Théorie des probabilités/*Probability Theory*

The St. Petersburg game and continued fractions[1]

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Abstract – The St. Petersburg game is a well known example of a sequence of i.i.d. random variables with infinite expectation and was termed a paradox since no single "fair" entry fee exists. This paper shows how the sequence of continued fraction digits of a random real number makes a reasonable choice of entry fees. Moreover, known results for continued fractions can be obtained for the St. Petersburg game using exactly the same proofs and these explain exactly how the player is favoured even with a fair entry fee (thus resolving a point made by Aaronson).

Le jeu de Saint–Pétersbourg et fractions continues

Résumé – Le jeu de Saint Pétersbourg est un exemple bien connu d'une suite de variables aléatoires i.i.d. d'espérance infinie (donc il n'y a pas de droit d'entrée "équitable"). Dans cette note il est démontré que la suite de chiffres du développement en fraction continue d'un nombre réel est p.p. un choix raisonnable de droits d'entrée. Ainsi, certains résultats connus pour fractions continues sont obtenus pour le jeu de Saint–Pétersbourg en utilisant les mêmes preuves et ces résultats expliquent exactement comment le joueur est avantagé, même avec un droit d'entrée équitable (la résolution d'une question d'Aaronson).

Version française abrégée – Un tour du jeu de Saint Pétersbourg consiste à jeter une pièce jusqu'à la première apparition de "face" et le joueur gagne 2^k, où k est le nombre de jets. L'espérance de ce jeu est infinie donc il n'existe pas de droit d'entrée équitable. Feller [8] a tout de même démontré la loi faible de grands nombres pour les gains accumulés S_n: pour chaque $\varepsilon > 0$,

$$\Pr\left\{\left|\frac{S_n}{n\log_2 n} - 1\right| > \varepsilon\right\} \to 0, \qquad \text{si } n \to \infty\,. \tag{1}$$

C'est à dire, le droit d'entrée $n\log_2 n$ pour n tours est équitable au sens classique. Aaronson a remis ce concept en question puisque les résultats $1/e \le \liminf S_n/n\log_2 n \le \limsup S_n/n\log_2 n = \infty$, p.s., de [1] et [4] démontrent que le joueur est p.s. avantagé. Adler [2] a démontré que $\liminf S_n/n\log_2 n = 1$ p.p., et Csörgő et Simons [6] ont précisé ce résultat en démontrant que S_n est p.p. asymptotique à $n\log_2 n$ si on ignore les m plus grands gains du joueur, $m \ge 1$ (le droit d'entrée est équitable à part les plus grands gains).

Dans [22] il a été noté que les suites de sommes cumulées des chiffres du développement en fraction continue satisfont aussi la propriété (1). Il est surprenant que le théorème de Csörgő et Simons pour $m = 1$ (le cas important) a été démontré pour fractions continues par Diamond et Vaaler [7] et que cette preuve peut être utilisée directement pour prouver ce résultat pour le jeu de Saint Pétersbourg

Théorème 1. *Avec probabilité 1,* $S_n - \max_{k\le n} a_k \sim n\log_2 n$, *où* $a_k = S_k - S_{k-1}$.

L'avantage du joueur correspond donc à son plus grand gain qui est caractérisé par

Théorème 2. *Avec probabilité 1: (a) Soit $\varphi_n > 0$ une suite de nombres réels. Alors $a_n > \varphi_n$ pour une infinité d'entiers n si $\sum 1/\varphi_n = \infty$, et un nombre fini autrement. (b) Soi $\alpha + \beta > 1$, il n'existe q'un nombre*

[1] Running title: "The St. Petersburg game".

fini de n tel que $a_p > n\log^\alpha n$, $a_q > n\log^\beta n$ et $p,q \le n$ sont inégaux. (c) Soi φ_n une suite d'entiers, $\sum 1/\varphi_n = \infty$, $\Phi(N) = \sum_{n\le N} 2^{1-\lceil \log_2 \varphi_n \rceil}$, et $A(N)$ le nombres d'entiers $n \le N$ tels que $a_n \ge \varphi_n$, alors $\limsup_{N\to\infty} \frac{|A(N)-\Phi(N)|}{\sqrt{2\Phi(N)\,\log\log\Phi(N)}} = 1$. (d) $\liminf_{n\to\infty} \dfrac{\max_{k\le n} a_k}{n/\log\log n} = 1$.

