Discrete derivatives of sequences

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1. Introduction.

Consider a sequence $F : \mathbb{N} \to \mathbb{N}$, which is nondecreasing and onto. Then the quantity defined by

$$F'(m) := \frac{1}{|F^{-1}(F(m))|}$$

(where as usual $F^{-1}(k)$ denotes the set $\{\ell | F(\ell) = k\}$) can be interpreted as the "slope" of F at m. Accordingly we call the function $F' : \mathbb{N} \to \mathbb{R}$ the *discrete* derivative of F. And developing further the analogy with usual differential equations we can see that the notion of "integration" of this differential equation makes sense: it corresponds to reconstructing the sequence F from the sequence $\tau(F) = \{|F^{-1}(k)|\}_{k=1}^{\infty}$ of its runs.

Let us illustrate this by an example. Consider the sequence $F : \mathbb{N} \to \mathbb{N}$ (nondecreasing and onto), uniquely defined by the condition $|F^{-1}(k)| = k^2$: we wish to give an explicit expression for the increasing sequence F in which k appears exactly k^2 times. Its discrete derivative is thus

$$F'(m) = \frac{1}{(F(m))^2},$$

and it is natural to expect it to behave like one of the solutions of the standard differential equation

$$f'(t) = \frac{1}{f(t)^2},$$

which are the inverse functions $f(t) = (3t+C)^{1/3}$ of the functions $t(f) = \int_0^f x^2 dx - C/3$. It is thus natural to expect

$$m \sim \frac{F(m)^3}{3}, \quad \text{that is,} \quad F(m) \sim (3m)^{\frac{1}{3}},$$

to hold. This is of course true and follows from

$$m + O(F(m)^2) = \sum_{k \le F(m)} k^2 = \int_0^{F(m)} x^2 dx + O(F(m)^2)$$

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More generally let us now consider the sequence $F : I\!N \to I\!N$ (nondecreasing and onto) uniquely defined by the condition $|F^{-1}(k)| = \gamma_0(k)$, where $\gamma_0 : I\!N \to I\!N$. Its discrete derivative is

$$F'(m) = \frac{1}{\gamma_0(F(m))}.$$

If we put $\gamma_0(s) := \gamma_0([s])$ we have

$$H_0(F(m)) := \int_1^{F(m)} \gamma_0(s)ds = \sum_{k \le F(m)} \gamma_0(k) = m + O(\gamma_0(F(m))), \tag{1}$$

whence

$$F(m) = H_0^{-1}(m + O(\gamma_0(F(m)))).$$

This equation is usually sufficient to compute the asymptotic behaviour of F. Thus, as with $\gamma_0(k) = k^2$ above, use of the solutions of the standard differential equation

$$f'(t) = \frac{1}{\gamma(f(t))}, \tag{2}$$

where

$$\gamma : [1, \infty) \to (0, \infty) \ \text{ is a continuous function} \tag{3}$$

can in many cases simplify this computation. In order to see this we write

$$e(t) := \gamma(t) - \gamma_0(t) \quad \text{and} \quad E(x) = \frac{1}{x}\left|\int_1^x e(t)dt\right| \le \sup_{t \le x} |e(t)|. \tag{4}$$

Then from (1) we have

$$H(F(m)) := \int_1^{F(m)} \gamma(s)ds = m + O(\gamma_0(F(m))) + O(E(F(m))F(m)).$$

Thus, writing $H^{-1} = f$, we proved the following.

Theorem 1. *Let $\gamma_0 : I\!N \to I\!N$, γ, e and E be as in (3) and (4), and F be the non decreasing and onto solution of*

$$F'(m) = \frac{1}{\gamma_0(F(m))}, \quad F(1) = 1.$$

Then F satisfies

$$F(m) = f(m + O(\gamma_0(F(m))) + O(E(F(m))F(m))), \tag{5}$$

where the function $f(t)$ is a solution of (2).

Consider for instance the case where $\gamma_0(k) = [\sqrt{k}]$, that is where F is the increasing sequence F in which k appears exactly $[\sqrt{k}]$ times. Then we put $\gamma(t) = \sqrt{t}$, and note that $F(m) = o(m)$ and $E(x) = 1$. Now the solutions of

$$f'(t) = \frac{1}{\sqrt{f(t)}}$$

are the functions $f(t) = (3t/2 + C)^{2/3}$, which satisfy $f(m + o(m)) \sim f(m) = (3m/2)^{2/3} + O(m^{-1/3})$: it follows from (5) that $F(m) \sim (3m/2)^{2/3}$. So we can use again (5) with the more precise estimate $F(m) = O(m^{2/3})$, and obtain in fact

$$F(m) = \left(\frac{3}{2}m\right)^{\frac{2}{3}} + O(m^{\frac{1}{3}}).$$

Theorem 1 applies in cases where the sequence of runs is oscillating. For instance if $\gamma_0(t) = [t(2 + \sin(\log t)] = [\gamma(t)]$, then $F(m) = f(m + O(F(m)))$, where f is a solution of (2) and thus satisfies

$$\int_0^{f(t)} \gamma(s)ds = t + C, \tag{6}$$

where C is some real constant. Hence with (6) it is not difficult to see that f satisfies

$$f^2(t)\left(1 + \frac{2}{5}\sin(\log(f(t)) - \frac{1}{5}\cos(\log(f(t))\right) = t + C.$$

By using (6) again we see that two solutions are $O(1/\sqrt{t})$ apart, and we may thus choose f with $C = 0$. It is a slowly oscillating function with $f(t) \asymp \sqrt{t}$ and (using (2)) $f'(t) \asymp 1/\sqrt{t}$. Now two successive applications of (5), first with $F(m) \ll m$ and then with $F(m) \ll \sqrt{m}$, yield

$$F(m) = f(m) + O(1).$$

We finally note that Theorem 1 even applies in cases where the sequence of runs contains a bounded subsequence. If for instance we slightly modify the last example and put $\gamma_0(t) = [t(1+\sin(\log t)]+1 = [\gamma(t)]+1$, then again $F(m) = f(m+O(F(m)))$ where f is a solution of (2) and satisfies

$$f^2(t)\left(\frac{1}{2} + \frac{2}{5}\sin(\log(f(t)) - \frac{1}{5}\cos(\log(f(t))\right) = t + C.$$

Thus again f is a slowly oscillating function with $f(t) \asymp \sqrt{t}$. By using (6) we see that two solutions are $O(1)$ apart, whence as before we may choose f with $C = 0$. Now as in the previous example we see with (5) and $F(m) \leq m$ that $F(m) \leq \sqrt{m}$. This time however we may not simply use $f'(t)$, which is $+\infty$ for infinitely many values of t, in order to evaluate $f(m + O(\sqrt{m})) - f(m)$. A direct use of (6) nevertheless yields as above

$$F(m) = f(m + O(\sqrt{m})) = f(m) + O(1).$$

One can express Theorem 1 informally by saying that, in the case where the sequence of runs is given by an explicit and unbounded function $\gamma_0(m)$, the sequence with such runs has an asymptotic behaviour given, via (5), by a solution of some standard differential equation (2), where γ is a continuous function close to γ_0. In this case, the sequence with a given set of runs is completely understood in an asymptotic sense. The rest of the paper will therefore be devoted to the case where the sequence of runs is given by an *implicit* function, i.e., one in which the length of the runs depend on the sequence itself.

