$

And similarly for Π_2. In all these cases we can apply the induction hypothesis because the sum of the sizes of the proofs above the cut has been decreased. Note also that in all three cases the total size of the proof is not changed. In the literature on cut-elimination, cases like (6), (7), and (8) are called *commutative cases*. Let us now look at the cases where the rules above the cut apply to the formulas introduced by the cut. In the literature on cut-elimination, such cases are called *key cases*. For MLL, there are three key cases:

$$\mathsf{cut}\,\dfrac{\mathsf{i}{\downarrow}\,\dfrac{}{\vdash a,a^\bot} \qquad \begin{array}{c}\Pi\\ \vdash a^\bot,\Delta\end{array}}{\vdash a^\bot,\Delta} \quad\rightarrow\quad \begin{array}{c}\Pi\\ \vdash a^\bot,\Delta\end{array} \tag{9}$$

$$\mathsf{cut}\,\dfrac{\mathsf{i}{\downarrow}\,\dfrac{}{\vdash \mathbf{1}} \qquad \bot\,\dfrac{\begin{array}{c}\Pi\\ \vdash \Delta\end{array}}{\vdash \bot,\Delta}}{\vdash \Delta} \quad\rightarrow\quad \begin{array}{c}\Pi\\ \vdash \Delta\end{array} \tag{10}$$

$$\mathsf{cut}\,\dfrac{\otimes\,\dfrac{\overset{\Pi'_1}{\vdash A,\Gamma'}\qquad \overset{\Pi''_1}{\vdash B,\Gamma''}}{\vdash A\otimes B,\Gamma',\Gamma''}\qquad \parr\,\dfrac{\overset{\Pi'_2}{\vdash A^{\perp},B^{\perp},\Delta}}{\vdash A^{\perp}\parr B^{\perp},\Delta}}{\vdash \Gamma',\Gamma'',\Delta}\quad\rightarrow\quad \mathsf{cut}\,\dfrac{\overset{\Pi'_1}{\vdash A,\Gamma'}\qquad \mathsf{cut}\,\dfrac{\overset{\Pi''_1}{\vdash B,\Gamma_2}\qquad \overset{\Pi'_2}{\vdash A^{\perp},B^{\perp},\Delta}}{\vdash A^{\perp},\Gamma'',\Delta}}{\vdash \Gamma',\Gamma'',\Delta} \tag{11}$$

Note that in all three cases the total size of the proof is strictly decreased. In the first two cases the cut disappears. In case (11), the cut is replaced by two cuts, which means we need a slightly more sophisticated argument: First, note that we can apply the induction hypothesis to the proof

$$\mathsf{cut}\,\dfrac{\overset{\Pi''_1}{\vdash B,\Gamma_2}\qquad \overset{\Pi'_2}{\vdash A^{\perp},B^{\perp},\Delta}}{\vdash A^{\perp},\Gamma'',\Delta}$$

because $\mathsf{size}(\Pi''_1)+\mathsf{size}(\Pi'_2)+1<\mathsf{size}(\Pi'_1)+\mathsf{size}(\Pi''_1)+\mathsf{size}(\Pi'_2)+3$. This gives us a proof

$$\overset{\Pi''_2}{\vdash A^{\perp},\Gamma'',\Delta}$$

with

$$\mathsf{size}(\Pi''_2)<\mathsf{size}(\Pi''_1)+\mathsf{size}(\Pi'_2)+1 \quad .$$

Hence, we also have

$$\mathsf{size}(\Pi'_1)+\mathsf{size}(\Pi''_2)+1<\mathsf{size}(\Pi'_1)+\mathsf{size}(\Pi''_1)+\mathsf{size}(\Pi'_2)+3 \quad .$$

This means we can apply the induction hypothesis again to

$$\mathsf{cut}\,\dfrac{\overset{\Pi'_1}{\vdash A,\Gamma'}\qquad \overset{\Pi''_2}{\vdash A^{\perp},\Gamma'',\Delta}}{\vdash \Gamma',\Gamma'',\Delta}$$

which gives us a cut-free proof

$$\overset{\Pi_3}{\vdash \Gamma,\Delta} \tag{12}$$

such that

$$\begin{array}{rcl}\mathsf{size}(\Pi_3) & < & \mathsf{size}(\Pi_1') + \mathsf{size}(\Pi_2'') + 1\\ & < & \mathsf{size}(\Pi_1') + \mathsf{size}(\Pi_1'') + \mathsf{size}(\Pi_2') + 3\\ & = & \mathsf{size}(\Pi_1) + \mathsf{size}(\Pi_2) + 1\end{array}$$

This completes the proof of the lemma.

Proof (of Theorem 1.2): The statement of the theorem now follows from Lemma 1.3 by an induction on the number of cuts in the proof of $\vdash \Gamma$. □

1.4 Remark The system MLL is an exceptionally simple case for cut elimination. In most other logics, the size of the proof *does not* decrease during cut elimination. Usually there is an exponential or even hyper-exponential blow-up of the proof when cut elimination is applied. This means one has to find more sophisticated induction measures.

2 Calculus of Structures for MLL

In the calculus of structures, multiplicative linear logic is given by the following system:

$$\mathsf{ai}\!\downarrow \frac{S\{\mathbf{1}\}}{S\{a \parr a^\perp\}} \qquad\qquad \mathsf{s}\,\frac{S\{[A \parr B] \otimes C\}}{S\{A \parr (B \otimes C)\}} \tag{13}$$

which we will call MLS. As before, we consider formulas equivalent modulo the following equations:

$$\begin{array}{ccc}(A \otimes (B \otimes C)) = ((A \otimes B) \otimes C) & (A \otimes B) = (B \otimes A) & (A \otimes \mathbf{1}) = A\\ [A \parr [B \parr C]] = [[A \parr B] \parr C] & [A \parr B] = [B \parr A] & [A \parr \bot] = A\end{array} \tag{14}$$

A *proof* in this system is a derivation with premise $\mathbf{1}$. A formula A is *provable* if there is a proof Π with conclusion A. We denote this by

$$\begin{array}{c}\mathbf{1}\\ \mathsf{MLS}\,\Big\Vert\,\Pi\\ A\end{array} \qquad \text{or simply by} \qquad \begin{array}{c}\mathsf{MLS}\,\Big\Vert\!\!\top\,\Pi\\ A\end{array}$$

The cut rule is

$$\mathsf{ai}\!\uparrow \frac{S\{a \otimes a^\perp\}}{S\{\bot\}} \tag{15}$$

The the calculus of structures, the cut can be reduced to atomic form, which is not possible in the sequent calculus. The general form of the rules $\mathsf{ai}\!\downarrow$ and $\mathsf{ai}\!\uparrow$ are

$$\mathsf{i}\!\downarrow \frac{S\{\mathbf{1}\}}{S\{A \parr A^\perp\}} \qquad \text{and} \qquad \mathsf{i}\!\uparrow \frac{S\{A \otimes A^\perp\}}{S\{\bot\}} \tag{16}$$

2.1 Proposition *The rule* $\mathsf{i}\!\downarrow$ *is derivable in* $\{\mathsf{ai}\!\downarrow, \mathsf{s}\}$, *and the rule* $\mathsf{i}\!\uparrow$ *is derivable in* $\{\mathsf{ai}\!\uparrow, \mathsf{s}\}$.

