

An extension of the Error Correcting Pairs algorithm

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Codes, Cryptology and Curves
Celebrating the influence of Ruud Pellikaan

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Error Correcting Pairs algorithm

PECP for Reed-Solomon codes

PECP for Algebraic Geometry codes

Algorithms for Reed Solomon codes

$$t \leq \left\lfloor \frac{d-1}{2} \right\rfloor \quad \text{Berlekamp-Welch [1]}$$

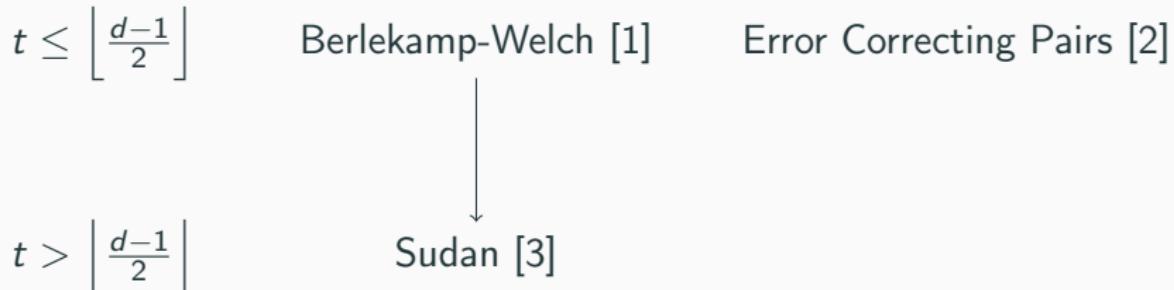
[1] L. R. Welch, E.R.Berlekamp. Error Correction for Algebraic Block Codes. United States Patent, 1986.

Algorithms for Reed Solomon codes

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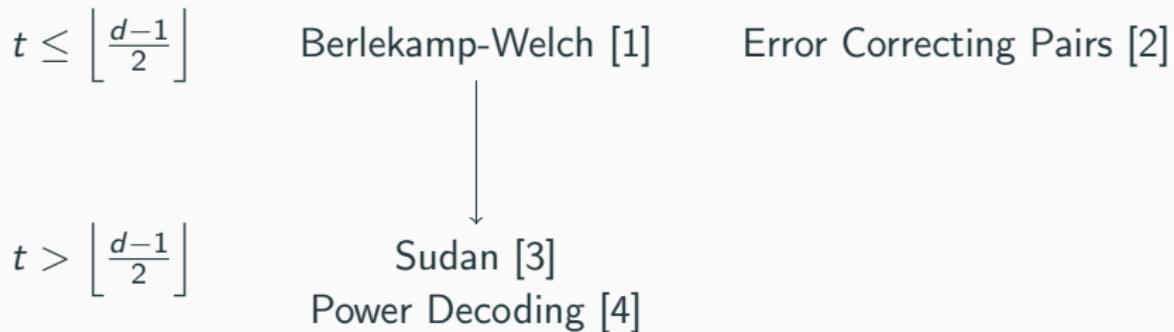
[2] R. Pellikaan. On decoding by error location and dependent sets of error positions. Discrete Mathematics, 1992.

Algorithms for Reed Solomon codes



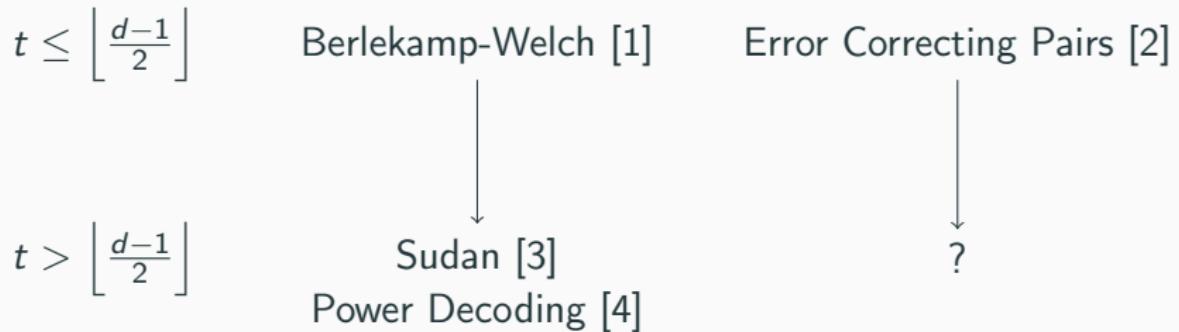
[3] M. Sudan. Decoding of Reed-Solomon codes beyond the error-correction bound. Journal of Complexity, 1997.