The canonical example is the sequence F defined by $|F^{-1}(k)| = F(k)$ (this time we want k to appear $F(k)$ times), i.e.

$$F'(m) = \frac{1}{(F \circ F)(m)} \tag{7}$$

It is known as Golomb's sequence (see [GMF], and also the more recent series of papers [V] [P1] [P2] [PEd] [R] [PR], devoted to this sequence),

$$\{F(m)\}_{m=1}^{\infty} = \{1, \underbrace{2, 2}_{2}, \underbrace{3, 3}_{2}, \underbrace{4, 4, 4}_{3}, \underbrace{5, 5, 5}_{3}, \underbrace{6, 6, 6, 6}_{4}, ...\}.$$

Note that F coincides with the sequence of its runs: in fact this property is the condition imposed in the usual definition of Golomb's sequence. In a previous paper [PRV] we considered the corresponding functional-differential equation

$$f'(t) = \frac{1}{(f \circ f)(t)}. \tag{8}$$

It was first studied by M.A. McKiernan[2] [K1]. Later, D. Marcus discovered a link between equation (8) and Golomb's sequence [GMF], which is clarified by the notation of the present paper via equation (7). He proposed a heuristic argument supporting the fact (proved using a different argument by N.J. Fine in the same issue of the A.M.S. Monthly) that the particular solution of (8)

$$g(t) = \phi^{2-\phi} t^{\phi-1}, \quad \text{where} \quad \phi = \frac{\sqrt{5}+1}{2},$$

describes the asymptotic behaviour of Golomb's sequence. In the present paper this fact can be verified as an application of Theorems 4 and 5.

More generally one can ask for k appearing $\Gamma_F(k)$ times, that is

$$F'(m) = \frac{1}{\Gamma_F(F(m))}, \tag{9}$$

where $\Gamma_F(1)$ is given, and where $\Gamma_F(\ell+1)$ can be computed in terms of $\{\Gamma_F(i) : i \leq \ell\}$. ¿From this perspective it is natural to consider positive integer valued operators $\Gamma_F(m)$ acting on the identity function $F_0(m)$, on $F_1(m) := F(m)$, and on iterated compositions

$$F_k(m) = F(F_{k-1}(m)) \quad (k \geq 1)$$

of $F(m)$. It is straightforward to see that the differential equation (9) can then be integrated and (with the initial condition $F(1) = 1$) provides a unique solution F satisfying

$$\{\Gamma_F(m)\}_{m=1}^{\infty} = \tau(F), \tag{10}$$

where we recall that $\tau(F) = \{|F^{-1}(m)|\}_{m=1}^{\infty}$ denotes the sequence of the runs of F.

We note in passing that the relation (10) can be used to construct F recursively.

[2]M.A. McKiernan's original encounter with this equation had apparently nothing to do with Golomb's sequence: "my interest in such equations", as he recalls it [K2], "started when students mistakenly interpreted $dy/dx = 1/(dx/dy)$ as meaning $Df = 1/Df^{-1}$. This led to asking if this last equation had any interesting solution, then to other functional differential equations."

Remarks. **(a)** Also note that an equation of the form "$F'(m) = 1/\gamma(m)$ ", where $\gamma(m)$ is a positive integer for every integer $m \geq 1$, does not necessarily have a solution. It is, for instance, easy to verify that the equation "$F'(m) = 1/m^k$ " (which is equivalent to "$|F^{-1}(F(m))| = m^k$") does not have a solution if $k \geq 1$.
(b) And also that an equation of the form "$\Gamma_F(m) = F(\gamma(m))$ ", where $\gamma(m)$ is a positive integer for every integer $m \geq 1$, does not necessarily have a solution, even if γ is a very regularly increasing function. For instance "$\Gamma_F(m) = F(2^m)$" has no solution (whereas "$\Gamma_F(m) = F(2^{2^{m+1}+1})$" has (at least) one solution, given by "$\Gamma_F(m) = 2^{m+1}$").

In particular, we will always take $\Gamma_F(k)$ of the form, or approximately of the form, $\Gamma(k, F_1(k), \ldots, F_{n-1}(k))$, where $\Gamma(x_1, x_2, \ldots, x_n)$ is a function from $I\!R_+^n$ into $I\!R_+$ (the abuse of notation should not cause confusion). Thus, in the case of Golomb's sequence, $\Gamma_F(k) = F(k)$, and $\Gamma(x_1, x_2) = x_2$. More precisely, we consider positive integer valued operators Γ satisfying

$$\Gamma_F(k)(1 + \epsilon(k)) = \Gamma(k, F(k), \cdots, F_{n-1}(k)), \tag{11}$$

with $\epsilon(k) \to 0$ as $k \to \infty$, and with $\Gamma : I\!R_+^n \to I\!R_+$ a differentiable function. In general, for any positive real function $f(t)$ we put

$$\Gamma[f](t) := \Gamma(f_1(t), f_2(t), \cdots, f_n(t)). \tag{12}$$

Thus by taking $k = F(m)$ in (11) we get $\Gamma_F(F(m))(1 + \epsilon(F(m))) = \Gamma[F](m)$, and the discrete differential equation (9) becomes

$$F'(m) = \frac{1 + \epsilon(F(m))}{\Gamma[F](m)},$$

and has the standard counterpart

$$f'(t) = \frac{1}{\Gamma[f](t)}. \tag{13}$$

For instance, if $\Gamma_F(m) = mF(m)^2F_2(m)^3$, then $\Gamma(x_1, x_2, x_3) = x_1x_2^2x_3^3$, the discrete differential equation (9) is $F'(m) = 1/(F(m)F_2(m)^2F_3(m)^3)$, its standard counterpart (13) is $f'(t) = 1/(f(t)f_2(t)^2f_3(t)^3)$, and the integral F of (9) is

$$1, \quad \underbrace{2, \cdots, 2,}_{2^6 = 2F(2)^2F_2(2)^3} \quad \underbrace{3, \cdots, 3,}_{3 \cdot 2^5} \cdots, \underbrace{2^6 + 1, \cdots, 2^6 + 1,}_{(2^6+1)2^5} \underbrace{2^6 + 2, \cdots, 2^6 + 2,}_{(2^6+2)3^2 2^3} \cdots. \tag{14}$$