Proof: The proof is very similar to the proof of Proposition 1.1. For $\mathsf{i}{\downarrow}$, the inductive cases are

$$\mathsf{i}{\downarrow}\,\frac{S\{\mathbf{1}\}}{S\{\bot \mathbin{⅋} \mathbf{1}\}} \qquad \rightarrow \qquad =\frac{S\{\mathbf{1}\}}{S\{\bot \mathbin{⅋} \mathbf{1}\}}$$

and

$$\mathsf{i}{\downarrow}\,\frac{S\{\mathbf{1}\}}{S\{(B\otimes C)\mathbin{⅋} B^{\bot}\mathbin{⅋} C^{\bot}\}} \qquad \rightarrow \qquad \mathsf{s}\,\frac{\mathsf{i}{\downarrow}\,\dfrac{=\dfrac{\mathsf{i}{\downarrow}\,\dfrac{S\{\mathbf{1}\}}{S\{C\mathbin{⅋} C^{\bot}\}}}{S\{(\mathbf{1}\otimes C)\mathbin{⅋} C^{\bot}\}}}{S\{([B\mathbin{⅋} B^{\bot}]\otimes C)\mathbin{⅋} C^{\bot}\}}}{S\{(B\otimes C)\mathbin{⅋} B^{\bot}\mathbin{⅋} C^{\bot}\}}$$

The cases for $\mathsf{i}{\uparrow}$ are dual. □

The system $\mathsf{MLS} + \mathsf{ai}{\uparrow}$ will be called SMLS. For this system, we have the cut elimination theorem:

2.2 Theorem *If a formula A is provable in SMLS, then it is provable in MLS.*

We can prove this theorem either by using the sequent calculus cut elimination, or by giving a direct proof in the calculus of structures. We show here both proofs. Before that, let us see some interesting consequences.

2.3 Corollary *The rule $\mathsf{i}{\uparrow}$ is admissible in MLS.*

Proof: Suppose we have a proof

$$\begin{array}{c} \\ \mathsf{MLS}\cup\{\mathsf{i}{\uparrow}\} \,\Big\Vert\, \Pi \\ A \end{array}$$

By Proposition 2.1, this can be transformed into a proof

$$\begin{array}{c} \\ \mathsf{SMLS} \,\Big\Vert\, \Pi' \\ A \end{array}$$

To this we apply Theorem 2.2. □

2.4 Corollary *For all formulas A and B, we have*

$$\begin{array}{c} A \\ \mathsf{SMLS}\,\Big\Vert\,\Pi_1 \\ B \end{array} \qquad \textit{if and only if} \qquad \begin{array}{c} \\ \mathsf{MLS}\,\Big\Vert\,\Pi_2 \\ [A^{\bot}\mathbin{⅋} B] \end{array}$$

Proof: From

$$\begin{array}{c} A \\ \mathsf{SMLS}\,\Big\Vert\,\Pi_1 \\ B \end{array}$$

we can obtain

$$\begin{array}{c} \mathsf{i}{\downarrow}\,\dfrac{\mathbf{1}}{[A^{\perp} \parr A]} \\ \mathsf{SMLS}\,\Big\Vert\,\Pi_1 \\ [A^{\perp} \parr B] \end{array}$$

Via Proposition 2.1, we obtain

$$\begin{array}{c} \mathsf{SMLS}\,\overline{\Big\Vert} \\ [A^{\perp} \parr B] \end{array}$$

By Theorem 2.2 we get

$$\begin{array}{c} \mathsf{MLS}\,\overline{\Big\Vert}\,\Pi_2 \\ [A^{\perp} \parr B] \end{array}$$

Conversely, from

$$\begin{array}{c} \mathsf{MLS}\,\overline{\Big\Vert}\,\Pi_2 \\ [A^{\perp} \parr B] \end{array}$$

we can construct

$$\begin{array}{c} =\,\dfrac{A}{(A \otimes \mathbf{1})} \\ \mathsf{MLS}\,\Big\Vert\,\Pi_2 \\ \mathsf{s}\,\dfrac{(A \otimes [A^{\perp} \parr B])}{\mathsf{i}{\uparrow}\,\dfrac{[(A \otimes A^{\perp}) \parr B]}{=\,\dfrac{[\perp \parr B]}{B}}} \end{array}$$

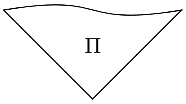
From which we get

$$\begin{array}{c} A \\ \mathsf{SMLS}\,\Big\Vert\,\Pi_1 \\ B \end{array}$$

by applying Proposition 2.1. ☐

Now, let us establish the relation between the systems MLL and MLS.

2.5 Proposition *If there is a proof*



$$\vdash A_1, \ldots, A_n$$

in MLL*, then there is a proof*

$$\begin{array}{c} \mathsf{MLS}\,\Big\|\,\Pi' \\ [A_1 \parr \cdots \parr A_n] \end{array} \quad .$$

Proof: We proceed by induction on the size of the proof Π, and make a case analysis on the bottommost rule instance in Π:

$$\mathsf{id}\,\frac{}{\vdash a, a^\perp} \qquad \rightarrow \qquad \mathsf{ai}\!\downarrow \frac{\mathbf{1}}{[a \parr a^\perp]}$$

$$\mathbf{1}\,\frac{}{\vdash \mathbf{1}} \qquad \rightarrow \qquad = \frac{\mathbf{1}}{\mathbf{1}}$$

$$\perp\,\frac{\begin{array}{c}\overset{\Pi_1}{\bigtriangledown} \\ \vdash A_2, \ldots, A_n\end{array}}{\vdash \perp, A_2, \ldots, A_n} \qquad \rightarrow \qquad = \frac{\begin{array}{c} \mathsf{MLS}\,\Big\|\,\Pi'_1 \\ [A_2 \parr \cdots \parr A_n] \end{array}}{[\perp \parr A_2 \parr \cdots \parr A_n]}$$

$$\parr\,\frac{\begin{array}{c}\overset{\Pi_1}{\bigtriangledown} \\ \vdash A'_1, A''_1, A_2, \ldots, A_n\end{array}}{\vdash [A'_1 \parr A''_1], A_2, \ldots, A_n} \qquad \rightarrow \qquad \begin{array}{c} \mathsf{MLS}\,\Big\|\,\Pi'_1 \\ [A'_1 \parr A''_1 \parr A_2 \parr \cdots \parr A_n] \end{array}$$

$$\otimes\,\frac{\begin{array}{c}\overset{\Pi_1}{\bigtriangledown} \\ \vdash A'_1, A_2, \ldots, A_k\end{array} \quad \begin{array}{c}\overset{\Pi_2}{\bigtriangledown} \\ \vdash A''_1, A_{k+1}, \ldots, A_n\end{array}}{\vdash (A'_1 \otimes A''_1), A_2, \ldots, A_k, A_{k+1}, \ldots, A_n} \qquad \rightarrow \qquad \mathsf{s}\,\frac{\begin{array}{c} \mathsf{MLS}\,\Big\|\,\Pi'_2 \\ = \dfrac{[A''_1 \parr A_{k+1} \parr \cdots \parr A_n]}{[(\mathbf{1} \otimes A''_1) \parr A_{k+1} \parr \cdots \parr A_n]} \\ \mathsf{MLS}\,\Big\|\,\Pi'_1 \\ [([A'_1 \parr A_2 \parr \cdots \parr A_k] \otimes A''_1) \parr A_{k+1} \parr \cdots \parr A_n] \end{array}}{[(A'_1 \otimes A''_1) \parr A_2 \parr \cdots \parr A_k \parr A_{k+1} \parr \cdots \parr A_n]}$$

