MEKH-MAT ENTRANCE EXAMINATIONS PROBLEMS

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The recent article of Anatoly Vershik and Alexander Shen [21] [19] describes discrimination against Jews in Soviet universities during the 1970's and 1980's. The article contains a report by Alexander Shen on the specific role of examinations in discrimination against Jewish applicants to the Mekh–mat at Moscow State University during the 1970's and 1980's. The article goes on to list "killer problems" that were given to Jewish candidates. However, solutions to the problems were not given in the article, so in order to judge their difficulty, one must try the problems for oneself. The aim of this note is to relieve readers of this time consuming task by providing a full set of solutions to the problems. Hopefully, this will help readers gain some insight into the ethical questions involved.

Section 2 consists of a personal evaluation of the problems in the style of a referee's report. It was written to provide a template for readers to make a similar evaluation of the problems. This evaluation also reflects the author's own mathematical strengths and weaknesses as well as his approach to problem solving. Readers are therefore encouraged to make up their own minds.

The problems are given exactly as in [19] with the names of the examiners and the year (A. Shen has explained that in his article, the name of the examiners and year is given by a set of problems ending with the name). Some inaccuracies of [19] both in the statement of the problems and attribution of examiners have have been corrected, see Section 4. Some of the statements are nevertheless incorrect. These errors are a reflection of either the examinations themselves, the report given by the students, or the article of [19]. In any case, this is further evidence for the need of a complete solution set.

These solutions were worked out during a six week period in July and August 1999. In order to retain some aspect of an examination, no sources were consulted. As a result, the solutions reflect gaps in the author's background. However, this might offer some insight into how one can deal with a wide range of elementary problems without the help of outside references. An effort was therefore made to explain how the solutions were found. The solutions are the most direct that the author could come up with, so some unobvious tricks may have been overlooked.

After completing these problems, the author discussed them with other mathematicians who, in some cases, found much better solutions. These solutions are therefore given along with the author's solutions in Section 3. Section 4 provides notes on the problems such as outside references and historical remarks.

The most egregious aspect of these problems is the fact that they are, to the author's knowledge, the only example in which mathematics itself has been used a political tool. It is important to note that there is absolutely no controversy about whether this discrimination actually took place–it appears that antisemitism at the Mekh–mat was accepted as a fact of life. It is the author's conviction that the best course of action

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now is to provide as much information as possible about what took place. A more detailed account of the political practices described by Vershik and Shen should follow.

Acknowledgment. I would like to thank Marcel Berger, Jean–Pierre Bourgignon, Ofer Gabber, Mikhail Grinberg, Leonid Polterovich, Igor Rivin, and Anatoly Vershik for helpful remarks. A. Shen was kind enough to supply supplementary information and explanations about the problems. I would like to thank the following persons for communicating their solutions: Marcel Berger (**1**), David Fried, Ofer Gabber (**22**), Georg Illies (**2**), Igor Rivin (**2**, **22**), David Ruelle (**2**), Pavol Severa (**1**, **2**), and Michel Waldschmidt (**4**).

Finally, I would like to thank Victor Kac for bringing these problems to my attention.

1 Problems

1. (Lawrentiew, Gnedenko, O.P. Vinogradov, 1973) (V.F. Maksimov, Falunin, 1974) *K is the midpoint of a chord AB. MN and ST are chords that pass through K. MT intersects AK at a point P and NS intersects KB at a point Q. Show that $KP = KQ$.*

2. (Maksimov, Falunin, 1974) *A quadrangle in space is tangent to a sphere. Show that the points of tangency are coplanar.*

3. (Nesterenko, 1974) *The faces of a triangular pyramid have the same area. Show that they are congruent.*

4. (Nesterenko, 1974) *The prime decompositions of different integers m and n involve the same primes. The integers $m+1$ and $n+1$ also have this property. Is the number of such pairs (m,n) finite or infinite?*

5. (Podkolzin, 1978) *Draw a straight line that halves the area and perimeter of a triangle.*

6. (Podkolzin, 1978) *Show that $(1/\sin^2 x) \leq (1/x^2) + 1 - 4/\pi^2$ for $0 < x < \pi/2$.*

7. (Podkolzin, 1978) *Choose a point on each edge of a tetrahedron. Show that the volume of at least one of the resulting tetrahedrons is $\leq 1/8$ of the volume of the initial tetrahedron.*

8. (Sokolov, Gashkov, 1978) *We are told that $a^2 + 4\,b^2 = 4$, $cd = 4$. Show that $(a-d)^2 + (b-c)^2 \geq 1.6$.*

9. (Fedorchuk, 1979; Filimonov, Proshkin, 1980) *We are given a point K on the side AB of a trapezoid $ABCD$. Find a point M on the side CD that maximizes the area of the quadrangle which is the intersection of the triangles AMB and CDK.*

10. (Pobedrya, Proshkin, 1980) *Can one cut a three–faced angle by a plane so that the intersection is an equilateral triangle?*