On the other hand if $\Gamma_F(m) = [2\sqrt{F(m)}]$, it is convenient to put $\Gamma(x_1, x_2) = 2x_2^{1/2}$. In this case equation (9) is $F'(m) = 1/[2\sqrt{F_2(m)}] = (1 + \epsilon(F(m)))/(2F_2(m)^{1/2})$, its counterpart (13) is $f'(t) = 1/(2f_2(t)^{1/2})$, and the integral F of (9) is

$$\underbrace{1, 1}_{[2\sqrt{1}]}, \underbrace{2, 2}_{[2\sqrt{1}]}, \underbrace{3, 3}_{[2\sqrt{2}]}, \underbrace{4, 4}_{[2\sqrt{2}]} \underbrace{5, 5, 5,}_{[2\sqrt{3}]} \underbrace{6, 6, 6,}_{[2\sqrt{3}]} \underbrace{7, 7, 7, 7,}_{[2\sqrt{4}]} \cdots. \tag{15}$$

In this paper we deal with operators Γ as in (11) such that the solution F of the discrete functional-differential equation (9) behaves asymptotically like one of the solutions of the *approximate* functional-differential equation

$$f'(t) = \frac{1+o(1)}{\Gamma[f](t)}. \tag{16}$$

Theorem 2 provides a simple test ensuring that this is the case.

When in addition a solution of (16) is asymptotically equivalent to every other solution, then we may choose a particular solution of the *exact* functional-differential equation (13), which then describes precisely the asymptotic behaviour of our sequence F. We give an illustration of such a situation by considering in Theorem 3 a class of operators for which equation (13) possesses a special solution having a simple expression. We assume that the function Γ is a monomial

$$\Gamma(x_1, x_2, \cdots, x_n) = K x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \quad (K > 0), \tag{17}$$

where the exponents a_i are real numbers. If the a_i are all positive, then Γ satisfies the hypotheses of Theorem 2. And if they satisfy the additional condition of Theorem 4, then all the solutions of (16) behave like the special solution provided by Theorem 3. This in particular is applicable to Golomb's sequence.

The proof of Theorem 4, however, requires a particular recursive argument, (R1) below, which so far we couldn't adapt to functions Γ of a different type than (17). But Theorem 2 has a considerably wider range of application than just functions of type (17): see for instance Example 1 in Section 5. Also there are many cases where it doesn't apply because the simple but rather restrictive conditions (C2) and (C3) don't hold, but where its conclusion, or possibly a slightly weaker property, is nevertheless satisfied: see Examples 2 and 3 in Section 5. So we also propose another recursive process (R2) having a wide range of application, but which doesn't usually yield an information as precise as that provided by Theorem 4. (In some cases it does: see Example 1). In general one can establish lower and upper bounds for all the solutions of (16) with this process. For instance, in the special case where Γ is of type (17), but this time with the only restriction that the a_i be positive, it yields Theorem 5. And finally, in Example 3, we apply (R2) to a sequence F satisfying a weaker form of Theorem 2.

2. Statement of the results.

Our first result in this section states that if the solution F of (9) increases slowly (C1) and not too irregularly (C2 and C3), then it asymptotically behaves like a solution of the corresponding appproximate (standard) functional differential equation (16).

Theorem 2. *Let F be the solution of (9), where Γ and Γ_F are as in (11), and put, for real positive t, $F(t) := F([t])$ and $\Gamma_F(t) := \Gamma_F([t])$. Assume that Γ satisfies the following conditions.*

(C1) $\Gamma[F](m) \to \infty$ *as* $m \to \infty$,

(C2) $\Gamma[F](m+1) \sim \Gamma[f](m))$ *as* $m \to \infty$, *and*

(C3) $\Gamma[F](t) \sim \Gamma[h](t)$ *as* $t \to \infty$ *if* $F(t) = h(t) + O(1)$ *as* $t \to \infty$.

Then there is an increasing and differentiable real function $f : [0, \infty) \to [0, \infty)$ satisfying

$$f(t) = F(t) + O(1) \quad (t \to \infty) \tag{18}$$

and the approximate functional-differential equation (16)

Remarks. 1. The following slightly more restrictive condition implies both (C2) and (C3).

(C4) $\Gamma[F](t) \sim \Gamma(F_1(t) + O(1), \cdots, F_n(t) + O(1))$ as $t \to \infty$.

2. A function Γ of type (17), with positive exponents a_i, as in Examples (14) and (15) of the introduction, satisfies the conditions (C1) and (C4).

3. The conditions (C2) and (C3) are very practical, both having a simple expression, and allowing the use of a simple argument in the proof below. But very clearly too they are not optimal. In Examples 2 and 3 of Section 5 below, the mere fact that $\Gamma[F](m)$ grows quicker than $e^{F_k(m)}$ for some k, even though it is extremely regular, makes Theorem 2 unapplicable.

4. If we replace the symbol "$\sim$" in (C2) and (C3) (or in (C4)) by the symbol "$\asymp$", then we get a weaker form of Theorem 2: we may conclude, as noted in Example 3, that there is an increasing and differentiable real function $f : [0, \infty) \to [0, \infty)$ satisfying $f(t) = F(t) + O(1) \quad (t \to \infty)$ and the approximate functional-differential relation $f'(t) \asymp (\Gamma[F](t))^{-1}$.

Now suppose the function Γ is as in (17), a monomial of "degree" $d(\Gamma) = a_1 + a_2 + \cdots + a_n$ where the exponents a_i are real numbers (not necessarily positive), and let P be the polynomial

$$P(X) = 1 - (a_1 + 1)X - a_2 X^2 - \cdots - a_n X^n. \tag{19}$$

Note that we have $P(1) = -d(\Gamma)$. We first obtain an explicit solution of (13) under these assumptions.

Theorem 3. *For each real positive root β of P the functional-differential equation (13) has the solution*

$$g_+(t) = \beta^{(1-\beta)(1+Q)/P(1)} t^\beta \quad (t \geq 0), \tag{20}$$

where $K = \beta^Q$ is the constant K of (17).

Remark. If for instance we consider the functional-differential equation (8) related to Golomb's sequence, then $\Gamma(x_1, x_2) = x_2$, so that $P(X) = 1 - X - X^2$ which has real positive root $\beta = \phi - 1$, and as mentioned $g_+(t) = \phi^{2-\phi} t^{\phi-1}$ $(t \geq 0)$ is a solution of (8). But $\beta = -\phi$ is also a root of P, and equation (8) has another simple solution related to this negative root, $g_-(t) = -\phi^{-\phi-1}(-t)^{-\phi}$ $(t < 0)$. In general, with the notation and under the assumptions of Theorem 3, there is a natural way to extend the definition of the operator Γ, in order to let it also act on negative real functions f of a negative argument t, and with which the functional differential equation (13) also has simple explicit solutions in case P possesses real negative roots. One sets $\Gamma(t, f(t), \cdots, f_{n-1}(t)) = -|t|^{a_1} |f(t)|^{a_2} \cdots |f_{n-1}(t)|^{a_n}$. Then for each real negative root β of P the functional-differential equation $f'(t) = 1/\Gamma[f](t)$ has the solution

$$g_-(t) = -(-\beta)^{(1-\beta)/P(1)} (-t)^\beta \quad (t \leq 0). \tag{21}$$

Of course these negative solutions have no direct relationship with our concern here.

