A positive perspective on term representation

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Outline

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- What to do with terms? Equality, substitution, evaluation, etc.

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- Sequent calculus: too little structure, too much non-essential information.
- Focused proof system *LJF*: large-scale rules, flexibility of polarization.

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- Large-scale rules (not phases!): *synthetic inference rules* and bipoles.

Two phases: an example in LL

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
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\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash B, B^\perp$}
\RightLabel{$\otimes$}
\BinaryInfC{$\vdash A^\perp, B^\perp, A \otimes B$}
\RightLabel{$\oplus_1$}
\UnaryInfC{$\vdash A^\perp, B^\perp \oplus (C^\perp \otimes D^\perp), A \otimes B$}
\AxiomC{}
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\RightLabel{$\parr$}
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\end{prooftree}
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  - Atomic formulas are either positive or negative.
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- A polarized theory is a theory together with an atomic bias assignment.
- Different polarizations do not affect provability in *LJF*, but give different forms of proofs.
  - ▷ If a sequent is provable in *LJF* for some polarization, then it is provable for all such polarizations.

LJF sequents and border sequents

Two kinds of sequents:

- $\Uparrow$-sequents, used with invertible rules

$$\Gamma \Uparrow \Theta \vdash \Delta \Uparrow \Delta'$$

- $\Downarrow$-sequents, used to specify the formula under focus

$$\Gamma \Downarrow B \vdash \Delta' \quad \text{left focus}$$

$$\Gamma \vdash B \Downarrow \quad \text{right focus}$$

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Border sequents:

$$\Gamma \Uparrow \cdot \vdash \cdot \Uparrow \Delta \rightsquigarrow \Gamma \vdash \Delta$$

▷ Inference rules are collected into large-scale rules (synthectic inference rules) by looking at border sequents in a proof.

The *LJF* system with only implication

Decide, Release, and Store Rules

$$\frac{N, \Gamma \Downarrow N \vdash A}{N, \Gamma \vdash A}\ D_l \qquad \frac{\Gamma \vdash P \Downarrow}{\Gamma \vdash P}\ D_r \qquad \frac{\Gamma \Uparrow P \vdash A}{\Gamma \Downarrow P \vdash A}\ R_l \qquad \frac{\Gamma \vdash N \Uparrow}{\Gamma \vdash N \Downarrow}\ R_r$$

$$\frac{\Gamma, C \Uparrow \Theta \vdash \Delta' \Uparrow \Delta}{\Gamma \Uparrow \Theta, C \vdash \Delta' \Uparrow \Delta}\ S_l \qquad \frac{\Gamma \Uparrow \Theta \vdash A}{\Gamma \Uparrow \Theta \vdash A \Uparrow}\ S_r$$

Initial Rules

$$\frac{A \text{ positive}}{A, \Gamma \vdash A \Downarrow}\ I_r \qquad \frac{A \text{ negative}}{\Gamma \Downarrow A \vdash A}\ I_l$$

Introduction Rules for Implication

$$\frac{\Gamma \vdash B \Downarrow \quad \Gamma \Downarrow B' \vdash A}{\Gamma \Downarrow B \supset B' \vdash A} \supset L \qquad \frac{\Gamma \Uparrow \Theta, B \vdash B' \Uparrow}{\Gamma \Uparrow \Theta \vdash B \supset B' \Uparrow} \supset R$$

Synthetic inference rules

Synthetic inference rule = large-scale rule = $\Downarrow$-phase + $\Uparrow$-phase

Definition

A *left synthetic inference rule* for B is an inference rule of the form

$$\dfrac{\Gamma_1 \vdash A_1 \quad \dots \quad \Gamma_n \vdash A_n}{\Gamma \vdash A}\ B$$

justified by a derivation (in LJF) of the form

$$\dfrac{\begin{array}{c}\Gamma_1 \vdash A_1 \qquad \dots \qquad \Gamma_n \vdash A_n \\ \vdots\ \Uparrow \text{phase} \\ \\ \vdots\ \Downarrow \text{phase} \\ \Gamma \Downarrow B \vdash A\end{array}}{\Gamma \vdash A}\ D_l$$

Bipoles

Bipoles:

A (left) bipole for a formula B is a (left) synthetic inference rule such that only atomic formulas are stored in its corresponding derivation (in LJF).