11. (Vavilov, Ugol'nikov, 1981) *Let H_1, H_2, H_3, H_4, be the altitudes of a triangular pyramid. Let O be an interior point of the pyramid and let h_1, h_2, h_3, h_4 be the perpendiculars from O to the faces. Show that $H_1^4 + H_2^4 + H_3^4 + H_4^4 \geq 1024\, h_1 \cdot h_2 \cdot h_3 \cdot h_4$.*

12. (Vavilov, Ugol'nikov, 1981) *Solve the system of equations $y(x+y)^2 = 9$, $y(x^3 - y^3) = 7$.*

13. (Dranishnikov, Savchenko, 1984) *Show that if a, b, c are the sides of a triangle and A, B, C are its angles, then*

$$\frac{a+b-2c}{\sin(C/2)} + \frac{b+c-2a}{\sin(A/2)} + \frac{a+c-2b}{\sin(B/2)} \geq 0\,.$$

14. (Dranishnikov, Savchenko, 1984) *In how many ways can one represent a quadrangle as the union of two triangles?*

15. (Bogatyi, 1984) *Show that the sum of the numbers $1/(n^3 + 3n^2 + 2n)$ for n from 1 to 1000 is $< 1/4$.*

16. (Evtushik, Lyubishkin, 1984) *Solve the equation $x^4 - 14x^3 + 66x^2 - 115x + 66.25 = 0$.*

17. (Evtushik, Lyubishkin, 1984) *Can a cube be inscribed in a cone so that 7 vertices of the cube lie on the surface of the cone?*

18. (Evtushik, Lyubishkin, 1986) *The angle bisectors of the exterior angles A and C of a triangle ABC intersect at a point of its circumscribed circle. Given the sides AB and BC, find the radius of the circle.* [From [19]: "The condition is incorrect: this doesn't happen."]

19. (Evtushik, Lyubishkin, 1986) *A regular tetrahedron $ABCD$ with edge a is inscribed in a cone with a vertex angle of $90°$ in such a way that AB is on a generator of the cone. Find the distance from the vertex of the cone to the straight line CD.*

20. (Smurov, Balsanov, 1986) *Let $\log(a, b)$ denote the logarithm of b to base a. Compare the numbers $\log(3, 4) \cdot \log(3, 6) \cdot \ldots \cdot \log(3, 80)$ and $2 \log(3, 3) \cdot \log(3, 5) \cdot \ldots \cdot \log(3, 79)$*

21. (Smurov, Balsanov, 1986) *A circle is inscribed in a face of a cube of side a. Another circle is circumscribed about a neighboring face of the cube. Find the least distance between points of the circles.*

22. (Andreev, 1987) *Given k segments in a plane, show that the number of triangles all of whose sides belong to the given set of segments is less than $C\,k^{3/2}$, for some positive constant C which is independent of k.*

23. (Kiselev, Ocheretyanskii, 1988) *Use ruler and compasses to construct, from the parabola $y = x^2$, the coordinate axes.*

24. (Tatarinov, 1988) *Find all a such that for all $x < 0$ we have the inequality $ax^2 - 2x > 3a - 1$.*

25. (Podol'skii, Aliseichik, 1989) *Let A, B, C be the angles and a, b, c the sides of a triangle. Show that*

$$60° \leq \frac{aA + bB + cC}{a + b + c} \leq 90° .$$

2 Evaluation of the Problems

I have classified the problems according to difficulty, inherent interest, and correctness. The first two criteria are subjective, however, the fact that the problems listed in **VIII** and **IX** have incorrect statements is proved in Section 3. This evaluation reflects the solutions presented in Section 3.

Since these problems appear to be at a level similar to Olympiad problems [19], it seems that Olympiad problems are an appropriate standard for comparison, see [7] [8] [13] [15] [16] [17].

However, it must be stressed that these problems were given in *oral* examinations. This makes the comparison to Olympiad problems valid only in the sense that given similar conditions, the problems have the same level of difficulty. Note that the International Mathematical Olympiad consists of a written examination given over two days, with a total of hours to solve 6 problems.

It should be noted that these problems also differ from Olympiad problems by being, in many cases, either false or poorly stated. Such defects have the side effect of making the problems more interesting in some cases, as they are less artificial than Olympiad problems in which a certain type of solution is often expected.

I. *Easy.* **15**, **24**.

By this, I mean problems which, once one has understood the statement, offer no conceptual or technical difficulty–there are is no idea or difficult computation to challenge the solver. I also include problems which require ideas which are completely standard and should be known to students wishing to pursue a college mathematics education.

II. *Tricky.* **4**, **7**, **8**, **13**, **14**, **18**, **23**.

By this, I mean problems which can be quite challenging until one has found a simple but not well motivated idea after which the result is immediate. This applies to the proof that the statement of problem **18** is false. Note that problem **7** is much more difficult than the others, see Remark 7.2. Problem **14** has a "trap" which caught some students [20], but the examiners themselves overlooked a trap, see Remark 14.1.