Corollary. *The solution g_+ of Theorem 3 has the fixed point $p_+ = \beta^{1/P(1)}$, and the solution g_- of Remark 1 has the fixed point $p_- = -(-\beta)^{1/P(1)}$.*

Now, assuming that the hypotheses of Theorem 2 are satisfied, what we need to do is to evaluate the rate of growth of this positive solution of the approximate equation (16) asymptotically behaving like F. We make use of two recursive processes.

(R1) When one explicit positive solution $g(t)$ of the exact equation (13) is known, like the function g_+ of Theorem 3, we write in the approximate equation (16) $f(t) = G(t)g_+(t)$ and get a functional-differential equation for the function G. It is sometimes possible to infer from this equation that $G(t) \to 1$ as $t \to \infty$. This idea works when Γ belongs to a certain subclass of the functions of type (17), as in Theorem 4 below. We could not so far exploit (R1) in other cases.

(R2) The second process exploits the relation

$$\int_c^T \Gamma(t, f(t), \ldots, f_{n-1}(t))dt \sim f^{-1}(T) \quad (T \to \infty) \tag{22}$$

which is the approximate integral equation equivalent to (16). To see this, note that the equalities

$$\begin{aligned}\int_c^T \Gamma(t, f(t), \cdots, f_{n-1}(t))dt &= \int_{f^{-1}(c)}^{f^{-1}(T)} \Gamma(f(u), f_2(u), \cdots, f_n(u))f'(u)du \\ &= \int_{f^{-1}(c)}^{f^{-1}(T)} \Gamma[f](u)f'(u)du = \int_{f^{-1}(c)}^{f^{-1}(T)} (1 + o(1))du\end{aligned}$$

are satisfied by a solution f of (16) if c is a large enough constant. This may yield, in case the process (R1) doesn't apply, and provided an upper (resp. lower) bound for Γ can be derived from an upper (resp. lower) bound for f, a recursive process producing not too bad estimates for f. In fact, it sometimes even provides an asymptotic equivalence for f, as in Example 1 of Section 5. On the other hand, when Γ is a function of type (17) to which Theorem 4 doesn't apply, it yields Theorem 5. And finally, even if we only have at our diposal for f the weak form of Theorem 2 mentioned in Remark 4, this can be translated into an integral equation, less precise than (22), but which can still be exploited to yield a recursive process (R2), as in Example 3.

Application of (R1) to some of the Γ of type (17).

Theorem 4. *With the notation of Theorem 3, if the real positive function f satisfies the approximate functional-differential equation (16), where Γ is as in (17), where each of the exponents a_i is non negative, one of them at least being non zero, and if*

$$\beta < \frac{2}{d(\Gamma) + 2}, \tag{23}$$

where β is the (unique) real positive root of P, then

$$f(t) \sim \beta^{(1-\beta)(1+Q)/P(1)} t^{\beta} \quad (t \to \infty).$$

This of course applies to the case where $\Gamma = x_2$ (corresponding to Golomb's sequence). It doesn't apply for instance to the case where $\Gamma = x_k$ if $k > 2$, but it does when $\Gamma = x_3^a$ if a belongs to the interval $(0, 2(\sqrt{2}-1))$. Concerning the likely optimality of the condition (23), see Section 6.

Applications of (R2) to the other Γ of type (17).
With the notation of Theorems 3 and 4, one sees that the unique positive root β of $P(X)$ (which satisfies $0 < \beta < 1$), is also the unique fixed point in $(0,1)$ of

$$g(X) := \frac{1}{1 + a_1 + a_2 X + \cdots + a_n X^{n-1}}.$$

In particular it is a fixed point of $h = g \circ g$.

Theorem 5.
(A) *If β is the only fixed point of h in $(0,1)$, then we have*

$$\log f(t) \sim \beta \log t \quad (t \to \infty).$$

(B) *If not, then there are in $(0,1)$ pairs of numbers (A_i, B_i) satisfying $g(A_i) = B_i$ and $g(B_i) = A_i$ (A_i and B_i being thus fixed points of h). Let A be the smallest A_i: then the corresponding B is the largest B_i, and we have*

$$t^{A-\epsilon} \ll f(t) \ll t^{B+\epsilon}$$

for every $\epsilon > 0$.
Moreover

$$A_0 \le \limsup \frac{\log f(t)}{\log t} \quad \text{and} \quad \liminf \frac{\log f(t)}{\log t} \le B_0.$$

where A_0, respectively B_0, is the positive root of

$$\frac{1}{A}(a_n x^{n+1} + \cdots + a_1 x^2) + x - 1 = 0, \quad \textit{respectively} \quad \frac{1}{B}(a_n x^{n+1} + \cdots + a_1 x^2) + x - 1 = 0.$$

And we have $0 < A < A_0 < \beta < B_0 < B < 1$.

Result (A) applies to the case where $\Gamma = x_k$ if $k = 3, 4$ and 5 But not when $\Gamma = x_k$ if $k \ge 6$, neither for instance when $\Gamma = x_2^2 x_3^8$. To these however, Result (B) applies; for the last sequence for instance we have $A_0 = 1/4$ and $B_0 = 1/2$.

3. Proofs.

Proof of Theorem 2. The proof being very similar to that of Proposition 1 in [P2] we will not go into all the details.

First we construct a continuous function f_0 whose graph consists in segments of straight lines, by setting $f_0(k) = F(k)$ for each integer k of the form

$$k = \sum_{\ell=1}^{m} \Gamma_F(\ell) \tag{24}$$

(note that k is the last integer satisfying $F(k) = m$), and for $k = 0$. Then we smooth off the angles in f_0, obtaining a differentiable function f satisfying (13), with $f(t) = f_0(t)$ everywhere except if t is in an interval of the form $(k-1/k, k+1/k)$ where the slope of f_0 changes. And if the slope of f_0 changes at k then we may ensure that the derivative $f'(t)$ is a monotone function of t on $(k-1/k, k+1/k)$. For two consecutive k and k' of the form (24) we have

$$\frac{1+\epsilon(F(k'))}{\Gamma[F](k')} = \frac{1}{\Gamma_F(F(k'))} = \frac{F(k')-F(k)}{k'-k} = \frac{f_0(k')-f_0(k)}{k'-k} = \frac{f(k'-\frac{1}{k'})-f(k+\frac{1}{k})}{k'-\frac{1}{k'}-k-\frac{1}{k}},$$

and this is $f'(t)$ for each t in the interval $[k+1/k, k'-1/k']$. For t in the interval $(k-1/k, k+1/k)$ we have $f'(t) = 1/(\Gamma_F(F(k)) + \delta(t))$, where $\delta(t)$ increases or decreases from 0 to $\Gamma_F(F(k')) - \Gamma_F(F(k))$ as t increases from $k-1/k$ to $k+1/k$. By the conditions (C1) and (C2) (and by (11)) we have thus $f'(t) = (1+o(1))/\Gamma[F](k)$ as $k \to \infty$ for all the t in the interval $(k-1/k, k'-1/k')$. An appeal to the condition (C3) concludes the proof of (18) and of Theorem 2.