Le résultat de 2(a) est bien connu pour les chiffres des fractions continues, et le resultat de 2(b) est une généralisation du résultat pour les fractions continues de [7, lemme 2]. Le résultat de 2(c) a été prouvé pour les fractions continues par Phillipp [18]. Le résultat de 2(d) pour les fractions continues (avec $n/\log\log n$ remplacé par $n/(\log 2\,\log\log n)$) parait être nouveau mais la même preuve peut être utilisée pour les deux cas.

Ces théorèmes démontrent exactement comment le joueur est avantagé: l'égalité asymptotique est très rarement interrompue par un grand gain qui lui donne un avantage de durée relativement courte. Plus précisément, disons que a_n est un *grand gain* si $a_n > n\,\log n/(\log\log n)^\varepsilon$, $\varepsilon > 0$ (par le théorème 1, l'égalité asymptotique n'est interrompue que par un grand gain).

Théorème 3. *Avec probabilité 1, (a) le nombre de grand gains $\le N$ est $O((\log\log N)^{1+\varepsilon})$. (b) Si a_n est un grand gain, le joueur n'est avantagé que pour un intervalle temporel contenu dans $[n, n\,(\log\log n)^{1+\varepsilon})$ et il y a égalité asymptotique pour tout $k \in [n/(\log n)^{1-\varepsilon}, n) \cup [n\,(\log\log n)^{1+\varepsilon}, n\,(\log n)^{1-\varepsilon}]$. (c) Le joueur n'est avantagé que pour un ensemble de densité logarithmique zéro.*

(Le théorème 3 vaut aussi pour les fraction continues.)

Corollaire 1. *Avec probabilité 1, l'ensemble des points limites de $\dfrac{S_n}{n\log_2 n}$ est $[1,\infty)$.*

Ces théorèmes démontrent que, avec probabilité 1, la suite des droits d'entrée correspondant aux chiffres du développement en fraction continue d'un nombre réel est équitable et que pour ces droits d'entrée, l'avantage et le désavantage du joueur ont le même comportement.

L'analogie avec les fraction continues est confirmée par le fait que les suites de gains du jeu de Saint Pétersbourg correspondent à un développement de nombres réels (similaire à ceux de [9] [13] [15]).

Théorème 4. *Chaque suite de gains $s = (a_1, a_2, \ldots)$ correspond à un nombre réel $f(s) = \frac{1}{a_1} + \frac{1}{a_1\,a_2} + \cdots$. Soi $\{x\}$ la partie fractionaire de x, alors $Tx = 2^{\{\log_2 x\}} - 1$ correspond à l'opérateur de décalage des suites de gains, et T est ergodique par rapport à la mesure de Lebesgue.*

1. Distribution of partial sums. A single trial of the St. Petersburg game consists in tossing a fair coin until heads appears and the player wins 2^k dollars if this happens on the kth toss. The expectation is seen to be infinite so this was termed a paradox since no single "fair" entry fee exists. Feller [8] showed that the law of large numbers still holds for this by proving that if there are n trials of the game and S_n is the total winnings, then for any $\varepsilon > 0$

$$\text{(1)} \qquad \Pr\left\{\left|\frac{S_n}{n\log_2 n} - 1\right| > \varepsilon\right\} \to 0, \qquad \text{as } n \to \infty\,,$$

where $\log_2 n = \frac{\log n}{\log 2}$. In other words, $n\log_2 n$ seems to be a fair entry fee if there are to be n trials. It was subsequently shown that (1) does not hold almost everywhere, and Aaronson even showed that the player has an advantage because both the following hold with probability one: $\limsup S_n/n\log_2 n = \infty$, and $\liminf S_n/n\log_2 n \ge 1/e$. The first relation is proved in [4] and the second is proved in [1]. Adler

[2] subsequently improved Aaronson's result to $\liminf S_n/n\log_2 n = 1$, with probability one, and this was further refined by Csörgő and Simons [6] who showed that, with probability one, S_n is asymptotic to $n\log_2 n$ if the m largest wins are ignored, $m \geq 1$ (the entry fee is fair except for the largest wins).

In [22] it was noted that partial sums of the continued fraction digits of a random real number also satisfy (1) (thus the house might consider using the sequence of entry fees $(3, 7, 15, 1, 292, \ldots)$ corresponding to the continued fraction digits of π). It is surprising that the result of Csörgő and Simons for $m = 1$ (the crucial case) was proved for continued fractions by Diamond and Vaaler [7] and that their proof can be used directly to prove the result for the St. Petersburg game.

Theorem 1. *With probability one,* $S_n - \max_{k\leq n} a_k \sim n\log_2 n$.