III. *Challenging and interesting.* **3**, **9**, **11**, **20**, **22**, **25**.

These are problems whose solutions require interesting ideas and whose statements are also of interest. In other words, these would make good Olympiad problems.

IV. *Straightforward and difficult.* **1**, **2**, **5**, **6**, **17**, **21**.

These are problems which can be solved by a direct computation which does not require any clever idea, though the computation may be quite involved. The problems have alternate solutions with interesting conceptual content and thus put them in category **III** (this applies to problems **1**, **2**, and **21**).

V. *Difficult and uninteresting.* **10**, **12**, **16**, **19**.

These are problems with an uninteresting statement and whose solution is a long and unmotivated computation.

VI. *Inaccurate statement.* **5**, **7**, **9**, **14**, **19**, **22**.

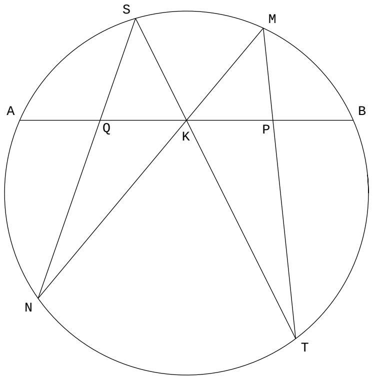
These problems have statements with alternative interpretations most likely not intended by the authors.

VII. *Completely wrong.* **18**, **19**. (So A. Shen's comment [19] about **18** is correct.)

This problem asks for conclusions about situations which cannot occur.

3 Solutions

Problem 1. *K is the midpoint of a chord AB. MN and ST are chords that pass through K. MT intersects AK at a point P and NS intersects KB at a point Q. Show that $KP = KQ$.*



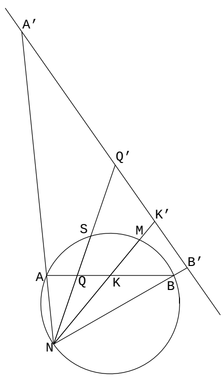
Solution S: The following solution is due to Pavol Severa.

The claim can be made obvious using Lobačevskij geometry. In the Klein disk, i.e., projective, model, the Lobačevskij plane is a disk and straight lines are chords. Let the notation be as in the statement of the problem. If K is an arbitrary point on the Lobačevskij line AB then QK and PK are congruent, since a $180°$ rotation about K preserves the picture, except that it exchanges P and Q. It follows that P and Q are equidistant to K in the Lobačevskij metric. Now let CD be the diameter of the circle passing through K, then the previous remark shows that the Lobačevskij reflection about CD takes P to Q. But since we chose K to be the Euclidean centre of AB, CD is perpendicular to AB, so that the Lobačevskij reflection about CD equals the Euclidean reflection about CD. It follows that $|QK| = |PK|$ in Euclidean sense as well.

Solution R: The following solution is due to David Ruelle. The idea is to use the cross ratio of four points A, B, C, D which can be defined by

$$[A, B, C, D] = \frac{|AC| \cdot |BD|}{|AD| \cdot |BC|}. \tag{1}$$

One simply notes that A, Q, K, B are the stereographic projections of A, S, M, B with pole N and that A, K, P, B are the stereographic projections of A, S, M, B with pole T. Since stereographic projection preserves cross ratios, it follows that $[A, Q, K, B] = [A, K, P, B]$. The result follows from a simple computation using the above algebraic definition of cross ratio.



It should be noted that the above stereographic projection is not the standard one but still preserves cross ratios. To see this, let L be a line perpendicular to a diameter through N and let A', Q', K', B' be the projections of A, Q, K, B onto L. This projection preserves the cross ratio. The stereographic projection of A, S, N, B onto A', Q', K', B' is now the standard one, i.e., is an inversion, and thus preserves cross ratios [10, Theorem 5.42].

Algebraic solution: The following argument uses an algebraic approach which seems to be the most direct, i.e., requires the least amount of ingenuity or knowledge.

Let the circle be $\{(x,y) : \ x^2+y^2=1\}$ and let $K=(0,\beta)$, so that $A=(-\sqrt{1-\beta^2},\beta)$ and $B=(\sqrt{1-\beta^2},\beta)$. One first excludes the trivial case $M=S$ and $N=T$ when $P=Q=K$ and the result holds. Otherwise, one considers two lines L_1 and L_2 passing through K which determine the chords. Assuming for the time being that neither L_1 nor L_2 is parallel to the y–axis, the lines L_1, L_2 can be defined by the equations $y=m_1x+\beta$, and $y=m_2x+\beta$, respectively.