In all cases the derivations Π'_1 and Π'_2 are obtained via the induction hypothesis from Π_1 and Π_2. □

2.6 Proposition *If there is a proof*

$$\begin{array}{c}\overset{\Pi}{\bigtriangledown} \\ \vdash A_1, \ldots, A_n\end{array}$$

in MLL + cut*, then there is a proof*

$$\begin{array}{c} \mathsf{SMLS}\,\Big\|\,\Pi' \\ [A_1 \parr \cdots \parr A_n] \end{array} \quad .$$

Proof: The proof is the same as the previous one. We only need to add the case for the cut:

$$\otimes\,\dfrac{\begin{array}{c}\Pi_1\\ \vdash B, A_1,\dots,A_k\end{array}\qquad\begin{array}{c}\Pi_2\\ \vdash B^\perp, A_{k+1},\dots,A_n\end{array}}{\vdash A_1,\dots,A_k,A_{k+1},\dots,A_n}\quad\rightarrow\quad \mathsf{i}\!\uparrow\dfrac{\mathsf{s}\,\dfrac{\begin{array}{c}=\dfrac{\begin{array}{c}\mathsf{MLS}\,\Big\Vert\,\Pi'_2\\ {[B^\perp\parr A_{k+1}\parr\cdots\parr A_n]}\end{array}}{[(\mathbf{1}\otimes B^\perp)\parr A_{k+1}\parr\cdots\parr A_n]}\\ \mathsf{MLS}\,\Big\Vert\,\Pi'_1\\ {[([B\parr A_1\parr\cdots\parr A_k]\otimes B^\perp)\parr A_{k+1}\parr\cdots\parr A_n]}\end{array}}{[(B\otimes B^\perp)\parr A_1\parr\cdots\parr A_k\parr A_{k+1}\parr\cdots\parr A_n]}}{[A_1\parr\cdots\parr A_k\parr A_{k+1}\parr\cdots\parr A_n]}$$

Finally, we need to apply Proposition 2.1. □

2.7 Proposition *If there is a proof*

$$\begin{array}{c}\mathsf{SMLS}\,\Big\Vert\,\Pi\\ Q\end{array}\quad,$$

then there is a proof

$$\begin{array}{c}\Pi'\\ \vdash Q\end{array}$$

in MLL + cut.

Proof: Again, we proceed by induction on the size of Π, and consider the bottommost rule instance in Π:

$$\rho\,\dfrac{\begin{array}{c}\Big\Vert\,\Pi_1\\ Q_1\end{array}}{Q}$$

By induction hypothesis, there is a proof

$$\begin{array}{c}\Pi'_1\\ \vdash Q_1\end{array}$$

in MLL + cut. Now we show that there is also a proof

$$\begin{array}{c}\Pi'_2\\ \vdash Q_1^\perp, Q\end{array}$$

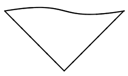
in MLL + cut, from which we can then construct Π′:

$$\mathsf{cut}\,\dfrac{\begin{array}{c}\Pi'_1\\ \vdash Q_1\end{array}\qquad\begin{array}{c}\Pi'_2\\ \vdash Q_1^\perp, Q\end{array}}{\vdash Q}$$

For constructing Π_2', we first show for every rule

$$\rho\,\frac{S\{A\}}{S\{B\}}$$

there is a proof



$$\vdash A^\perp, B$$

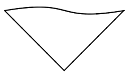
For $\mathsf{ai}{\downarrow}$ and $\mathsf{ai}{\uparrow}$, we have

$$\bot\,\dfrac{\parr\,\dfrac{\mathsf{id}\,\dfrac{}{\vdash a, a^\perp}}{\vdash a \parr a^\perp}}{\vdash \bot, a \parr a^\perp}$$

For s, we have

$$\parr\,\dfrac{\parr\,\dfrac{\otimes\,\dfrac{\mathsf{id}\,\dfrac{}{\vdash A^\perp, A} \qquad \otimes\,\dfrac{\mathsf{id}\,\dfrac{}{\vdash B^\perp, B} \qquad \mathsf{id}\,\dfrac{}{\vdash C^\perp, C}}{\vdash B^\perp, C^\perp, A, B \otimes C}}{\vdash A^\perp \otimes B^\perp, C^\perp, A, B \otimes C}}{\vdash A^\perp \otimes B^\perp, C^\perp, A \parr (B \otimes C)}}{\vdash (A^\perp \otimes B^\perp) \parr C^\perp, A \parr (B \otimes C)}$$

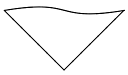
Similarly, we have to show for the equations in (14) that whenever $A = B$, then there is a proof



$$\vdash A^\perp, B$$

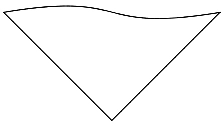
We leave this as an exercise. Finally, it remains to show that for every positive context $S\{\ \}$, we have

If



$$\vdash A^\perp, B$$

then



$$\vdash S\{A\}^\perp, S\{B\}$$

For this, we proceed by induction on the structure of $S\{\ \}$. The inductive case is

$$
\parr\,\dfrac{\otimes\,\dfrac{\mathsf{id}\,\dfrac{}{\vdash C^{\perp}, C} \qquad \overset{\Pi''}{\vdash S'\{A\}^{\perp}, S'\{B\}}}{\vdash C^{\perp} \otimes S'\{A\}^{\perp}, C, S\{B\}}}{\vdash C^{\perp} \otimes S'\{A\}^{\perp}, C \parr S\{B\}}
$$

where Π'' exists by induction hypothesis. □

Now we are ready for the first proof of Theorem 2.2:

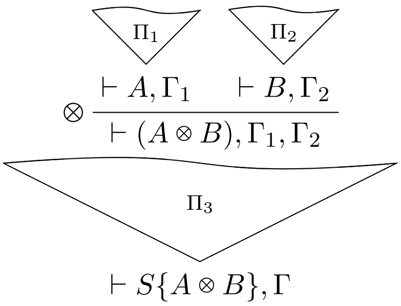
Proof (First proof of Theorem 2.2): A given proof in SMLS is first transformed into a proof in MLL + cut (by Proposition 2.7). To this proof we apply cut-elimination in the sequent calculus (Theorem 1.2). The result is translated into a proof in MLS (via Proposition 2.5). □

3 Splitting

The key argument for proving cut elimination in the sequent calculus (Theorem 1.2) relies on the following property: when the principal formulas in a cut are active in both branches, they determine which rules are applied immediately above the cut. This is a consequence of the fact that formulas have a root connective, and logical rules only hinge on that, and nowhere else in the formula.