Algorithms for Reed Solomon codes



[4] G. Schmidt, V. R. Sidorenko, M. Bossert. Syndrome Decoding of Reed-Solomon Codes Beyond Half of the Minimum Distance based on Shift-Register Synthesis. IEEE Transactions on Information Theory, 2010.

Algorithms for Reed Solomon codes



Error Correcting Pairs algorithm

Problem

Let $C \subseteq \mathbb{F}_q^n$ be a linear code and $y \in \mathbb{F}_q^n$. Given $t \in \mathbb{N}$, find a codeword c such that

$$d(y, c) \leq t.$$

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Hypothesis

There exist $c \in C$ and $e = (e_1, \dots, e_n) \in \mathbb{F}_q^n$ with $w(e) = t$ such that

$$y = c + e.$$

We denote the support of the error vector by

$$I = \{i \in \{1, \dots, n\} \mid e_i \neq 0\}.$$

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Error Correcting Pairs (ECP)

Given a linear code $C \subseteq \mathbb{F}_q^n$, a pair of linear codes (A, B) with $A, B \subseteq \mathbb{F}_q^n$ is a t -error correcting pair for C if

- $A * B \subseteq C^\perp$;
- $\dim(A) > t$;
- $d(B^\perp) > t$;
- $d(A) + d(C) > n$.

Theorem (R. Pellikaan, 1992)

Let $C \subseteq \mathbb{F}_q^n$ be a linear code. If there exists a t -error correcting pair for C , then for all $y \in \mathbb{F}_q^n$ such that

$$y = c + e,$$

with $c \in C$ and $w(e) \leq t$, the ECP algorithm recovers c with complexity $O(n^3)$.

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Proposition

If a linear code C has a t -error correcting pair, then

$$t \leq \left\lfloor \frac{d(C) - 1}{2} \right\rfloor.$$

Let $J = \{j_1, \dots, j_s\} \subset \{1, \dots, n\}$ and $x = (x_1, \dots, x_n) \in \mathbb{F}_q^n$.

We denote

- $x_J := (x_{j_1}, \dots, x_{j_s})$ (puncturing);
- $Z(x) := \{i \in \{1, \dots, n\} \mid x_i = 0\}$.

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Moreover, if $A \subseteq \mathbb{F}_q^n$ we will indicate

- $A_J := \{a_J \mid a \in A\} \subseteq \mathbb{F}_q^{|J|}$;
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- $A(J) := \{a \in A \mid a_J = 0\} \subseteq \mathbb{F}_q^n$ (shortening).

Localisation of errors

We define $M := \{a \in A \mid \langle a * y, b \rangle = 0 \quad \forall b \in B\}$.

Lemma

Let y , $I = \text{supp}(e)$ and M as above. If $A * B \subseteq C^\perp$, then

- $A(I) \subseteq M \subseteq A$;

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Proof of $A(I) \subseteq M$: given $a \in A(I)$, we get for all $b \in B$

$$\langle a * y, b \rangle = \underbrace{\langle a * c, b \rangle}_{\langle a * b, c \rangle} + \underbrace{\langle a * e, b \rangle}_{\langle 0, b \rangle} = 0.$$

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→ we take $J := Z(M)$.

Recovering e

Let $H \in \mathcal{M}(n, m)$, and H^i its columns. Given $J \subseteq \{1, \dots, m\}$, we define

$$H_J = (H^j)^{j \in J}.$$

Let us consider a full rank parity check matrix H for C .

Lemma

If $d(A) + d(C) > n$ and $I \subseteq J$, then there exists a unique solution for the system

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PECP for Reed-Solomon codes

Let $C \subseteq \mathbb{F}_q^n$ be a RS[n,k] code. There exists $f \in \mathbb{F}_q[x]_{<k}$ such that $c = (f(x_1), \dots, f(x_n))$. Let us take

$$A = RS[n, t+1], \quad B^\perp = RS[n, t+k].$$

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Proposition

We have that $d(B^\perp) > t$ if and only if

$$t \leq \left\lfloor \frac{d(C) - 1}{2} \right\rfloor.$$

Berlekamp-Welch key equations and the choice of M

Berlekamp Welch algorithm's key equation (Roth)