Without loss of generality, one can assume that $m_1>0$. First, one considers the case when $m_2<0$. Let L_1 intersect the circle at $M=(x_1,y_1)$ and $N=(x_3,y_3)$, where $x_1>x_3$, and $y_1>y_3$, and L_2 intersect the circle at $T=(x_2,y_2)$ and $S=(x_4,y_4)$, where $x_2>x_4$, $y_2<y_4$.

One now computes P, i.e., one finds the x–coordinate of the point on the line segment MT which has y–coordinate equal to β. The line segment is represented by $\lambda M+(1-\lambda)T$, $0\le\lambda\le 1$, so $\lambda y_1+(1-\lambda)y_2=\beta$, and one gets $\lambda=(\beta-y_2)/(y_1-y_2)$. Letting $P=(\alpha,\beta)$, one has

$$\alpha=\frac{\beta\,(x_1-x_2)+y_1x_2-y_2x_1}{y_1-y_2}\,. \tag{2}$$

One observes that

$$y_1x_2-y_2x_1=x_1x_2\left(\frac{y_1}{x_1}-\frac{y_2}{x_2}\right)=x_1x_2\left(m_1+\frac{\beta}{x_1}-m_2-\frac{\beta_2}{x_2}\right)=x_1x_2(m_1-m_2)+\beta x_2-\beta x_1\,.$$

Substituting this into (2) gives

$$\alpha=\frac{m_1-m_2}{\dfrac{m_1}{x_2}-\dfrac{m_2}{x_1}}\,. \tag{3}$$

Since M and T lie on the unit circle, one has $x_1^2+(m_1x_1^2+\beta)^2=1$ and $x_2^2+(m_2x_2^2+\beta)^2=1$, so with the above assumptions,

$$x_1=\frac{-m_1\beta+\sqrt{1+m_1^2-\beta^2}}{1+m_1^2}\,,\quad x_2=\frac{-m_2\beta+\sqrt{1+m_2^2-\beta^2}}{1+m_2^2}\,.$$

This implies that

$$\begin{aligned}\frac{m_1}{x_2}-\frac{m_2}{x_1}&=\frac{m_1(1+m_2^2)(-m_2\beta-\sqrt{1+m_2^2-\beta^2})}{m_2^2\beta^2-(1+m_2^2-\beta^2)}-\frac{m_2(1+m_1^2)(-m_1\beta-\sqrt{1+m_1^2-\beta^2})}{m_1^2\beta^2-(1+m_1^2-\beta^2)}\\&=\frac{-m_1\sqrt{1+m_2^2-\beta^2}+m_2\sqrt{1+m_1^2-\beta^2}}{\beta^2-1}\,,\end{aligned}$$

and gives

$$\alpha=\frac{(m_1-m_2)(1-\beta^2)}{m_1\sqrt{1+m_2^2-\beta^2}-m_2\sqrt{1+m_1^2-\beta^2}}\,.$$

One now observes that the value of α is invariant under $\beta\mapsto-\beta$. This in fact proves the result in this case. To see this, note that $\beta\mapsto-\beta$ corresponds to a $180°$ rotation which interchanges M and N and interchanges S and T, and therefore interchanges P and Q. Moreover, this preserves the slopes of L_1 and L_2 since $\angle BKM = \angle AKN$ and $\angle BKT = \angle AKS$. Since the value of α does not change, this shows that $|KP| = |KQ|$.

Next, one considers the case in which $m_2 > 0$. Without loss of generality, assume that $m_2 > m_1$. One then lets $M = (x_1, y_1)$ be the intersection of L_1 with the circle and $T = (x_2, y_2)$ be the intersection of the circle with L_2, where $x_1, y_1 > 0$ and $x_2, y_2 < 0$. Arguing exactly as above one lets $P = (\alpha, \beta)$ and once again (3) holds. Solving for T, M on the unit circle one now obtains

$$x_1 = \frac{-m_1\beta + \sqrt{1+m_1^2-\beta^2}}{1+m_1^2}, \quad x_2 = \frac{-m_2\beta - \sqrt{1+m_2^2-\beta^2}}{1+m_2^2}.$$

Substituting this into (3) yields

$$\alpha = \frac{(m_2 - m_1)(1-\beta^2)}{m_1\sqrt{1+m_2^2-\beta^2} + m_2\sqrt{1+m_1^2-\beta^2}}.$$

Once again, α is invariant under $\beta \mapsto -\beta$ and the result for this case follows as above.

Finally, there remain the cases when L_1 or L_2 are parallel to the x–axis or y–axis. Since $|PK|$ and $|QK|$ are obviously continuous functions of M and S, the result follows by continuity from the previous cases.

Elementary geometry solution: After much effort the following elementary "geometric" argument was found. However, this proof seems more difficult, as some of the intermediate results appear to be at least as deep as the main result. On the other hand, this argument does not require knowledge of hyperbolic geometry.