This property does not necessarily hold in the calculus of structures. Further, since rules can be applied anywhere deep inside structures, everything can happen above a cut. This complicates considerably the task of proving cut elimination. On the other hand, a great simplification is made possible in the calculus of structures by the reduction of cut to its atomic form, which happens simply and independently of cut elimination. The remaining difficulty is actually understanding what happens, while going up in a proof, *around* the atoms produced by an atomic cut. The two atoms of an atomic cut can be produced inside any structure, and they do not belong to distinct branches, as in the sequent calculus: complex interactions with their context are possible. The solution that we show here is called *splitting*.

It can be best understood by looking again at the sequent calculus. If we have an MLL-proof of the sequent $\vdash S\{A \otimes B\}, \Gamma$, where $S\{A \otimes B\}$ is a formula that contains the subformula $(A \otimes B)$, we know for sure that somewhere in the proof there is one and only one instance of the $\otimes$ rule, which splits A and B along with their context. This is indicated below:



corresponds to

$$\begin{array}{c} \Vert\, \Pi_1 \\ [A \parr \Gamma_1] \\ \Vert\, \Pi_2 \\ \mathsf{s}\,\dfrac{([A \parr \Gamma_1] \otimes [B \parr \Gamma_2])}{\mathsf{s}\,\dfrac{[([A \parr \Gamma_1] \otimes B) \parr \Gamma_2]}{[(A \otimes B) \parr \Gamma_1 \parr \Gamma_2]}} \\ \Vert\, \Pi_3 \\ [S\{A \otimes B\} \parr \Gamma] \end{array} \tag{17}$$

We can consider, as shown at the left, the proof for the given sequent as composed of three pieces, Π_1, Π_2 and Π_3. In the calculus of structures, many different proofs correspond to the sequent calculus one: they differ for the possible sequencing of rules, and because rules in the calculus of structures have smaller granularity and larger applicability. But, among all these proofs, there must also be one that fits the scheme at the right in (17). This precisely illustrates the idea behind the splitting technique.

The derivation Π_3 above implements a *context reduction* and a proper splitting. We can state, in general, these principles separately as follows:

1. Context reduction: The idea of context reduction is to reduce a problem that concerns an arbitrary (deep) context $S\{\ \}$ to a problem that concerns only a shallow context $[\{\ \} \parr U]$. In the case of cut elimination, for example, we will then be able to apply splitting. In the example above, $[S\{\ \} \parr \Gamma]$ is reduced to $[\{\ \} \parr \Gamma']$, for some Γ'.

2. Splitting: In the example above Γ' is reduced to $[\Gamma_1 \parr \Gamma_2]$. More generally, if $[(A \otimes B) \parr K]$ is provable, then K can be reduced to $[K_A \parr K_B]$, such that $[A \parr K_A]$ and $[B \parr K_B]$ are provable.

Context reduction is proved by splitting, which is at the core of the matter.

3.1 Lemma (Splitting) *Let A, B, K be formulas. If there is a derivation*

$$\begin{array}{c} \mathsf{MLS}\,\Vert\, \Pi \\ [(A \otimes B) \parr K] \end{array}$$

then there are formulas K_A and K_B such that

$$\begin{array}{c} [K_A \parr K_B] \\ \mathsf{MLS}\,\Vert\, \Pi_K \\ K \end{array} \quad \textit{and} \quad \begin{array}{c} \mathsf{MLS}\,\Vert\, \Pi_A \\ [A \parr K_A] \end{array} \quad \textit{and} \quad \begin{array}{c} \mathsf{MLS}\,\Vert\, \Pi_B \\ [B \parr K_B] \end{array}$$

where $\mathsf{size}(\Pi_A) + \mathsf{size}(\Pi_B) < \mathsf{size}(\Pi)$.

Proof: We proceed by induction on the size of Π. We consider the bottommost rule instance ρ in the proof Π. There are three diffent types of cases:

(a) Assume ρ is applied inside A. Then Π is

$$\begin{array}{c}\mathsf{MLS}\,\Big\|\,\Pi' \\ \rho\,\dfrac{[(A'\otimes B)\parr K]}{[(A\otimes B)\parr K]}\end{array}$$

and we can apply the induction hypothesis to Π' because it has shorter length than Π. Hence, we get

$$\begin{array}{c}[K_{A'}\parr K_B] \\ \mathsf{MLS}\,\Big\|\,\Pi_K \\ K\end{array} \qquad \text{and} \qquad \begin{array}{c}\mathsf{MLS}\,\Big\|\,\Pi_{A'} \\ \rho\,\dfrac{[A'\parr K_{A'}]}{[A\parr K_{A'}]}\end{array} \qquad \text{and} \qquad \begin{array}{c}\mathsf{MLS}\,\Big\|\,\Pi_B \\ [B\parr K_B]\end{array}$$

We have

$$\begin{array}{rcl}\mathsf{size}(\Pi_A)+\mathsf{size}(\Pi_B) & = & \mathsf{size}(\Pi_{A'})+1+\mathsf{size}(\Pi_B) \\ & < & \mathsf{size}(\Pi')+1 \\ & = & \mathsf{size}(\Pi)\end{array}$$

If ρ applies inside B or inside K, the situation is similar.

(b) The second type of case appears when the subformula $(A\otimes B)$ remains untouched by ρ. This means ρ is s. The most general form of this case is

$$\begin{array}{c}\mathsf{MLS}\,\Big\|\,\Pi' \\ \mathsf{s}\,\dfrac{[([(A\otimes B)\parr K_1\parr K_3]\otimes K_2)\parr K_4]}{[(A\otimes B)\parr (K_1\otimes K_2)\parr K_3\parr K_4]}\end{array}$$

Since the length of Π' is smaller than the length of Π, we can apply the induction hypothesis to Π'. This gives us

$$\begin{array}{c}[Q_1\parr Q_2] \\ \mathsf{MLS}\,\Big\|\,\Pi_1 \\ K_4\end{array} \qquad \text{and} \qquad \begin{array}{c}\mathsf{MLS}\,\Big\|\,\Pi_2 \\ [(A\otimes B)\parr K_1\parr K_3\parr Q_1]\end{array} \qquad \text{and} \qquad \begin{array}{c}\mathsf{MLS}\,\Big\|\,\Pi_3 \\ [K_2\parr Q_2]\end{array}$$

where $\mathsf{size}(\Pi_2)+\mathsf{size}(\Pi_3)<\mathsf{size}(\Pi')$. In particular, we have $\mathsf{size}(\Pi_2)<\mathsf{size}(\Pi')$. Hence we can apply the induction hypothesis to Π_2. From this we get

$$\begin{array}{c}[K_A\parr K_B] \\ \mathsf{MLS}\,\Big\|\,\Pi_4 \\ [K_1\parr K_3\parr Q_1]\end{array} \qquad \text{and} \qquad \begin{array}{c}\mathsf{MLS}\,\Big\|\,\Pi_A \\ [A\parr K_A]\end{array} \qquad \text{and} \qquad \begin{array}{c}\mathsf{MLS}\,\Big\|\,\Pi_B \\ [B\parr K_B]\end{array}$$

where $\mathsf{size}(\Pi_A) + \mathsf{size}(\Pi_B) < \mathsf{size}(\Pi_2) < \mathsf{size}(\Pi)$ and we can build Π_K as follows:

$$\begin{array}{c}
[K_A \parr K_B] \\
\mathsf{MLS}\,\Big\Vert\,\Pi_4 \\
= \dfrac{[K_1 \parr K_3 \parr Q_1]}{[(K_1 \otimes \mathbf{1}) \parr K_3 \parr Q_1]} \\
\mathsf{MLS}\,\Big\Vert\,\Pi_3 \\
\mathsf{s}\,\dfrac{[(K_1 \otimes [K_2 \parr Q_2]) \parr K_3 \parr Q_1]}{[(K_1 \otimes K_2) \parr K_3 \parr Q_1 \parr Q_2]} \\
\mathsf{MLS}\,\Big\Vert\,\Pi_1 \\
[(K_1 \otimes K_2) \parr K_3 \parr K_4]
\end{array}$$

"Morally", this case is similar to the commutative cases in the sequent calculus.