The advantage to the player thus corresponds to his largest win which is characterized by

Theorem 2. *With probability 1: (a) Let φ_n be a sequence of positive real numbers, then $a_n > \varphi_n$ occurs infinitely if and only if $\sum 1/\varphi_n$ diverges. (b) Let $\alpha+\beta > 1$, then there are at most finitely many n for which $a_p > n\log^\alpha n$ and $a_q > n\log^\beta n$ for two distinct $p, q \leq n$. (c) Let φ_n be a sequence of integers with $\sum 1/\varphi_n = \infty$, $\Phi(N) = \sum_{n\leq N} 2^{1-\lceil \log_2 \varphi_n\rceil}$, and $A(N)$ the number of times that $a_n \geq \varphi_n$, $n \leq N$, then $\limsup_{N\to\infty} \frac{|A(N)-\Phi(N)|}{\sqrt{2\Phi(N)\log\log\Phi(N)}} = 1$. (d) $\liminf_{n\to\infty} \dfrac{\max_{k\leq n} a_k}{n/\log\log n} = 1$.*

These results show exactly how the game is favourable to the player, even with the "fair" entry fee suggested by Feller: Asymptotic equality is very rarely interrupted by a large win that puts him ahead for a relatively short period. More precisely, say that a_n is a *large win* if $a_n > n\log n/(\log\log n)^\varepsilon$ (by Theorem 1, asymptotic equality can only be interrupted by a large win).

Theorem 3. *With probability 1, (a) The number of large wins $\leq N$ is $O((\log\log N)^{1+\varepsilon})$. (b) Around a large win, the player is only favoured in an interval contained in $[n, n(\log\log n)^{1+\varepsilon})$ since asymptotic equality holds for all $k \in [n/(\log n)^{1-\varepsilon}, n) \cup [n(\log\log n)^{1+\varepsilon}, n(\log n)^{1-\varepsilon}]$. (c) The player is favoured only on a set of logarithmic density zero since $\frac{1}{\varepsilon}\log\log\log\log N \gg \frac{1}{\log\log N}\sum_{\substack{n\leq N\\ S_n/n\log n>1+\varepsilon}} \frac{1}{n} \gg \frac{1}{\varepsilon}$.*

Proof: (a) Since $2^{1-\lceil\log_2 x\rceil} = O(1/x)$, Theorem 2(c) implies that the number of large wins $\leq N$ is $\ll \sum_{n\leq N}\frac{(\log\log n)^\varepsilon}{n\log n} = O((\log\log N)^{1+\varepsilon})$. (b) Theorem 2(a) implies that, with probability 1, $a_n < n\log n(\log\log n)^{1+\varepsilon/2}$ for all sufficiently large n, so that by 2(b), for all $k \in [n(\log\log n)^{1+\varepsilon}, n(\log n)^{1-\varepsilon}]$, one has $\max_{m\leq k} a_m = a_n = O(k\log k(\log\log k)^{-\varepsilon/2})$ and by Theorem 1, $S_k \sim k\log_2 k$ for this range of k, and similarly for $k \in [n/(\log n)^{1-\varepsilon}, n)$. Note that this is optimal in the sense that, by Theorem 2(a), there are a.e. an infinite number of n such that $S_k/k\log k > \log\log\log k$ for all $k \in [n, n\log\log n]$. (c) It follows from 2(c) and 3(b) that the sum is of the same order as $\sum_{a_n/n\log n>\varepsilon} \log\left(\frac{a_n}{\varepsilon n\log n}\right)$. The lower bound comes from applying 2(c) to the subsum $\sum_{10\varepsilon > a_n/n\log n>\varepsilon} \log\left(\frac{a_n}{\varepsilon n\log n}\right)$. The upper bound follows by breaking the sum up as

$$\sum_{\substack{n\leq N\\ g(n)\geq\frac{a_n}{n\log n}>\varepsilon}} \log\left(\frac{a_n}{\varepsilon n\log n}\right) + \sum_{\substack{n\leq N\\ \frac{a_n}{n\log n}>g(n)}} \log\left(\frac{a_n}{\varepsilon n\log n}\right) \ll \frac{1}{\varepsilon}(\log\log N)\log g(N) + \frac{(\log\log N)\,\log\log\log N}{g(N)},$$

where the second estimate follows from $a_n < \varepsilon n\log n(\log\log n)^2$ for all sufficiently large n, and then letting $g(n) = \log\log\log n$. (Compare this result with [7, Corollary 4].) □

Corollary 1. *With probability one, the set of limit points of* $\dfrac{S_n}{n \log_2 n}$ *is* $[1, \infty)$.