A trivial case occurs if one of the chords equals AB, so it will be assumed that this is not the case. The first observation is that the result follows from

$$\frac{\operatorname{area} KMT}{\operatorname{area} KMBT} = \frac{\operatorname{area} KSN}{\operatorname{area} KSAN}. \tag{4}$$

To see why this is the case, define h_1 to be the distance between M and KB, i.e., the altitude of KMB, and similarly let h_2 the distance between T and KB, h_3 the distance between S and AK, and h_4 the distance between N and AK. It follows that

$$\begin{aligned} &\operatorname{area} KMT = |KP|\,(h_1+h_2), \quad \operatorname{area} KMBT = |KB|\,(h_1+h_2), \\ &\operatorname{area} KSN = |KQ|\,(h_3+h_4), \quad \operatorname{area} KSAN = |KA|\,(h_3+h_4). \end{aligned} \tag{5}$$

Since $|KA| = |KB|$, equation (4) implies that $|KQ| = |KP|$, which is the statement of the result.

The proof of (4) begins by recalling that if two chords XY and ZW of a circle intersect at T, then $|XT| \cdot |TY| = |ZT| \cdot |TW|$. Since the proof of this result is much easier than what is to follow, it is left as a preparatory exercise for the reader.

The intersection of chords implies that triangles KMT and KSN are similar and it will be convenient to let $\rho = |KM|/|KS|$ be the common ratio between the corresponding sides. Since $\angle TKM = \angle NKS$, the area of triangle KMT equals $\rho^2 \operatorname{area} NKS$ (this follows from the same argument as Lemma 7.1 below). Equation (4) is therefore equivalent to $\operatorname{area} KMBT = \rho^2 \operatorname{area} KSAN$, and (5) shows that this is in turn equivalent to

$$h_1 + h_2 = \rho^2\,(h_3 + h_4). \tag{6}$$

The proof of this follows from

Lemma 1.1. *Let the notation be as above, then (a)* $h_1 h_2 = \rho^2 h_3 h_4$, *(b)* $\dfrac{1}{h_1} + \dfrac{1}{h_2} = \dfrac{1}{h_3} + \dfrac{1}{h_4}$.

Assuming this for the moment, one has

$$\frac{h_3 + h_4}{h_3 h_4} = \frac{1}{h_3} + \frac{1}{h_4} = \frac{1}{h_1} + \frac{1}{h_2} = \frac{h_1 + h_2}{h_1 h_2} = \frac{h_1 + h_2}{\rho^2 h_3 h_4},$$

which implies (6) and the main result.

Proof of Lemma 1.1: In order to prove part (a) one uses the above result about chords which shows that triangle KMB is similar to triangle KAN and triangle KTB is similar to triangle KAS. As before, one has

$$\operatorname{area} KMB = \left(\frac{|KM|}{|AK|}\right)^2 \operatorname{area} KAN\,, \quad \operatorname{area} KTB = \left(\frac{|KB|}{|KS|}\right)^2 \operatorname{area} KAS\,.$$

Since $|KB| = |KA|$ it follows that

$$(\operatorname{area} KMB)\,(\operatorname{area} KTB) = \rho^2\,(\operatorname{area} KAN)\,(\operatorname{area} KAS)\,. \tag{7}$$

But one also has

$$\operatorname{area} KMB = h_1\,|KB|, \quad \operatorname{area} KTB = h_2\,|KB|, \quad \operatorname{area} KAN = h_4\,|AK|, \quad \operatorname{area} KAS = h_3\,|AK|\,,$$

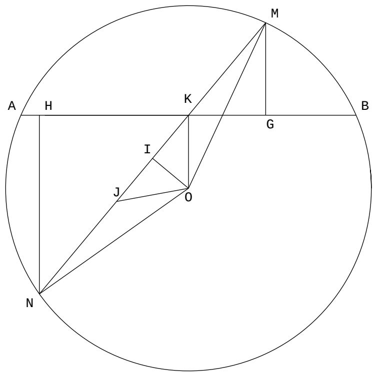
which, upon using $|AK| = |KB|$, gives part (a) of the lemma. Part (b) is a consequence of the following surprising result:

Lemma 1.2. *Given a chord AB of a circle, let MN be any other chord bisecting AB, x the distance from N to the line AB and y the distance from M to the line AB. Then $1/x - 1/y$ is independent of MN except for its sign which only depends on the side of AB that M lies on.*

Assuming this, one notes that M and N lie on different sides of AB, so applying Lemma 1.2 gives

$$\frac{1}{h_1} - \frac{1}{h_4} = -\frac{1}{h_2} + \frac{1}{h_3}\,,$$

which is equivalent to part (b) of Lemma 1.1.