(c) Finally, we have consider the situations where the subformula $(A \otimes B)$ is destroyed by ρ. Again this means ρ is s. The most general form of this case is

$$\begin{array}{c}
\mathsf{MLS}\,\Big\Vert\,\Pi' \\
\mathsf{s}\,\dfrac{[([(A_1 \otimes B_1) \parr K_1] \otimes A_2 \otimes B_2) \parr K_2]}{[(A_1 \otimes A_2 \otimes B_1 \otimes B_2) \parr K_1 \parr K_2]}
\end{array}$$

For the same reasons as before, we can apply the induction hypothesis to Π':

$$\begin{array}{c}
[Q_1 \parr Q_2] \\
\mathsf{MLS}\,\Big\Vert\,\Pi_1 \\
K_2
\end{array}
\qquad\text{and}\qquad
\begin{array}{c}
\mathsf{MLS}\,\Big\Vert\,\Pi_2 \\
[(A_1 \otimes B_1) \parr K_1 \parr Q_1]
\end{array}
\qquad\text{and}\qquad
\begin{array}{c}
\mathsf{MLS}\,\Big\Vert\,\Pi_3 \\
[(A_2 \otimes B_2) \parr Q_2]
\end{array}$$

where $\mathsf{size}(\Pi_2) + \mathsf{size}(\Pi_3) < \mathsf{size}(\Pi')$. In particular, we have $\mathsf{size}(\Pi_2) < \mathsf{size}(\Pi)$ and $\mathsf{size}(\Pi_3) < \mathsf{size}(\Pi)$, which allows us to apply the induction hypothesis to Π_2 and Π_3. We get:

$$\begin{array}{c}
[K_{A_1} \parr K_{B_1}] \\
\mathsf{MLS}\,\Big\Vert\,\Pi_4 \\
[K_1 \parr Q_1]
\end{array}
\qquad\text{and}\qquad
\begin{array}{c}
\mathsf{MLS}\,\Big\Vert\,\Pi_5 \\
[A_1 \parr K_{A_1}]
\end{array}
\qquad\text{and}\qquad
\begin{array}{c}
\mathsf{MLS}\,\Big\Vert\,\Pi_6 \\
[B_1 \parr K_{B_1}]
\end{array}$$

where $\mathsf{size}(\Pi_5) + \mathsf{size}(\Pi_6) < \mathsf{size}(\Pi_2)$ and

$$\begin{array}{c}
[K_{A_2} \parr K_{B_2}] \\
\mathsf{MLS}\,\Big\Vert\,\Pi_7 \\
Q_2
\end{array}
\qquad\text{and}\qquad
\begin{array}{c}
\mathsf{MLS}\,\Big\Vert\,\Pi_8 \\
[A_2 \parr K_{A_2}]
\end{array}
\qquad\text{and}\qquad
\begin{array}{c}
\mathsf{MLS}\,\Big\Vert\,\Pi_9 \\
[B_2 \parr K_{B_2}]
\end{array}$$

where $\mathsf{size}(\Pi_8) + \mathsf{size}(\Pi_9) < \mathsf{size}(\Pi_3)$. We let $K_A = [K_{A_1} \parr K_{A_2}]$ and $K_B = [K_{B_1} \parr K_{B_2}]$, and we can put all the bits and pieces together as follows:

$$
\begin{array}{c}
= \dfrac{[K_{A_1} \parr K_{A_2} \parr K_{B_1} \parr K_{B_2}]}{[K_{A_1} \parr K_{B_1} \parr K_{A_2} \parr K_{B_2}]} \\
\mathsf{MLS} \Big\Vert \Pi_4 \\
[K_1 \parr Q_1 \parr K_{A_2} \parr K_{B_2}] \\
\mathsf{MLS} \Big\Vert \Pi_7 \\
[K_1 \parr Q_1 \parr Q_2] \\
\mathsf{MLS} \Big\Vert \Pi_1 \\
[K_1 \parr K_2]
\end{array}
\qquad \text{and} \qquad
\begin{array}{c}
\mathsf{MLS} \overline{\Big\Vert} \Pi_5 \\
= \dfrac{[A_1 \parr K_{A_1}]}{[(A_1 \otimes \mathbf{1}) \parr K_{A_1}]} \\
\mathsf{MLS} \Big\Vert \Pi_8 \\
\mathsf{s} \dfrac{[(A_1 \otimes [A_2 \parr K_{A_2}]) \parr K_{A_1}]}{[(A_1 \otimes A_2) \parr K_{A_1} \parr K_{A_2}]}
\end{array}
$$

and similarly we get a proof of $[(B_1 \otimes B_2) \parr K_{B_1} \parr K_{B_2}]$. This gives us

$$
\mathsf{size}(\Pi_A) = \mathsf{size}(\Pi_5) + \mathsf{size}(\Pi_8) + 1 \qquad \text{and} \qquad \mathsf{size}(\Pi_B) = \mathsf{size}(\Pi_6) + \mathsf{size}(\Pi_9) + 1 \quad .
$$

Note that we also have

$$
\mathsf{size}(\Pi_5) + \mathsf{size}(\Pi_6) + 1 \leq \mathsf{size}(\Pi_2) \qquad \text{and} \qquad \mathsf{size}(\Pi_8) + \mathsf{size}(\Pi_9) + 1 \leq \mathsf{size}(\Pi_3) \quad .
$$

Hence, we have

$$
\begin{array}{rcl}
\mathsf{size}(\Pi_A) + \mathsf{size}(\Pi_B) & = & \mathsf{size}(\Pi_5) + \mathsf{size}(\Pi_8) + \mathsf{size}(\Pi_6) + \mathsf{size}(\Pi_9) + 2 \\
& \leq & \mathsf{size}(\Pi_2) + \mathsf{size}(\Pi_3) \\
& < & \mathsf{size}(\Pi)
\end{array}
$$

as desired. ☐

3.2 Lemma (Atomic "splitting") *Let a be an atom and let K be a formula. If $[a \parr K]$ is provable in* MLS*, then there is a derivation*

$$
\begin{array}{c}
a^{\perp} \\
\mathsf{MLS} \Big\Vert \\
K
\end{array}
$$

Proof: Exercise. ☐

3.3 Lemma (Context Reduction) *Let A be a formula, and let $S\{\ \}$ be a context. If $S\{A\}$ is provable in* MLS*, then there is a formula K_A, such that*

$$
\begin{array}{c}
[\{\ \} \parr K_A] \\
\mathsf{MLS} \Big\Vert \Pi_S \\
S\{\ \}
\end{array}
\qquad \textit{and} \qquad
\begin{array}{c}
\mathsf{MLS} \overline{\Big\Vert} \Pi_A \\
[A \parr K_A]
\end{array}
$$