The proof of Theorem 3 (and of Remark 1) is straightforward.

Proof of Theorem 4. The argument is inspired by the proof of Proposition 2 in [P2]. Let the function f satisfy the hypotheses of Theorem 4. If we put

$$f(t) = \beta^{(1-\beta)(Q+1)/P(1)} t^\beta G(t) =: ct^\beta G(t),$$

then we have

$$\begin{aligned}&(G(t) + \frac{t}{\beta}G'(t))G(t)^{a_1+a_2\beta+\cdots+a_n\beta^{n-1}} \qquad (25)\\ =&(1+\eta(t))G(f_{n-1}(t))^{-a_n}G(f_{n-2}(t))^{-a_{n-1}-a_n\beta}\cdots G(f(t))^{-a_2-a_3\beta-\cdots-a_n\beta^{n-2}},\end{aligned}$$

where $\eta(t) = o(1)$. We want to show that $G(t) \to 1$ as $t \to \infty$. We dispose first of an easy (and unlikely) case.

Lemma. *If $G-1$ eventually stays of the same sign, then $G(t) \to 1$ as $t \to \infty$.*

Proof. Suppose for instance that $G(t) \leq 1$ for all t large enough, and first assume that $G(t) \leq q$ for some $q < 1$ (and for all t large enough). Then from (25) we see that

$$\frac{t}{\beta}G'(t) \geq (1+o(1))q^{-\delta} - q \geq k(1+o(1)),$$

where δ and k are positive constant, whence $G(t) \gg \log t$ as $t \to \infty$, a contradiction. Thus G takes values arbitrarily close to (and not exceeding) 1. Now for $\epsilon > 0$ let T be large enough to ensure that

$$\begin{aligned}&(G(t) + \frac{t}{\beta}G'(t))G(t)^{a_1+\cdots+a_n\beta^{n-1}}\\ &\qquad > (1-\epsilon)G(f_{n-1}(t))^{-a_n}G(f_{n-2}(t))^{-a_{n-1}-a_n\beta}\cdots G(f(t))^{-a_2-\cdots-a_n\beta^{n-2}} \qquad (26)\end{aligned}$$

when $t \geq T$ (such a T exists by (25)). There is a $t_* \geq T$ with $G(t_*) \geq 1-\epsilon$, and any $t > t_*$ with $G(t) < 1-\epsilon$ would, by (26) and the fact that $G(f_k(t)) < 1$ for

$k = 1, \cdots, n-1$, satisfy $G'(t) > 0$. So there cannot be such a t, and $G(t) \geq 1 - \epsilon$ for every $t \geq t_*$. This concludes the proof of the lemma.

We return to the proof of Theorem 4, and we may now assume that $G(t) - 1$ changes sign for arbitrarily large values of t. We first note that an easy argument (by contradiction) shows that the increasing function $f(t)$ takes arbitrarily large values, whence $f'(t) \to 0$ as $t \to \infty$, whence for all t large enough we have $f(t) < t/2$. We denote by E_- the set of local minima less than 1 of G, by E_+ the set of local maxima exceeding 1 of G, and we put $E = E_- \cup E_+$. If $0 < \epsilon < 1/2$ we consider an interval $[T_1, T_2]$, where T_1 is sufficiently large to ensure that the factor $1 + \eta(t)$ of (25) is in the interval $(1 - \epsilon, 1 + \epsilon)$ when $t \geq T_1$, and where $T_2 > T_1$ is large enough to ensure that

(i) $T_1 < f_{n-1}(t) < f_{n-2}(t) < \cdots < f(t) < t/2$ when $t > T_2$, and that

(ii) the interval $(T_1, f_{n-1}(T_2))$ contains at least an element of E_- and one of E_+.

We put

$$G_- = \min_{t \in [T_1, T_2]} G(t), \quad G_+ = \max_{t \in [T_1, T_2]} G(t),$$

and

$$A = \max(G_+, 1/G_-) \quad (\text{so that } 1/A \leq G_- \leq G_+ \leq A).$$

Theorem 4 will be proved if we ensure that

$$\liminf_{\substack{t \to \infty \\ t \in E_-}} G(t) = 1 \quad \text{and} \quad \limsup_{\substack{t \to \infty \\ t \in E_+}} G(t) = 1.$$

We prove the first equality. For an element m_0 of E_- larger than T_2 we have from (25)

$$G(m_0) = (1 + \eta(m_0))^\beta G(f(m_0))^{-A_1} \cdots G(f_{n-1}(m_0))^{-A_{n-1}},$$

where $|\eta(m_0)| < \epsilon$, where we use $(1 + a_1 + a_2\beta + \cdots + a_n\beta^{n-1})^{-1} = \beta > 0$, and where by the condition (23) we have $A_i \geq 0$ $(1 \leq i \leq n-1)$ and $A_1 + A_2 + \cdots + A_{n-1} =: A_0 < 1$ (which in fact is equivalent to (25)).

We now choose a number m_{1,k_1} as follows. If $G(f_{k_1}(m_0)) \leq 1$, or if $G(f_{k_1}(m_0)) > 1$ and $G'(f_{k_1}(m_0)) \leq 0$, then m_{1,k_1} is the last element of E_+ not exceeding $f_{k_1}(m_0)$. If $G(f_{k_1}(m_0)) > 1$ and $G'(f_{k_1}(m_0)) > 0$, then $m_{1,k_1} = f_{k_1}(m_0)$. We have $G(m_{1,k_1}) \geq G(f_{k_1}(m_0))$ and thus

$$G(m_0) \geq (1 + \eta(m_0))^\beta G(m_{1,1})^{-A_1} \cdots G(m_{1,n-1})^{-A_{n-1}}.$$

Moreover the choice of m_{1,k_1} ensures that

$$G(m_{1,k_1}) \leq (1 + \eta_1)^\beta G(f(m_{1,k_1}))^{-A_1} \cdots G(f_{n-1}(m_{1,k_1}))^{-A_{n-1}},$$

where $\eta_1 = \max \eta(m_{1,k_1}) \leq \epsilon$. (The number η_1 is positive; at the next step the number η_2 will be negative, with $|\eta_2| < \epsilon$).