Proof of Lemma 1.2: Let O be the center of the circle and assume that N and O lie on the same side of AB. Draw a perpendicular from O to KN meeting KN at I Since N and M lie on a circle with center O, one has $|OM| = |ON|$ and so the triangle OIM is congruent to the triangle OIN. It follows that I bisects

NM. Now let J lie on the ray IN be such that $|JI| = |IK|$. By the previous argument $|IK| < |IM|$ so J lies strictly between N and I. Since I also bisects JK, it follows that $|NJ| = |KM|$ and thus

$$|KN| = 2\,|KI| + |KM|\,. \tag{8}$$

Now drop a perpendicular from N to AK meeting the line AK at H, so that the signed length of NH is η. Since $\angle KHN = \angle HK0 = 90°$ and $\angle HKN + \angle IKO = 90°$, it follows that triangle KHN is similar to triangle OIK. From similar triangles, one gets

$$\frac{|KI|}{x} = \frac{|OK|}{|KN|}\,.$$

Substituting this into (8) gives

$$\frac{|KN|}{|KM|} = 1 + \frac{2x\,|OK|}{|NK| \cdot |KM|} = 1 + \frac{2x\,|OK|}{R^2 - |OK|^2}\,,$$

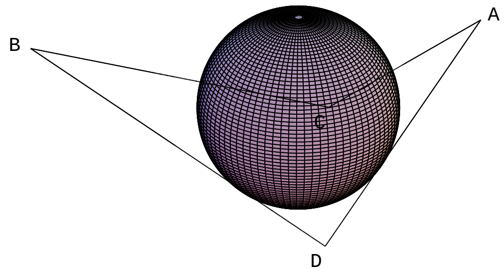
where R is the radius of the circle. The last equality follows from the fact that NK intersects the diameter DE containing K so by intersection of chords $|NK| \cdot |KM| = |DK| \cdot |KE| = (R + |OK|)(R - |OK|)$. One therefore gets

$$\frac{1}{x} - \frac{|KN|}{x\,|KM|} = \frac{-2|OK|}{R^2 - |OK|^2}\,. \tag{9}$$

One now drops a perpendicular from M to KB meeting KB at G, so that the length of MG is y, since G and O lie on opposite sides of AB. The triangle KGM is similar to KHN so that $y = x|KM|/|KN|$. Plugging this into (9) shows that $1/x - 1/y$ has the constant value $-2\,|OK|/(R^2 - |OK|^2)$, which proves the result in this case.

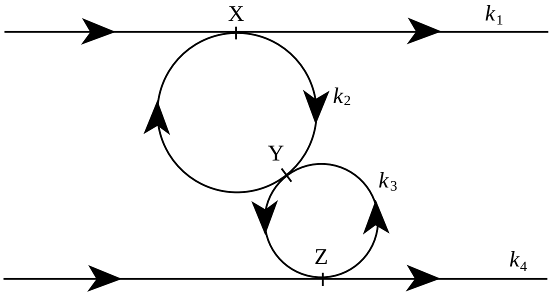
Similarly, if it is M and O which lie on the same side of AB, then one replaces N with M in the above argument, i.e., one lets I lie on KM, etc., and the result carries through in the same way and one arrives at (9) with x and y interchanged, which proves that value of $1/x - 1/y$ changes sign if N lies on the opposite side of AB. □

Problem 2. *A quadrangle in space is tangent to a sphere. Show that the points of tangency are coplanar.*



Solution: This solution was independently found by Pavol Severa and Igor Rivin.

Let $A_1, \dots, A_4$ be the vertices of the quadrangle given in cyclic order. For each point A_i let k_i be the circle on the sphere where the tangents passing through A_i touch the sphere. Notice that the cyclic order of the vertices induces a cyclic order on the k_i's, in particular, these can be oriented so that their orientations at the four points of tangency agree. Now we make a stereographic projection from one of these points of tangency so the picture looks like



We have to prove that X, Y and Z lie on a line. This is visually obvious, but just to help: the homothety with centre at Y that maps k_2 to k_4, maps k_1 to a tangent of k_3 parallel to k_4. Actually, since the orientations agree, it has to map k_1 to k_4 and therefore X to Z.

Alternate solution: The following solution was found by Georg Illies.

Excluding trivial special cases we assume that the points T_1, T_3, T_4 and T_2 in which the sides of the quadrangle ABCD touch the sphere (with center M) lie in the interior of the sides, i.e. $T_1 \in AB$, $T_1 \neq A, B$ and so on. We also assume that A, B, C, D are not coplanar.

Consider the plane $\mathcal{E}$ determined by T_1, T_3 and T_4. If one edge of $ABCD$ were in $\mathcal{E}$ the others would also, by our assumptions. So by the above, A and C lie on different sides of $\mathcal{E}$ as do C and B as well as B and D. Thus also A and D lie on different sides of $\mathcal{E}$.

Let the points A', B', C', D' in $\mathcal{E}$ be such that $AA' \perp \mathcal{E}$, $BB' \perp \mathcal{E}$ etc. Let Q be the intersection point of $\mathcal{E}$ and AD, it is thus the point in which AD and $A'D'$ intersect. (Observe that $AA' \| DD'$ so A, A', D, D' are coplanar; the same argument shows that T_1 is the point in which AC and $A'C'$ intersect and so on.) We have to show $T_2 = Q$.