Proof: We proceed by induction on the size of $S\{\ \}$. There is only one case to consider, namely, $S\{\ \}$ is of the shape $[(S'\{\ \} \otimes B) \parr C]$ where $B \neq \mathbf{1}$ (but we allow $C = \bot$). Then we apply splitting (Lemma 3.1) to the proof of $[(S'\{A\} \otimes B) \parr C]$, which gives us

$$\begin{array}{c}[C_S \parr C_B]\\ \mathsf{MLS}\,\Big\Vert\,\Pi_1\\ C\end{array} \qquad\text{and}\qquad \begin{array}{c}\mathsf{MLS}\,\overline{\Big\Vert}\,\Pi_2\\ [S'\{A\} \parr C_S]\end{array} \qquad\text{and}\qquad \begin{array}{c}\mathsf{MLS}\,\overline{\Big\Vert}\,\Pi_3\\ [B \parr C_B]\end{array}$$

Because $B \neq \mathbf{1}$, we can now apply the induction hypothesis to Π_2. This gives us

$$\begin{array}{c}[\{\ \} \parr K_A]\\ \mathsf{MLS}\,\Big\Vert\,\Pi_4\\ [S'\{\ \} \parr C_S]\end{array} \qquad\text{and}\qquad \begin{array}{c}\mathsf{MLS}\,\overline{\Big\Vert}\,\Pi_A\\ [A \parr K_A]\end{array}$$

From this we can get Π_S as follows:

$$\begin{array}{c}[\{\ \} \parr K_A]\\ \mathsf{MLS}\,\Big\Vert\,\Pi_4\\ [S'\{\ \} \parr C_S]\\ \mathsf{MLS}\,\Big\Vert\,\Pi_3\\ \mathsf{s}\,\dfrac{[(S'\{\ \} \otimes [B \parr C_B]) \parr C_S]}{[(S'\{\ \} \otimes B) \parr C_S \parr C_B]}\\ \mathsf{MLS}\,\Big\Vert\,\Pi_1\\ [(S'\{\ \} \otimes B) \parr C]\end{array}$$

□

Now we can put the pieces together.

Proof (Second proof of Theorem 2.2): Let a proof Π of a formula A in SMLS be given. We proceed by induction on the number of instances of $\mathsf{ai}\!\uparrow$ in Π. If this number is zero, then Π is in MLS, and we are done. So, let us assume there is at least one $\mathsf{ai}\!\uparrow$ in Π. Let us consider the topmost instance of $\mathsf{ai}\!\uparrow$ in Π, i.e., for us Π looks as follows:

$$\begin{array}{c}\mathsf{MLS}\,\overline{\Big\Vert}\,\Pi_1\\ \mathsf{ai}\!\uparrow\,\dfrac{S\{a \otimes a^{\bot}\}}{S\{\bot\}}\\ \mathsf{SMLS}\,\Big\Vert\,\Pi_2\\ A\end{array}$$

To Π_1, we can apply context reduction (Lemma 3.3). This gives us a K such that

$$\begin{array}{c}[\{\ \} \parr K]\\ \mathsf{MLS}\,\Big\Vert\,\Pi_3\\ S\{\ \}\end{array} \qquad\text{and}\qquad \begin{array}{c}\mathsf{MLS}\,\overline{\Big\Vert}\,\Pi_4\\ [(a \otimes a^{\bot}) \parr K]\end{array}$$

From Π_3 we get

$$\begin{array}{c} K \\ \mathsf{MLS}\,\Big\Vert\,\Pi'_3 \\ S\{\bot\} \end{array}$$

and to Π_4 we can apply splitting (Lemma 3.1), which gives us

$$\begin{array}{c} [K_1 \parr K_2] \\ \mathsf{MLS}\,\Big\Vert\,\Pi_5 \\ K \end{array} \qquad \text{and} \qquad \begin{array}{c} \mathsf{MLS}\,\overline{\Big\Vert}\,\Pi_6 \\ [a \parr K_1] \end{array} \qquad \text{and} \qquad \begin{array}{c} \mathsf{MLS}\,\overline{\Big\Vert}\,\Pi_7 \\ [a^\bot \parr K_2] \end{array}$$

To Π_6 and Π_7, we can apply atomic splitting (Lemma 3.2), which gives us

$$\begin{array}{c} a^\bot \\ \mathsf{MLS}\,\Big\Vert\,\Pi_8 \\ K_1 \end{array} \qquad \text{and} \qquad \begin{array}{c} a \\ \mathsf{MLS}\,\Big\Vert\,\Pi_9 \\ K_2 \end{array}$$

Now we simply put all the bits and pieces together to get a proof Π' of A in which one instance of $\mathsf{ai}{\uparrow}$ is removed:

$$\begin{array}{c} \mathsf{ai}{\downarrow}\,\dfrac{\mathbf{1}}{[a^\bot \parr a]} \\ \mathsf{MLS}\,\Big\Vert\,\Pi_8,\Pi_9 \\ [K_1 \parr K_2] \\ \mathsf{MLS}\,\Big\Vert\,\Pi_5 \\ K \\ \mathsf{MLS}\,\Big\Vert\,\Pi'_3 \\ S\{\bot\} \\ \mathsf{SMLS}\,\Big\Vert\,\Pi_2 \\ A \end{array}$$

Hence, we can apply the induction hypothesis. □

4 Exponentials

Now we reintroduce contraction and weakening in a restricted form, by using *modalities*. These are unary connectives. In linear logic, they are denoted by ? and !, i.e., if A is a formula, then so are $?A$ and $!A$. They are dual to each other, i.e., for defining negation for all formulas, the equations in (3) have to be extended by

$$(!A)^\bot = ?A^\bot \qquad (?A)^\bot = !A^\bot \tag{18}$$

The sequent calculus rules for these modalities are:

$$\mathsf{?w}\,\frac{\vdash \Gamma}{\vdash ?A, \Gamma} \qquad \mathsf{?c}\,\frac{\vdash ?A, ?A, \Gamma}{\vdash ?A, \Gamma} \qquad \mathsf{?d}\,\frac{\vdash A, \Gamma}{\vdash ?A, \Gamma} \qquad \mathsf{!p}\,\frac{\vdash A, ?B_1, \ldots, ?B_n}{\vdash !A, ?B_1, \ldots, ?B_n} \tag{19}$$

where in the $\mathsf{!p}$-rule we have that $n \geq 0$. The system consisting of set of rules in (1), (2) and (19) is called MELL (without the rules in (2) it is denoted by MELL^-). The logic is called *multiplicative exponential linear logic*. For MELL, we have the cut elimination result:

4.1 Theorem *If a sequent* $\vdash \Gamma$ *is provable in* $\mathsf{MELL}+\mathsf{cut}$*, then it is provable in* MELL *without cut.*

The proof is much more involved than for MLL, and we do not show it here. The main problem is finding the right induction measure, since one cut reduction case is as follows:

$$\mathsf{cut}\,\dfrac{\mathsf{?c}\,\dfrac{\overset{\Pi_1}{\vdash \Gamma, ?A, ?A}}{\vdash \Gamma, ?A} \qquad \mathsf{!p}\,\dfrac{\overset{\Pi_2}{\vdash A, ?B_1,\ldots,?B_n}}{\vdash !A, ?B_1,\ldots,?B_n}}{\vdash \Gamma, ?B_1,\ldots,?B_n}$$

is reduced to

$$\mathsf{?c}\,\dfrac{\begin{array}{c}\mathsf{?c}\,\dfrac{\mathsf{cut}\,\dfrac{\mathsf{cut}\,\dfrac{\overset{\Pi_1}{\vdash \Gamma, ?A, ?A} \qquad \mathsf{!p}\,\dfrac{\overset{\Pi_2}{\vdash A, ?B_1,\ldots,?B_n}}{\vdash !A, ?B_1,\ldots,?B_n}}{\vdash \Gamma, ?A, ?B_1,\ldots,?B_n} \qquad \mathsf{!p}\,\dfrac{\overset{\Pi_2}{\vdash A, ?B_1,\ldots,?B_n}}{\vdash !A, ?B_1,\ldots,?B_n}}{\vdash \Gamma, ?B_1,\ldots,?B_n, ?B_1,\ldots,?B_n}}{} \\ \vdots \end{array}}{\vdash \Gamma, ?B_1,\ldots,?B_n}$$

where the proof Π_2 has been duplicated.

For the equivalent system in the calculus of structures, we add the following rules to MLS:

$$\mathsf{e}{\downarrow}\,\dfrac{S\{\mathbf{1}\}}{S\{!\mathbf{1}\}} \qquad \mathsf{p}{\downarrow}\,\dfrac{S\{![A \parr B]\}}{S\{!A \parr ?B\}} \qquad \mathsf{w}{\downarrow}\,\dfrac{S\{\bot\}}{S\{?A\}} \qquad \mathsf{b}{\downarrow}\,\dfrac{S\{?A \parr A\}}{S\{?A\}} \qquad \mathsf{g}{\downarrow}\,\dfrac{S\{??A\}}{S\{?A\}} \tag{20}$$

We use the same equational theory as before, and we write ELS to denote the system MLS extended by the rules in (20). To get the symmetric version SELS of that system, we need to add the duals of these rules as well:

$$\mathsf{e}{\uparrow}\,\dfrac{S\{?\bot\}}{S\{\bot\}} \qquad \mathsf{p}{\uparrow}\,\dfrac{S\{?A \otimes !B\}}{S\{?(A \otimes B)\}} \qquad \mathsf{w}{\uparrow}\,\dfrac{S\{!A\}}{S\{\mathbf{1}\}} \qquad \mathsf{b}{\uparrow}\,\dfrac{S\{!A\}}{S\{!A \parr A\}} \qquad \mathsf{g}{\uparrow}\,\dfrac{S\{!A\}}{S\{!!A\}} \tag{21}$$

As before, the general versions of $\mathsf{i}{\downarrow}$ and $\mathsf{i}{\uparrow}$ can be reduced to their atomic version:

4.2 Proposition *The rule* $\mathsf{i}{\downarrow}$ *is derivable in* $\{\mathsf{ai}{\downarrow}, \mathsf{s}, \mathsf{e}{\downarrow}, \mathsf{p}{\downarrow}\}$*, and the rule* $\mathsf{i}{\uparrow}$ *is derivable in* $\{\mathsf{ai}{\uparrow}, \mathsf{s}, \mathsf{e}{\uparrow}, \mathsf{p}{\uparrow}\}$*.*

The proof is similar to the one for Proposition 2.1 where ! and ? where not in the language. The cut elimination theorem also holds:

4.3 Theorem *If a formula A is provable in* SELS*, then it is provable in* ELS*.*

As before, we can prove this theorem either by using the sequent calculus cut elimination, or by giving a direct proof in the calculus of structures. We will not go into further details here, but note that we have the same corollaries as for MLS, and they can be proved in exactly the same way:

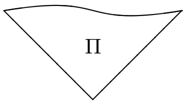
4.4 Corollary *The rule* $\mathsf{i}{\uparrow}$ *is admissible in* ELS*.*

4.5 Corollary *For all formulas A and B, we have*

$$\begin{array}{c} A \\ \mathsf{SELS}\,\Big\Vert\,\Pi_1 \\ B \end{array} \qquad \text{if and only if} \qquad \begin{array}{c} \mathsf{ELS}\,\overline{\Big\Vert}\,\Pi_2 \\ [A^{\perp} \mathbin{⅋} B] \end{array}$$

The relation between the systems MELL in the sequent calculus and ELS in the calculus of structures is as expected.

4.6 Proposition *If there is a proof*

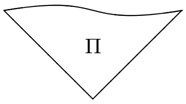


$$\vdash A_1, \ldots, A_n$$

in MELL*, then there is a proof*

$$\begin{array}{c} \mathsf{ELS}\,\overline{\Big\Vert}\,\Pi' \\ [A_1 \mathbin{⅋} \cdots \mathbin{⅋} A_n] \end{array} \quad .$$

4.7 Proposition *If there is a proof*



$$\vdash A_1, \ldots, A_n$$

in $\mathsf{MELL} + \mathsf{cut}$*, then there is a proof*

$$\begin{array}{c} \mathsf{SELS}\,\overline{\Big\Vert}\,\Pi' \\ [A_1 \mathbin{⅋} \cdots \mathbin{⅋} A_n] \end{array} \quad .$$

4.8 Proposition *If there is a proof*

$$\begin{array}{c} \mathsf{SELS}\,\Vert\,\Pi \\ Q \end{array} ,$$

then there is a proof

$$\begin{array}{c} \bigtriangledown_{\Pi'} \\ \vdash Q \end{array}$$

in $\mathsf{MELL} + \mathsf{cut}$.

All three propositions are proved in the same way as for MLL and MLS.