If $m_{1,k_1} > T_2$ we similarly choose numbers $m_{2,k_1,k_2} \leq f_{k_2}(m_{1,k_1})$ $(1 \leq k_2 \leq n-1)$ for which $G(m_{2,k_1,k_2}) \leq G(f_{k_2}(m_{1,k_1}))$ and

$$G(m_{1,k_1}) \leq (1 + \eta_1)^\beta G(m_{2,k_1,1})^{-A_1} \cdots G(m_{2,k_1,n-1})^{-A_{n-1}}.$$

And thus we have

$$G(m_0) \geq (1+\eta(m_0))^{\beta}(1+\eta_1)^{-\beta A_0} \prod_{\substack{k_1 \leq n-1 \\ k_2 \leq n-1 \\ m_{1,k_1} > T_2}} G(m_{2,k_1,k_2})^{A_{k_1}A_{k_2}}.$$

Continuing in this way we obtain recursively estimates of type

$$G(m_0) \geq (1+\eta(m_0))^{\beta}(1+\eta_1)^{-\beta A_0}(1+\eta_2)^{\beta A_0^2} \cdots (1+\eta_{\ell-1})^{(-1)^{\ell-1}\beta A_0^{\ell-1}} \times$$

$$\times \prod_{\substack{k_1 \leq n-1 \\ k_2 \leq n-1 \\ \cdots \\ k_{\ell-1} \leq n-1 \\ m_{\ell-1,k_1,\cdots,k_{\ell-1}} > T_2}} G(m_{\ell,k_1,\cdots,k_\ell})^{(-1)^{\ell}A_{k_1}A_{k_2}\cdots A_{k_\ell}}. \quad (27)$$

Since $f(t) < t/2$ for $t > T_2$, the process must end after a finite number of steps. If the last step is the $\ell-$th one all the arguments of G in the last product of (27) lie in the interval $[T_1, T_2]$. Thus we have

$$G(m_0) \geq (1-\epsilon)^{\beta \sum_{i\geq 0} A_0^i} \prod_{\substack{k_1 \leq n-1 \\ k_2 \leq n-1 \\ \cdots \\ k_{\ell-1} \leq n-1 \\ m_{\ell-1,k_1,\cdots,k_{\ell-1}} > T_2}} A^{-A_{k_1}A_{k_2}\cdots A_{k_\ell}}.$$

Hence,

$$G(m_0) \geq (1-\epsilon)^{\beta(1-A_0)^{-1}} A^{-A_0^{\ell'}},$$

where $\ell' \leq \ell$ is the last integer for which $m_{\ell'-1,k_1,\cdots,k_{\ell'-1}} > T_2$ holds for every choice of $(k_1, \cdots, k_{\ell'-1})$ in $\{1, 2, \cdots, n-1\}^{\ell'-1}$. Now as $m_0 \to \infty$, this number ℓ' of steps performed must also tend to infinity, and thus

$$\liminf_{\substack{t\to\infty \\ t \in E_-}} G(t) \geq (1-\epsilon)^{\beta(1-A_0)^{-1}}.$$

Letting $\epsilon \to 0$ finishes the proof.

Proof of Theorem 5. If f satisfies an approximate functional-differential equation of type (16), with Γ as in (12) and (17), and with nonnegative exponents a_i (one of them at least being non zero), then clearly $f(t)$ is unbounded and eventually increasing, whence $f'(t) \to 0$ and $f(t) = o(t)$ as $t \to \infty$. And in particular there is a positive constant c, such that the integrals used to establish formula (22) are well defined, and such that (22) holds. Hence, putting $\alpha_0 = 1$ we have

$$f(t) \ll t^{\alpha_0} \quad \text{as} \quad t \to \infty$$

and

$$f^{-1}(T) = (1+o(1)) \int_c^T \Gamma(t, f(t), \cdots, f_{n-1}(t))dt$$

$$\ll \int_c^T t^{a_1+a_2\alpha_0+\cdots+a_n\alpha_0^{n-1}} dt \ll T^{a_1+a_2\alpha_0+\cdots+a_n\alpha_0^{n-1}+1},$$

whence

$$f(t) \gg t^{g(\alpha_0)} \quad \text{as} \quad t \to \infty.$$

Repeating the process with $\alpha_1 = g(\alpha_0)$ (> 0) we obtain

$$f(t) \ll t^{g(\alpha_1)} \quad \text{as} \quad t \to \infty.$$

If we put $\alpha_n = g(\alpha_{n-1})$ ($n \geq 1$) we obtain recursively, for every integer $k \geq 0$,

$$f(t) \ll t^{\alpha_{2k}} \quad \text{and} \quad f(t) \gg t^{\alpha_{2k-1}} \quad \text{as} \quad t \to \infty.$$

It is easy to check, recursively, that the sequence $\{\alpha_{2k}\}_{k=0}^{\infty}$ is decreasing, and that the sequence $\{\alpha_{2k-1}\}_{k=0}^{\infty}$ is increasing. They thus converge, say respectively to B and to $A \leq B$. Since $\alpha_{n+2} = (g \circ g)(\alpha_n) = h(\alpha_n)$, both A and B are fixed points of h. This proves (A) and the first display of (B).

In order to prove the second display of (B) we proceed as follows. Suppose for instance that for some number b_0 we have $f(t) \gg t^{b_0}$. It follows that

$$\Gamma[f](t) \gg t^{a_1 b_0 + \cdots + a_n b_0^n}.$$

Now since $f(t) \ll t^{B+\epsilon}$ for every positive number ϵ, if we denote by B_ϵ the exponent $B + \epsilon$ we have

$$\Gamma[f](t) \gg f(t)^{\frac{1}{B_\epsilon}(a_1 b_0 + \cdots + a_n b_0^n)}.$$

With (16) we thus see that

$$\frac{d}{dt}\left(f(t)^{\frac{1}{B_\epsilon}(a_1 b_0 + \cdots + a_n b_0^n)+1}\right) \asymp f'(t) f(t)^{\frac{1}{B_\epsilon}(a_1 b_0 + \cdots + a_n b_0^n)} \ll 1,$$

whence

$$f(t) \ll t^{\left(\frac{1}{B_\epsilon}(a_1 b_0 + \cdots + a_n b_0^n)+1\right)^{-1}}.$$

But this is compatible with $f(t) \gg t^{b_0}$ only if

$$\frac{1}{B_\epsilon}(a_1 b_0^2 + \cdots + a_n b_0^{n+1}) + b_0 \leq 1.$$

If we let $\epsilon \to 0$ it becomes clear that b_0 cannot exceed B_0.

Thus we proved that when $\epsilon > 0$, $f(t) \gg t^{B_0+\epsilon}$ cannot hold; the proof that $f(t) \ll t^{A_0-\epsilon}$ cannot hold either is exactly similar.

Finally, the proof of $A_0 < \beta < B_0$ (in Theorem 5(B)) is straightforward; the other inequalities are trivial.