Now we have $|AT_2| = |AT_1|$, $|CT_1| = |CT_3|$ and so on (as the right triangles AMT_2 and AMT_1 are congruent etc.). Thus

$$\frac{|AT_2|}{|DT_2|} = \frac{|AT_1|}{|CT_1|} \cdot \frac{|CT_3|}{|BT_3|} \cdot \frac{|BT_4|}{|DT_4|} = \frac{|AA'|}{|CC'|} \cdot \frac{|CC'|}{|BB'|} \cdot \frac{|BB'|}{|DD'|} = \frac{|AA'|}{|DD'|} = \frac{|AQ|}{|DQ|},$$

where the second and fourth equality follows by considering the similar right triangles $AA'T_1$ and $CC'T_1$. One therefore gets $T_2 = Q$, as claimed.

Algebraic solution: The following approach puts the problem into purely algebraic form and minimizes geometric intuition.

Let the quadrangle be $ABCD$ and the sphere S. Assume, without loss of generality, that the points of tangency lie on AC, AD, BC, BD. If the quadrangle lies in the plane, then the result is trivial, so it will be assumed that this is not the case.

Let K be the center of S, then A, B, and K lie on a plane. Without loss of generality, one can assume to be the x–y plane, that A and B lie on the x–axis and that K lies on the y–axis, so that $A = (a, 0, 0)$, $B = (b, 0, 0)$, and $K = (0, k, 0)$. One can also assume that $b > a$, $k \geq 0$, and that the sphere has radius 1. Let the points of tangency be at $T_i = (x_i, y_i, z_i)$, $i = 1, \ldots, 4$, where T_1 lies on AC, T_2 on AD, T_3 on BC, and T_4 on BD.

The approach begins by noticing that A, B, C, T_1, T_3 lie in a plane and the same holds for A, B, D, T_2, T_4. In fact, choose a plane P containing the x–axis and of slope m with respect to the x–y plane, and two points $T, T' \in S \cap P$ so that AT and BT' are tangent to the sphere. Then, generically, there is an interval of slopes m such that AT and BT' meet at a point. One concludes that characterizing T and T' in terms of m will lead to all possible quadrangles with 4 points of tangency on S.

Thus, consider the plane P of slope m with respect to the x axis, so that (x, y, z) lies on P if and only if $z = my$. Now let $T = (x, y, z)$ be a point on S such that TA is tangent to S. Thus $(T - K) \cdot (T - A) = 0$, so that

$$x(x-a) + y(y-k) + z^2 = 0\,.$$

Moreover, since S has radius 1 and center K, one gets

$$x^2 + (y-k)^2 + z^2 = 1\,. \tag{10}$$

Subtracting these equations yields

$$x = \frac{ky + 1 - k^2}{a}\,, \tag{11}$$

where one assumes for the time being that $a \neq 0$. It follows that

$$T = \left(\frac{ky + 1 - k^2}{a}, y, my\right)\,, \tag{12}$$

for some y. One can now use (10) to solve for y and this leads to

$$y^2\left(\frac{k^2 + a^2m^2 + a^2}{k^2 + a^2 - 1}\right) - 2ky + k^2 - 1 = 0\,. \tag{13}$$

Note that this equation is well defined since the conditions of the problem imply that the distance from A to the center of the circle K is greater than the radius of the circle, i.e., $a^2 + k^2 > 1$.

Instead of solving directly for the y_i's, it seems more efficient continue by using (12) to gain information about $T_1, \ldots, T_4$. Thus, assume that A, B, C lie on a plane of slope m with respect to the x–y plane, and that A, B, D lie on a plane of slope n with respect to the x–y plane. Applying (12) yields

$$T_1 = \left(\frac{ky_1 + 1 - k^2}{a}, y_1, my_1\right)\,, \quad T_2 = \left(\frac{ky_2 + 1 - k^2}{a}, y_2, ny_2\right)\,,$$
$$T_3 = \left(\frac{ky_3 + 1 - k^2}{b}, y_3, my_3\right)\,, \quad T_4 = \left(\frac{ky_4 + 1 - k^2}{b}, y_4, ny_4\right)\,.$$

In order to tell whether $T_1, \ldots, T_4$ lie in a plane one checks to see if $T_2 - T_1$, $T_3 - T_1$, and $T_4 - T_1$ form a linearly independent set. Since $T_4 - T_1 = (T_4 - T_1) - (T_3 - T_1)$, this is equivalent to verifying whether the following determinant vanishes

$$D = \begin{vmatrix} \frac{k}{a}(y_2 - y_1) & y_2 - y_1 & ny_2 - my_1 \\ k\left(\frac{y_3}{b} - \frac{y_1}{a}\right) + (1-k^2)\left(\frac{1}{b} - \frac{1}{a}\right) & y_3 - y_1 & m(y_3 - y_1) \\ \frac{k}{b}(y_4 - y_3) & y_4 - y_3 & ny_4 - my_3 \end{vmatrix}$$