Order of a formula:

- $ord(A) = 0$ for A atomic.
- $ord(B_1 \supset B_2) = max(ord(B_1) + 1, ord(B_2))$.

Theorem

Let B be a negative polarized formula. If $ord(B) \leq 2$, then the left synthetic rule for B is a bipole.

Axioms as rules

Definition

Let $\mathcal{T}$ be a finite polarized theory of order 2 or less, We define $LJ\langle\mathcal{T}\rangle$ to be the extension of LJ with the left synthetic inference rules for the formulas in $\mathcal{T}$. More precisely, for every left synthetic inference rule

$$\frac{\color{red}{B}, \Gamma_1 \vdash A_1 \quad \dots \quad \color{red}{B}, \Gamma_n \vdash A_n}{\color{red}{B}, \Gamma \vdash A}\ B$$

with $B \in \mathcal{T}$, the inference rule

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Theorem

$\mathcal{T}, \Gamma \vdash B$ provable in $LJ \Leftrightarrow \Gamma \vdash B$ provable in $LJ\langle\mathcal{T}\rangle$.

An example

Let $\mathcal{T}$ be the collection of formulas
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and the inference rules in $LJ\langle\mathcal{T}\rangle$ include

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- ▷ a shortest proof of linear size

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- each formula on the left hand side is given a label.

Untyped λ-terms

We fix a theory $\mathcal{T} = \{\Phi : D \supset D \supset D, \Psi : (D \supset D) \supset D\}$ with D atomic and consider proofs of sequents of the form $\mathcal{T}, x_1 : D, \cdots, x_k : D \vdash t : D$

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Here we use the $\lambda\kappa$-calculus[1] to annotate terms.

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Untyped λ-terms

We fix a theory $\mathcal{T} = \{\Phi : D \supset D \supset D, \Psi : (D \supset D) \supset D\}$ with D atomic and consider proofs of sequents of the form $\mathcal{T}, x_1 : D, \cdots, x_k : D \vdash t : D$

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Two different polarity assignments give **two different term structures**:

- D is negative:

$x^\frown\epsilon$	`nvar x`	x
$\Phi^\frown(\lfloor t \rfloor :: \lfloor u \rfloor :: \epsilon)$	`napp t u`	tu
$\Psi^\frown(\lfloor \lambda x.t \rfloor :: \epsilon)$	`nabs (x\ t)`	$\lambda x.t$

  $\rightarrow$ Top-down / tree-like structure

- D is positive:

$\lceil x \rceil$	`pvar x`	x
$\Phi^\frown(x :: y :: \kappa z.t)$	`papp x y (z\ t)`	*name* $z = xy$ *in* t
$\Psi^\frown(\lfloor \lambda x.t \rfloor :: \kappa y.s)$	`pabs (x\ t) (y\ s)`	*name* $y = \lambda x.t$ *in* s

  $\rightarrow$ Bottom-up / DAG structure

Some examples for the positive-bias syntax

```
name y = app x x in name z = app y y in z
```

- Arguments of `app` are all names

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```
name z = abs (x\ name y1 = app y y in y1) in z
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- Parallel naming

Cut-elimination for $LJ\langle\mathcal{T}\rangle$

The following theorem[2] states that cut is admissible for the extensions of LJ with polarized theories based on synthetic inference rules.

Theorem (Cut admissibility for $LJ\langle\mathcal{T}\rangle$)

Let $\mathcal{T}$ be a finite polarized theory of order 2 or less. Then the cut rule is admissible for the proof system $LJ\langle\mathcal{T}\rangle$.