Proof of Theorem 2: The result of Theorem 2(a) is well known for continued fraction coefficients, e.g., see [12] (compare with [8, Problem 8.7.5]). Theorem 2(b) is a generalization of the continued fractions result [7, Lemma 2] and that proof generalizes to both cases. Theorem 2(c) was proved for continued fractions by Philipp [18] and the proof for the St. Petersburg game is identical except that $\Pr\{a_n > \varphi_n\} = 2^{1-\lceil \log_2 \varphi_n \rceil}$ in this case. The analogue of Theorem 2(d) does not seem to been known for continued fractions (in that case substitute $n/\log\log n$ with $n/(\log 2 \log\log n)$), see [11], but the proofs for both cases are similar (the methods of [9] are required to adapt the proof for continued fractions). To show that the $\liminf \le 1$, let $0 < c < 1$ and let $d(k)$ be such that $d(k)/(c \log\log d(k))$ is the power of 2 closest to k^{4k}. Then

$$\mu\left(\left\{s:\ a_m \le \frac{d(k)}{c \log\log d(k)},\ \ d(k-1) < m \le d(k)\right\}\right) \sim \left(1 - \frac{c \log\log d(k))}{d(k)}\right)^{d(k)} \gg (k \log k)^{-c}.$$

The result follows from the Borel–Cantelli Lemma and the fact that $d(k-1)\log^2 d(k-1) = O(d(k)/\log d(k))$. The proof that the $\liminf \ge 1$ is similar. □

Remarks: The theorem of Csörgő and Simons for $m > 1$ follows immediately from Theorem 1 and Theorem 2(b) (in fact, one can let $m = (\log n)^{1/2-\varepsilon}$, while Adler's result, included in Corollary 1, follows directly from Theorems 1 and 2(d). The results of Theorem 3 also hold for continued fractions. It would be intersting to find out whether Theorem 3(c) can be strenghtened to a limiting distribution result, i.e., that $\frac{1}{\log\log N} \sum\limits_{\substack{n \le N \\ S_n/n \log n > 1+\varepsilon}} \frac{1}{n} \to h(\varepsilon)$, for some function $h(x)$. Other results from the metric theory of continued fractions follow directly for the St. Petersburg game. For example, one can prove a law of the iterated logarithm by using the method of Szüz for continued fractions [21].

To illustrate exactly how continued fractions techniques yield direct proofs for the St. Petersburg game, I will give a complete proof of Theorem 1 based on [7] (compare with [6]). Let $n' = n(\log n)^{1/2+\varepsilon}$, where $0 < \varepsilon < 1/7$, and let $x^* = x$ if $x \le n'$ and $x^* = 0$ otherwise, with $S_n^* = \sum_{k \le n} a_k^*$.

Proposition 1. *With probability one* $S_n^* \sim n \log_2 n$.

Proof: Consider $M_n = \int S_n^*(s) d\mu$ and $V_n = \int (S_n^*(s) - M_n)^2 d\mu$. It follows that $M_n \sim n \log_2 n$ and $V_n = O(n\, n')$. Now let $c(k) = \exp(k^{1-\varepsilon})$, then the previous estimate gives

$$\int \sum_{k=1}^{\infty} \frac{\left(S_{c(k)}^*(s) - M_{c(k)}\right)^2}{c(k)c(k)'\, k^{1+\varepsilon}} d\mu \ll \sum_{k=1}^{\infty} \frac{1}{k^{1+\varepsilon}} < \infty.$$

This implies that the integrand is finite a.e., so $S_{c(k)}^* - M_{c(k)} = o\left(\left\{c(k)\, c(k)'\, k^{1+\varepsilon}\right\}^{1/2}\right)$ for almost all s. The assumption that $\varepsilon < 1/7$ then implies that for almost all s, $S_{c(k)}^* - M_{c(k)} = o(M_{c(k)})$. To complete the proof of Proposition 1, note that if $c(k-1) < n \le c(k)$, then $S_{c(k-1)}^*(s) \le S_n^*(s) \le S_{c(k)}^*(s)$, and that $c(k) \sim c(k-1)$ which implies $M_{c(k)} \sim M_{c(k-1)}$. □

By Theorem 2(b), there is at most one value excluded by the truncation so that $S_n^* \sim S_n - \max\limits_{k \le n} a_k$ and Theorem 1 follows directly from Proposition 1.