Let $\Lambda(x) := \prod_{i \in I} (x - x_i)$ and $N(x) := \Lambda(x)f(x)$. Then

$$(\Lambda(x_i))_i * y = (N(x_i))_i.$$

We get

- $(N(x_1), \dots, N(x_n)) \in B^\perp = RS[t+k];$
- $(\Lambda(x_1), \dots, \Lambda(x_n)) \in A(I) = RS[t+1](I);$

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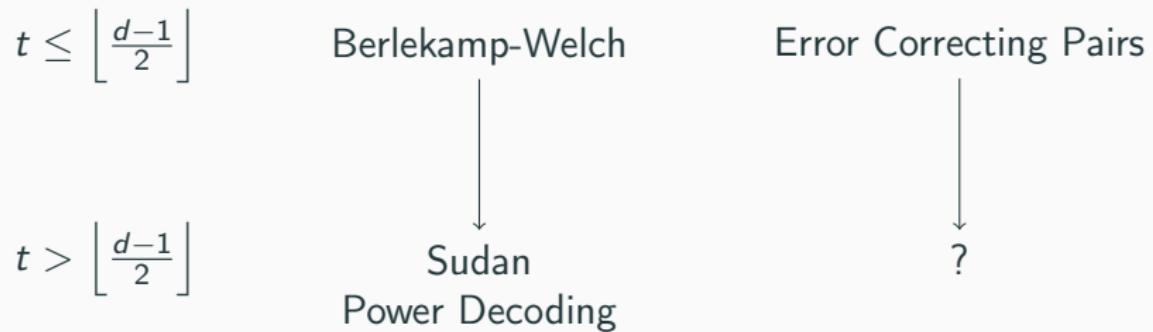
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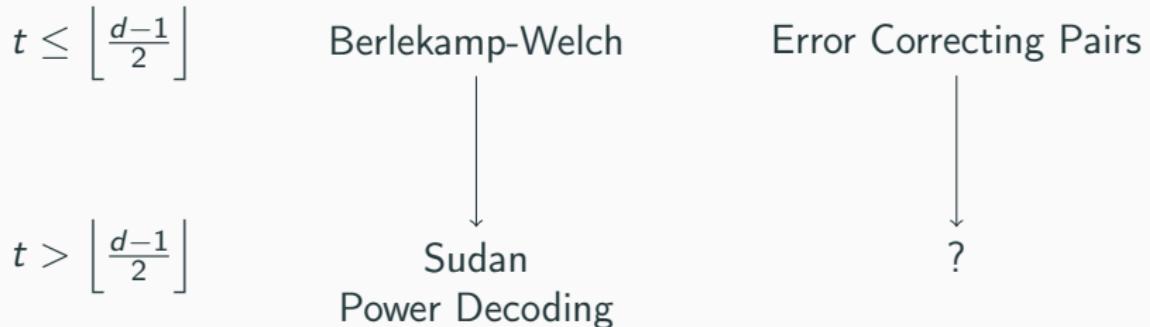
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- $(\Lambda(x_1), \dots, \Lambda(x_n)) \in \underbrace{\{a \in A \mid \langle a * y, b \rangle = 0 \quad \forall b \in B\}}_M.$

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Power Error Correcting Pairs algorithm with power $\ell = 2$

Error Locating Pair

Given A, B, C linear codes of length n , (A, B) is a t -error locating pair (QECP) for C if

- $A * B \subseteq C^\perp$;
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Pellikaan, 1992:

If I is an independent t -set of error positions with respect to B , where (A, B) is a t -error locating pair for C , then the algorithm corrects any word with error supported at I .

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Before we used “If $A * B \subseteq C^\perp$ and $d(B^\perp) > t$, then $A(I) = M$.”

Let us define e' this way

$$y^{*2} = c^{*2} + \underbrace{2c * e + e^{*2}}_{e'}.$$

Lemma

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Power Decoding algorithm's key equations

If $\Lambda(x) = \prod_{i \in I} (x - x_i)$ and $N_j(x) := \Lambda(x) f^j(x)$ for $j = 1, 2$, then

$$\begin{cases} (\Lambda(x_i))_i * y = (N_1(x_i))_i \\ (\Lambda(x_i))_i * y^{*2} = (N_2(x_i))_i \end{cases}.$$

Hence, if we consider $A = RS[n, t + 1]$, $B^\perp = RS[n, t + k]$ as before, we get

- $(N_1(x_1), \dots, N_1(x_n)) \in B^\perp$;
- $(N_2(x_1), \dots, N_2(x_n)) \in B^\perp * C$;

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- $(\Lambda(x_1), \dots, \Lambda(x_n)) \in A(I)$, $M_1 \cap M_2$.

where M_1 and M_2 are defined this way

$$M_1 := \{a \in A \mid \langle a * y, b \rangle = 0 \quad \forall b \in B\},$$

$$M_2 := \{a \in A \mid \langle a * y^{*2}, v \rangle = 0 \quad \forall v \in (B^\perp * C)^\perp\}.$$

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→ we take $M = M_1 \cap M_2$.