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- This defines the notion of substitution for terms.

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The proof is based on a cut elimination procedure for *LJF*

- ▹ This defines the notion of substitution for terms.

When we restrict to atomic cut formulas, the cut elimination procedure can be presented in a big-step style.

- ▹ Cuts are permuted with synthetic rules instead of *LJF* rules.

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Untyped λ-terms (substitution)

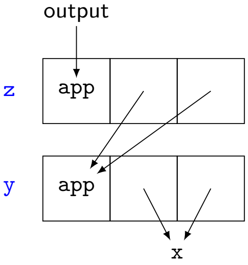
The cut-elimination procedure of *LJF* gives us the following definitions of substitutions.

```
type nsubst, psubst    tm -> (val -> tm) -> tm -> o.
```

```
nsubst T (x\ nvar x) T.
nsubst T (x\ nvar Y) (nvar Y).
nsubst T (x\ napp (R x) (S x)) (napp R' S') :-
   nsubst T R R', nsubst T S S'.
nsubst T (x\ nabs y\ R x y) (nabs y\ R' y) :-
   pi y\ nsubst T (x\ R x y) (R' y).
```

```
psubst (papp U V K) R (papp U V H) :- pi x\ psubst
    (K x) R (H x).
psubst (pabs S K)   R (pabs S H)   :- pi x\ psubst
    (K x) R (H x).
psubst (pvar U)     R (R U).
```

An example

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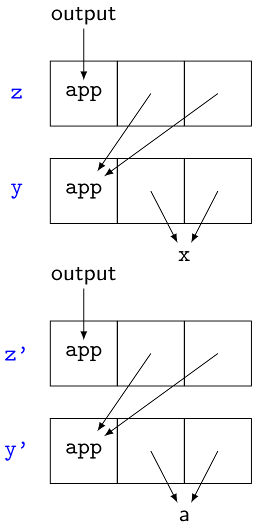


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```
name y' = app a a in
name z' = app y' y' in z'
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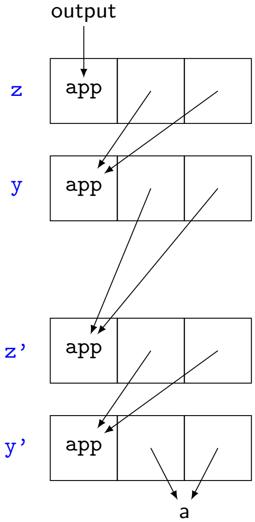


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We have now two different formats for untyped λ-terms.

When should two such expressions be considered the same?

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"White box" approach:

- Look at the actual syntax of proof expressions.
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"White box" approach:

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"Black box" approach:

- Describe *paths* by probing a term.

Path equality

We use λProlog programs to illustrate the idea.

- ▷ `npath T P` (resp. `ppath T P`) if `P` is a path in the `T`.

```
type npath, ppath   tm -> path -> o.
```

```
npath (napp M _) (left  P) :- npath M P.
npath (napp _ N) (right P) :- npath N P.
npath (nabs R)   (bnd   P) :- pi x\pi p\ npath (nvar x) p =>
                                      npath (R x) (P p).

ppath (papp U V K) P :-
   pi x\ (pi P\ ppath (pvar x) (left  P) :- ppath (pvar U) P) =>
         (pi P\ ppath (pvar x) (right P) :- ppath (pvar V) P) =>
   ppath (K x) P.
ppath (pabs R K) P :-
   pi x\  (pi Q\ ppath (pvar x) (bnd Q) :-
               pi p\ pi u\ ppath (pvar u) p => ppath (R u) (Q p))
          => ppath (K x) P.
```

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- Generalize to full LJF.

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- Multi-focusing:
  - ▷ Parallel actions (parallel `name` introductions).
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- Proof-theoretic methods for checking term equality.