4. On Theorems 4 and 5.

There are two objects to this section. First we show that the hypothesis in Theorem 5(A) is satisfactory, in the following sense: if Theorem 5(A) is not applicable, then neither is Theorem 4. Then, conversely, we derive from the hypothesis of Theorem 5(A) an inequality involving the positive root β of P which is fairly easy to verify: when it is not satisfied only Theorem 5(B) applies.

When Γ is of the form (17), with nonnegative exponents a_i, Theorem 5 gives us a good estimate of the order of growth of F if there are no numbers a and b with $0 < a < \beta < b < 1$ satisfying

$$(a_1 + 1) + a_2 a + \cdots + a_n a^{n-1} = \frac{1}{b} \tag{28}$$

and

$$(a_1 + 1) + a_2 b + \cdots + a_n b^{n-1} = \frac{1}{a}, \tag{29}$$

where β is the positive solution of

$$(a_1 + 1)\beta + a_2\beta^2 + \cdots + a_n\beta^n = 1. \tag{30}$$

Now on the one hand if there *are* numbers a and b with $0 < a < \beta < b < 1$ satisfying (28) and (29), then the estimates we can prove on F are not as precise, and on the other hand if (23) holds then we can prove an asymptotic equivalence for F. So it appears legitimate to ask whether there are functions Γ of the form (17), with nonnegative exponents a_i, such that there are numbers a and b with $0 < a < \beta < b < 1$ satisfying (28) and (29), *but also* such that the inequality (23) is satisfied. The answer to this question is no.

Lemma. *If, for nonnegative real numbers a_i, the equations (28), (29) and (30) hold for real numbers a, β and b with $0 < a < \beta < b < 1$, then the inequality*

$$\beta \leq \frac{2}{d(\Gamma) + 2}, \tag{23'}$$

cannot hold.

Proof. From (28) and (29) we have

$$a_2(b - a) + \cdots + a_n(b^{n-1} - a^{n-1}) = \frac{1}{a} - \frac{1}{b},$$

that is, since $a < b$,

$$a_2 + a_3(a + b) + \cdots + a_n(a^{n-2} + \cdots + b^{n-2}) = \frac{1}{ab}. \tag{31}$$

Now since $a < \beta$ and $b < 1$ we have $a^j + a^{j-1}b + \cdots + ab^{j-1} + b^j < \beta^j + \beta^{j-1} + \cdots + \beta + 1 = (1 - \beta^{j+1})/(1 - \beta)$, whence, from (31) and (30),

$$\frac{1}{ab} < \frac{1}{1-\beta}(a_1 + a_2 + \cdots + a_n - (a_1 + a_2\beta + \cdots + a_n\beta^{n-1})) = \frac{1}{1-\beta}(d(\Gamma) - (1-\beta)/\beta).$$

If we assume that (23') holds, this immediately yields $1/ab < 1/\beta$ or $ab > \beta$, a contradiction with $a < \beta$ and $b < 1$.

This concludes the proof of the lemma, which can be rephrased as

Lemma A. *If*

$$\beta \leq \frac{2}{d(\Gamma) + 2},$$

then the function $h = g \circ g$ of Theorem 5 has a unique positive fixed point (which is β).

Conversely we have the following.

Lemma B. *If the function $h = g \circ g$ of Theorem 5 has a unique positive fixed point (which then must be β), then we have*

$$Q'(\beta) \leq \frac{1}{\beta^2}, \tag{32}$$

where

$$Q(X) = \frac{1 - P(X)}{X} = (a_1 + 1) + a_2 X + \cdots + a_n X^{n-1}.$$

Proof. The hypothesis of the lemma implies that there are no numbers a and b with $0 < a < \beta < b < 1$ satisfying (28) and (29), that is $Q(a) = 1/b$ and $Q(b) = 1/a$. This in turn implies that for $0 < a < \beta$ we have $Q(1/Q(a)) \neq 1/a$. By continuity, and since $Q(1/Q(a)) - 1/a < 0$ when a is small enough, we must have $Q(1/Q(a)) < 1/a$ for $0 < a < \beta$. So if we put $Q(a) = 1/b$ the inequality

$$a_2(b - a) + \cdots + a_n(b^{n-1} - a^{n-1}) < \frac{1}{a} - \frac{1}{b}$$

is satisfied for $0 < a < \beta$, whence

$$a_2 + a_3(a + b) + \cdots + a_n(a^{n-2} + \cdots + b^{n-2}) < \frac{1}{ab}.$$

Now letting $a \to \beta$ we have $b \to \beta$ and the lemma.

Application. Let $F = {}_kF$ be the increasing solution of

$$F'(m) = \frac{1}{F_k(m)} \quad \text{with} \quad F(1) = 1.$$

Note that ${}_2F$ is Golomb's sequence. The corresponding polynomial ${}_kQ = Q$ is

$$Q(X) = 1 + X^{k-1},$$

and the condition (32) can be rephrased as

$$\beta \geq \frac{k-2}{k-1}.$$

this is true for $k \leq 5$ and false for $k \geq 6$. This means that only Theorem 5(B) is applicable to $k \geq 6$. One can easily check that Theorem 5(A) is applicable to $k \leq 5$.

5. Remarks on Theorems 1 and 2, and on the recursive process (R2).

For a sequence F whose derivative satisfies $F'(m) = 1/\Gamma_F(F(m))$, with $\Gamma_F(F(m)) + O(1) = \Gamma[F](m) = \Gamma(F(m), F_2(m), \cdots, F_n(m))$ for a differentiable function Γ, the second technique (R2) we described above can at best yield an estimate as in Theorem 5(A), when the function Γ is of the form (17), i.e. $\Gamma(x_1, x_2, \cdots, x_n) =$

$Kx_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$, with positive exponents a_i. Whereas the first technique (R1) sometimes yield an asymptotic equivalence through Theorem 4. But Γ doesn't need to be as in (17) for the recursive process (R2) to be applicable. And the example we now choose to describe shows that very precise estimates can sometimes be obtained with this method.

Example 1. We let F be the nondecreasing and onto solution of

$$F'(m) = \frac{1}{[\log(F\circ F)(m)]+1}, \quad F(1)=1,$$

that is

$$1, 2, \quad \underbrace{3, 3,}_{[\log F(3)]+1} \quad \underbrace{4, 4}_{[\log F(4)]+1} \quad \cdots \quad \underbrace{12, 12,}_{[\log F(12)]+1} \quad \underbrace{13, 13, 13,}_{[\log F(13)]+1} \cdots .$$