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- compute $M = M_1 \cap M_2$ (linear system);
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This algorithm can be run on all codes with an ELP.

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Lemma

If $A * B \subseteq C^\perp$, then $A(I) \subseteq M = M_1 \cap M_2 \subseteq A$.

We look for a **necessary condition** to have $M = A(I)$.

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Lemma

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We look for a **necessary condition** to have $M = A(I)$.

Since $M(I) = A(I)$, we get the implications:

$$M = A(I) \iff M(I) = M \iff M_I = \{0\}.$$

Given $a \in A$, we have by definition of M_1

$$a \in M_1 \iff \langle a * y, b \rangle = 0 \quad \forall b \in B.$$

If $A * B \subseteq C^\perp$, this is equivalent to $a_I \in (e * B)_I^\perp$.

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In the same way, given $a \in A$, it holds

$$a \in M_2 \iff a_I \in (e' * (B^\perp * C)^\perp)_I^\perp.$$

Lemma

We have $(M_1 \cap M_2)_I = (e * B)_I^\perp \cap (e' * (B^\perp * C)^\perp)_I^\perp \cap A_I$.

Remark

Since $A = RS[n, t + 1]$ is MDS, then $A_I = \mathbb{F}_q^t$.

Hence $(M_1 \cap M_2)_I = (e * B)_I^\perp \cap (e' * (B^\perp * C)^\perp)_I^\perp$.

Remark

Since $A = RS[n, t + 1]$ is MDS, then $A_I = \mathbb{F}_q^t$.

$$\text{Hence } (M_1 \cap M_2)_I = (e * B)_I^\perp \cap (e' * (B^\perp * C)^\perp)_I^\perp.$$

A necessary condition for $(M_1 \cap M_2)_I$ to be the null space is

$$\dim((e * B)_I^\perp) + \dim((e' * (B^\perp * C)^\perp)_I^\perp) \leq t.$$

This inequality implies the following

Necessary condition

$$\dim(B) + \dim((B^\perp * C)^\perp) \geq t.$$

Decoding radius for Reed-Solomon codes and $\ell = 2$

We get, as for the Power Decoding algorithm with power 2,

$$t \leq \frac{2n - 3k + 1}{3}.$$

It is possible to write the algorithm for a general power ℓ .

Decoding radius for Reed-Solomon codes and $\ell = 2$

We get, as for the Power Decoding algorithm with power 2,

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For **Reed-Solomon codes**, PECP has the same decoding radius as the Power Decoding algorithm, that is $t_{pow} = \frac{2n\ell - k\ell(\ell+1) + \ell(\ell-1)}{2(\ell+1)}$.

Complexity

PECP(ℓ):

- (i) find $M = \bigcap_{i=1}^{\ell} M_i$;
- (ii) given J , find c .

The main cost is the one of step (i), which reduces to a linear system of $O(n\ell)$ equations in

$$t + 1 = O\left(\frac{2n\ell + \ell(\ell + 1) + 2}{2(\ell + 1)}\right) = O(n)$$

unknowns. Hence we get the cost $O(n^3\ell)$.

PECP for Algebraic Geometry codes

Let χ be a smooth projective curve, $\mathcal{P} = \{P_1, \dots, P_n\} \subseteq \chi$, G a divisor for χ with $\text{supp}(G) \cap \mathcal{P} = \emptyset$ and

$$C = C_L(\chi, \mathcal{P}, G).$$

Theorem

There exists a t -error locating pair for C such that the necessary condition gives the correcting radius

$$t \leq \underbrace{\frac{2n\ell - \ell(\ell + 1)\deg(G) - 2\ell}{2(\ell + 1)}}_{t_{basic}, t_{pow} [SW98]} - g + \frac{g}{\ell + 1}.$$

Future tasks:

- study of the failure cases of the Power Decoding algorithm and the PECP algorithm for Reed-Solomon codes;
- examine the possibility to improve PECP algorithm's decoding radius for algebraic-geometry codes;
- is it possible to design a multiplicity version of ECP algorithm?

Thanks for your attention!