2. Expansion properties In the following it will be useful to specify the probability space being used, namely the *St. Petersburg space* of infinite sequences $\mathcal{S} = \{(a_1, a_2, \ldots),\ a_1, a_2, \ldots \text{powers of } 2\}$, with measure

μ generated by $\mu(\{s \in \mathcal{S} : s_1 = a_1, \ldots, s_r = a_r\}) = \frac{1}{a_1 \cdots a_r}$. To each $s \in \mathcal{S}$, I will associate a real number $f(s) \in (0,1)$ given by $f(a_1, a_2, \ldots) = \frac{1}{a_1} + \frac{1}{a_1\, a_2} + \cdots$, for example $\pi = 3 + f(2, 8, 32, 2, 2, 2, 2, 2, 4, 2, \ldots)$. This expansion is related to the ordinary base–2 expansion of real numbers and is a type of *Oppenheim expansion* discussed in [10] [13] [15] (Feller's result (1) also holds for some of these generalized expansions [13]), but is slightly different from the expansions considered in [19]. Note that f is invertible whenever $x \in (0,1)$ and is not a rational number with denominator a power of 2, i.e., it is invertible on a set of measure 1.

Theorem 4. *Let $\{x\}$ be the fractional part of x, then the map $Tx \stackrel{\text{def}}{=} \{2^{\{\log_2 x\}}\} = 2^{\{\log_2 x\}} - 1$ corresponds to the map of sequences in the St. Petersburg space, i.e., $(0,1)$ with the map T is isomorphic to the St. Petersburg space with left shift. Finally, T is ergodic under Lebesgue measure (and in fact is mixing).*

Proof: It is easily checked that T corresponds to the shift map of $\mathcal{S}$, i.e., if $x = \frac{1}{a_1} + \frac{1}{a_1\, a_2} + \cdots$, then $Tx = a_1\, x - 1$. The proof that T is ergodic is analogous to the one for continued fractions, e.g., see [3]. □

Corollary 2. *The ergodic theorem implies that, with probability one,*

(3) $$\lim_{n\to\infty} \sqrt[n]{a_1 a_2 \cdots a_n} = 2^{\sum_{k=1}^{\infty} k/2^k} = 4\,, \quad \textit{(the analogue of Khinchin's constant [14])},$$

$$\lim_{n\to\infty} \left(\frac{a_1^t + a_2^t + \cdots + a_n^t}{n} \right)^{1/t} = \left(\sum_{k=1}^{\infty} \frac{2^{kt}}{2^k} \right)^{1/t} = (2^{1-t} - 1)^{-1/t}, \qquad t < 1,\ t \neq 0\,.$$

The above results have some consequences about possible fair entry fees for the St. Petersburg game. It is undesirable to have variable entry fees, so Steinhaus [20] (see also [5]) considered the single sequence $h = (2, 4, 2, 8, \ldots)$ whose nth element is twice the highest power of 2 dividing n, since this sequence has partial sums asymptotic to $N \log_2 N$ (note that the Steinhaus sequence corresponds to the real number $f(h) = -1 + \prod_{n=1}^{\infty} \left(1 + \frac{1}{2^{2^n} - 1}\right)$). It is seen that the Steinhaus sequence has the correct average behaviour of Corollary 2, so is a good choice except that it does not include the large values predicted by Theorem 2(a) or the small ones predicted by 2(d) (in particular, the house never has a large advantage similar to the player's).

On the other hand, the continued fraction digits of a random real number (and π, conjecturally) also satisfy (1) and, used as an entry fee, will give the house an advantage with behaviour similar to Theorems 2 and 3, and thus similar to the player's advantage. This is therefore a good choice except that it does not satisfy the averages of Corollary 2, and has a different limiting distribution [22].

Continuing the analogy with continued fractions, one can study the fractional dimension of sets of real numbers with missing digits.

Theorem 5. *Given a set of positive integers B, let $C(B) \subset (0,1)$ be the set of numbers whose St. Petersburg digits are restricted to the values 2^b, $b \in B$. The Hausdorff dimension of $C(B)$ is $-\log_2 \xi$, where $1/2 \leq \xi \leq 1$ is the root of $\sum_{b \in B} x^b = 1$.* (For example, the Hausdorff dimension of the set of real numbers with all St. Petersburg digits ≤ 4 is the same as the ones with all digits ≥ 4, and both are given by $\log_2 \frac{\sqrt{5}+1}{2}$.)

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