Finally, we have for SELS a property, that has no counterpart in the sequent calculus:

4.9 Theorem *Every derivation*

$$\begin{array}{c} P \\ \mathsf{SELS}\,\Vert \\ Q \end{array}$$

can be decomposed into

$$\begin{array}{cccc}
P & P & P & P \\
\mathsf{e}{\downarrow}\,\Vert & \mathsf{g}{\uparrow}\,\Vert & \mathsf{e}{\downarrow}\,\Vert & \mathsf{g}{\uparrow}\,\Vert \\
P_1 & U_1 & W_1 & T_1 \\
\mathsf{g}{\uparrow}\,\Vert & \mathsf{b}{\uparrow}\,\Vert & \mathsf{g}{\uparrow}\,\Vert & \mathsf{b}{\uparrow}\,\Vert \\
P_2 & U_2 & W_2 & T_2 \\
\mathsf{b}{\uparrow}\,\Vert & \mathsf{e}{\downarrow}\,\Vert & \mathsf{b}{\uparrow}\,\Vert & \mathsf{w}{\uparrow}\,\Vert \\
P_3 & U_3 & W_3 & T_3 \\
\mathsf{ai}{\downarrow}\,\Vert & \mathsf{w}{\downarrow}\,\Vert & \mathsf{w}{\uparrow}\,\Vert & \mathsf{e}{\downarrow}\,\Vert \\
P_4 & U_4 & W_4 & T_4 \\
\mathsf{w}{\downarrow}\,\Vert & \mathsf{ai}{\downarrow}\,\Vert & \mathsf{ai}{\downarrow}\,\Vert & \mathsf{ai}{\downarrow}\,\Vert \\
P_5 & U_5 & W_5 & T_5 \\
\mathsf{s},\mathsf{p}{\downarrow},\mathsf{p}{\uparrow}\,\Vert & \mathsf{s},\mathsf{p}{\downarrow},\mathsf{p}{\uparrow}\,\Vert & \mathsf{s},\mathsf{p}{\downarrow},\mathsf{p}{\uparrow}\,\Vert & \mathsf{s},\mathsf{p}{\downarrow},\mathsf{p}{\uparrow}\,\Vert \\
Q_5 & V_5 & Z_5 & R_5 \\
\mathsf{w}{\uparrow}\,\Vert & \mathsf{ai}{\uparrow}\,\Vert & \mathsf{ai}{\uparrow}\,\Vert & \mathsf{ai}{\uparrow}\,\Vert \\
Q_4 & V_4 & Z_4 & R_4 \\
\mathsf{ai}{\uparrow}\,\Vert & \mathsf{w}{\uparrow}\,\Vert & \mathsf{w}{\downarrow}\,\Vert & \mathsf{e}{\uparrow}\,\Vert \\
Q_3 & V_3 & Z_3 & R_3 \\
\mathsf{b}{\downarrow}\,\Vert & \mathsf{e}{\uparrow}\,\Vert & \mathsf{b}{\downarrow}\,\Vert & \mathsf{w}{\downarrow}\,\Vert \\
Q_2 & V_2 & Z_2 & R_2 \\
\mathsf{g}{\downarrow}\,\Vert & \mathsf{b}{\downarrow}\,\Vert & \mathsf{g}{\downarrow}\,\Vert & \mathsf{b}{\downarrow}\,\Vert \\
Q_1 & V_1 & Z_1 & R_1 \\
\mathsf{e}{\uparrow}\,\Vert & \mathsf{g}{\downarrow}\,\Vert & \mathsf{e}{\uparrow}\,\Vert & \mathsf{g}{\downarrow}\,\Vert \\
Q & Q & Q & Q
\end{array}$$

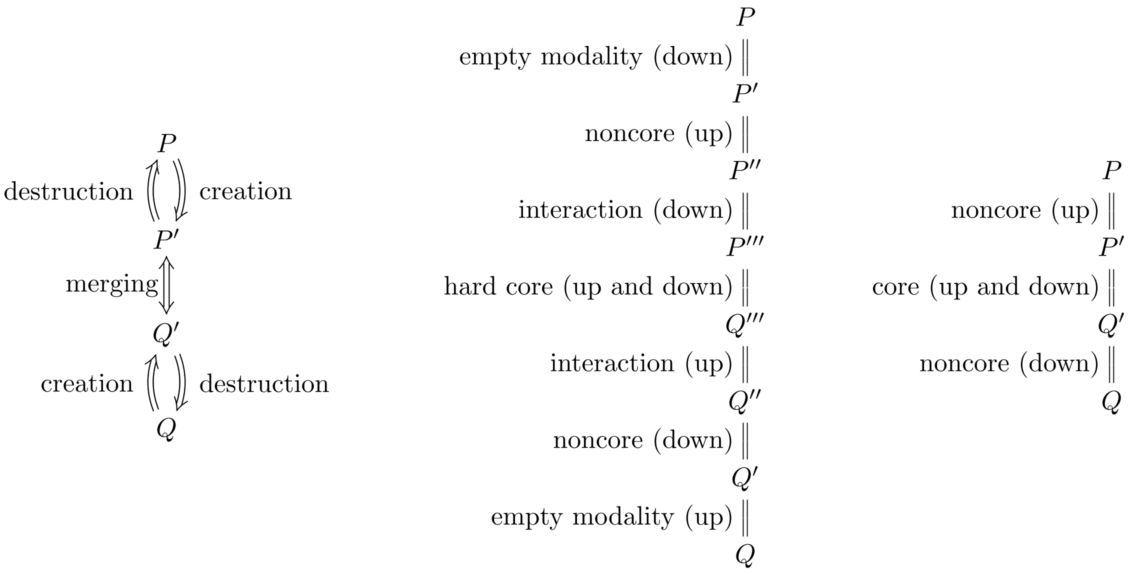


Figure 1: Readings of the decompositions

The four statements are called first, second, third, and fourth decomposition (from left to right).

Apart from a decomposition into eleven subsystems, the first and the second decomposition can also be read as a decomposition into three subsystems that could be called *creation*, *merging* and *destruction*. In the creation subsystem, each rule increases the size of the structure; in the merging system, each rule does some rearranging of substructures, without changing the size of the structures; and in the destruction system, each rule decreases the size of the structure. Here, the size of the structure incorporates not only the number of atoms in it, but also the modality-depth for each atom. In a decomposed derivation, the merging part is in the middle of the derivation, and (depending on your preferred reading of a derivation) the creation and destruction are at the top and at the bottom, as shown in the left of Figure 1. In system SELS the merging part contains the rules s, p↓ and p↑. In the top-down reading of a derivation, the creation part contains the rules e↓, g↑, b↑, w↓ and ai↓, and the destruction part consists of e↑, g↓, b↓, w↑ and ai↑. In the bottom-up reading, creation and destruction are exchanged.

This kind of decomposition (creation, merging, destruction) is quite typical for logical systems presented in the calculus of structures. It also hold for classical logic, for full propositional linear logic, and for non-commutative variants of linear logic.

The third decomposition allows a separation between hard core and noncore of the system[1], such that the up fragment and the down fragment of the noncore are not merged, as it is the case in the first and second decomposition. More precisely, we can separate the seven subsystems shown in the middle of Figure 1. The fourth decompostion is even stronger in this respect: it allows a complete separation between core and noncore, as shown on the right of Figure 1. This decomposition also plays a crucial rule for the cut elimination argument. Recall

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[1] We call *core* the set of rules needed to reduce the general i↓ and i↑ to their atomic versions, and *noncore* all others. The *hard core* are those core rules that are not e↓, e↑, ai↓, or ai↑.

that cut elimination means to get rid of the entire up-fragment. Because of the decomposition, the elimination of the non-core up-fragment is now trivial. Furthermore, recall that for cut elimination in the sequent calculus, the most problematic cases are usually the ones where cut interacts with rules like contraction and weakening, and that in our system these rules appear as the non-core down rules. In the third decomposition these are *below* the actual cut rules (i.e., the core up rules, cf. Proposition 4.2) and can therefore no longer interfere with the cut elimination. This considerably simplifies the cut elimination argument.