We see that the hypotheses of Theorem 2 are satisfied, and this ensures that F doesn't lie farther than a constant apart from a solution of the approximate functional differential equation

$$f'(t) = \frac{1+o(1)}{\log(f\circ f)(t)}. \tag{33}$$

Now a first application of the recursive process (R2) yields, after the easy observation that a solution must satisfy $f(t) \ll t$, the estimate $f^{-1}(T)(1+o(1)) = \int_c^T \log f(t)dt \ll T\log T$ or

$$f(S) \gg \frac{S}{\log S},$$

whence a second application yields $f^{-1}(T)(1+o(1)) = \int_c^T \log(t/\log t)dt \gg T\log T$ or

$$f(S) \ll \frac{S}{\log S}.$$

So $f(t)$ is of the order $t/\log t$ as $t\to\infty$, whence $\log f(t) = \log t - \log\log t + O(1)$. Finally a third application of the process yields $f^{-1}(T)(1+o(1)) = T\log T$ or

$$f(S) \sim \frac{S}{\log S}.$$

We cannot say more concerning a solution of (33). But a close look at the proof of Theorem 2 shows that in this case, instead of the condition (C3), the more precise following condition holds: "For every function h with $h(t) = F(t)+O(1)$ we have $\Gamma[F](t) = \log(h\circ h)(t) + O(1)$". This ensures that F in fact doesn't lie farther than a constant apart from a solution of the approximate functional differential equation

$$f'(t) = \frac{1+O\left(\frac{1}{\log(f\circ f)(t)}\right)}{\log(f\circ f)(t)} = \frac{1+O\left(\frac{1}{\log t}\right)}{\log(f\circ f)(t)}. \tag{34}$$

With (34) instead of (33) the third application of the process (R2) yields the more precise $f^{-1}(T)(1+O(1/\log T)) = T\log T - T\log\log T + O(T)$ or

$$f(S) = \frac{S}{\log S} + \frac{2S\log\log S}{\log^2 S} + O\left(\frac{S}{\log^2 S}\right).$$

Example 2. In this example and in the next one, we consider sequences F which, although they are very regularly increasing, don't satisfy the hypotheses of Theorem 2: (C1) is satisfied, but not (C2) and (C3). We let F be the nondecreasing and onto solution of

$$F'(m) = \frac{1}{[e^{F(m)}]}, \quad F(1) = 1,$$

that is

$$\underbrace{1,1}_{[e]}, \underbrace{2,2,2,2,2,2,2}_{[e^2]}, \underbrace{3,\cdots,3}_{[e^3]=20}, \cdots.$$

We have $\Gamma_F(k) = [e^k]$, and clearly condition (C4) is thus not satisfied. (In fact it is not difficult too see that neither (C2) nor (C3) is satisfied). But the first and last N such that $F(N) = k$ are respectively given by $N_f = e^k/(e-1) + O(k)$ and $N_\ell = e^{k+1}/(e-1) + O(k)$ (this shows for instance that (C2) is not satisfied), so that $F(n) = \log n + O(1)$. And since the function $f(t) = \log t$ is a solution of $f'(t) = 1/e^{f(t)}$, we see that the conclusion of Theorem 2 is nevertheless satisfied by F.

Note that in this case, the sequence of the runs of F being given by the explicit function $\gamma_0(k) = [e^k]$, we may also appeal to Theorem 1, which in fact yields a better estimate. With $\gamma(t) = e^t$, the solutions of (2) are the functions $f(t) = \log(t + C)$, and the $E(x)$ of (4) is a $O(1)$. Thus (5) with the trivial estimate $F(m) \leq m$ yields $F(m) = \log m + O(1)$, and then (5) again yields

$$F(m) = \log m + O\left(\frac{\log m}{m}\right).$$

Example 3. Now we let F be the nondecreasing and onto solution of

$$F'(m) = \frac{1}{[e^{F\circ F(m)}]}, \quad F(1) = 1,$$

that is

$$\underbrace{1,1}_{[e]}, \underbrace{2,2}_{[e]}, \underbrace{3,\cdots,3}_{[e^2]=7}, \underbrace{4,\cdots,4}_{[e^2]=7}, \underbrace{5,\cdots,5}_{[e^3]=20}, \cdots.$$

We have $\Gamma_F(k) = [e^{F(k)}]$, and (C2) and (C3) are not satisfied in this case either. However, the weaker conditions

(C2') $\Gamma[F](m+1) \asymp \Gamma[F](m))$ as $m \to \infty$, and

(C3') $\Gamma[F](t) \asymp \Gamma[h](t)$ as $t \to \infty$ if $F(t) = h(t) + O(1)$ as $t \to \infty$,

hold. And with these the argument in the proof of Theorem 2 shows that there is an increasing and differentiable real function $f : [0,\infty) \to [0,\infty)$ with $f(t) = F(t) + O(1) \quad (t \to \infty)$ satisfying the approximate functional-differential equation

$$f'(t) \asymp \frac{1}{\Gamma[f](t)}. \tag{16'}$$

The integral equation equivalent to (16') is

$$\int_c^T \Gamma(t, f(t))dt = \int_c^T e^{f(t)}dt \asymp f^{-1}(T) \qquad (22'),$$

and is fit to be used in a recursive process of type (R2). By using the fact that $f(t)$ becomes eventually smaller than t, we obtain with the first step of this process $f(t) \geq \log t + O(1)$. Then the second step yields $f(t) \ll \sqrt{t}$, the third step $f(t) \gg \log^2 t$, and the fourth and fifth steps

$$e^{c_1(\log\log s)^2} \leq f(t) \leq e^{c_2\sqrt{\log s}},$$

for some positive constants c_1 and c_2. Each step will increase the precision of one of the asymptotic bounds obtained so far.

6. Last remark and thanks.

We are so far unable to prove or disprove the optimality of hypothesis (23) in Theorem 4, i.e to decide whether there is a function f satisfying (16), with Γ as in (17), $a_i \geq 0$ $(i = 1, \cdots, n)$, $\sum a_i > 0$ and

$$\beta = \frac{2}{d(\Gamma) + 2} \tag{35}$$

where β is the real positive root of P, and such that

$$f(t) \sim \beta^{(1-\beta)(1+Q)/P(1)} t^{\beta} \quad (t \to \infty) \tag{36}$$

is not satisfied. Extensive computation made on the self-described sequence $F(n)$ defined by $F(1) = 1$ and $|F^{-1}(n)| = F(n)^2$, however, provide a very strong indication that such a function exists. Theorem 2 ensures that there is a function f, with $f(t) = F([t]) + O(1)$, satisfying (16) with $\Gamma(x_1, x_2) = x_2^2$, and thus satisfying (35) (with $\beta = 1/2$). For this function f the right hand side of (36) is $2^{1/4}t^{1/2}$. But the computation seems to indicate that $R(n) := F(n)2^{-1/4}n^{-1/2} \sim 1$ is not true, although the function $R(n)$ appears to have a periodic behaviour when represented on the scale $u = \log\log n/\log 2$. This can be seen on the figure below, where $R(n)$ is plotted for 1001 regularly spaced (on the $\log\log(n)/\log 2$ scale) values of n between 34981 and $.2541 \times 10^{141}$. Our thanks to Catherine Gallice, Tony Sickler and Guillaume Urban, students at the ESIAL, the engineering school of the University of Nancy 1, who implemented this computation as part of their "Initiation to research" module.

Figure

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