Assuming for the moment that $y_1 \neq y_2, y_3$ and $y_4 \neq y_3$, one gets

$$D = (y_2 - y_1)(y_3 - y_1)(y_4 - y_3)\begin{vmatrix} \frac{k}{a} & 1 & \frac{ny_2 - my_1}{y_2 - y_1} \\ \frac{k}{y_3 - y_1}\left(\frac{y_3}{b} - \frac{y_1}{a}\right) + \frac{1-k^2}{y_3 - y_1}\left(\frac{1}{b} - \frac{1}{a}\right) & 1 & m \\ \frac{k}{b} & 1 & \frac{ny_4 - my_3}{y_4 - y_3} \end{vmatrix}\,.$$

Subtracting the first row from the other two rows yields

$$D=(y_2-y_1)(y_3-y_1)(y_4-y_3)\begin{vmatrix} \frac{k}{a} & 1 & \frac{ny_2-my_1}{y_2-y_1} \\ \frac{\left(\frac{1}{b}-\frac{1}{a}\right)(ky_3+1-k^2)}{y_3-y_1} & 0 & \frac{(m-n)y_2}{y_2-y_1} \\ k\left(\frac{1}{b}-\frac{1}{a}\right) & 0 & (m-n)\,\frac{y_1y_4-y_2y_3}{(y_2-y_1)(y_4-y_3)} \end{vmatrix}.$$

Expanding from the top row and factoring out common terms in rows and columns yields

$$D=-(y_3-y_1)(y_4-y_3)(m-n)\left(\frac{1}{b}-\frac{1}{a}\right)\begin{vmatrix} \frac{ky_3+1-k^2}{y_3-y_1} & y_2 \\ k & \frac{y_1y_4-y_2y_3}{y_4-y_3} \end{vmatrix}.$$

This last determinant equals

$$\frac{(ky_3+1-k^2)(y_1y_4-y_2y_3)}{(y_3-y_1)(y_4-y_3)}-ky_2=\frac{k(-y_2y_3y_4+y_1y_3y_4+y_1y_2y_4-y_1y_2y_3)+(1-k^2)(y_1y_4-y_2y_3)}{(y_3-y_1)(y_4-y_3)}.$$

It follows that

$$D=(m-n)\left(\frac{1}{a}-\frac{1}{b}\right)\left[k(y_2y_3y_4-y_1y_3y_4-y_1y_2y_4+y_1y_2y_3)+(k^2-1)(y_1y_4-y_2y_3)\right]. \tag{14}$$

It is easily seen that this formula holds also if any of $y_1=y_2$, $y_1=y_3$, or $y_3=y_4$ holds, since both sides of (14) are analytic in $y_1,\ldots,y_4$.

Now let $w_i=1/y_i$, $i=1,\ldots,4$. Since A and B both lie on the x–axis, it is clear that to lie on a quadrangle, none of the points of tangency can satisfy $y_i=0$, so the w_i's are well defined. Substituting this in (14) results in

$$D=\frac{m-n}{w_1w_2w_3w_4}\left(\frac{1}{a}-\frac{1}{b}\right)\left[k(w_1-w_2-w_3+w_4)+(k^2-1)(w_2w_3-w_1w_4)\right]. \tag{15}$$

One now solves for the w_i's by applying (13) which gives

$$(k^2-1)w^2-2kw+\left(\frac{k^2+a^2m^2+a^2}{k^2+a^2-1}\right)=0,$$

so that, assuming that $k\neq 1$,

$$\begin{aligned} w &= \frac{k\pm\sqrt{k^2-(k^2-1)(k^2+a^2m^2+a^2)/(k^2+a^2-1)}}{k^2-1}, \\ &= \frac{k\pm\sqrt{(m^2-k^2m^2+1)\,a^2/(k^2+a^2-1)}}{k^2-1}, \\ &= \frac{k\pm f(m)g(a)}{k^2-1}, \end{aligned} \tag{16}$$

where

$$f(m)=\sqrt{m^2-k^2m^2+1},\quad g(a)=\sqrt{a^2/(k^2+a^2-1)}.$$

One thus gets

$$w_1=\frac{k\pm f(m)g(a)}{k^2-1},\quad w_2=\frac{k\pm f(n)g(a)}{k^2-1},\quad w_3=\frac{k\pm f(m)g(b)}{k^2-1},\quad w_4=\frac{k\pm f(n)g(b)}{k^2-1}.$$

Once again, one can make a further simplification before using this formula. By writing $w_i = (k+s_i)/(k^2-1)$, one gets

$$\begin{aligned} &k(w_1 - w_2 - w_3 + w_4) + (k^2-1)(w_2 w_3 - w_1 w_4) \\ &= \frac{k}{k^2-1}\,(s_1 - s_2 - s_3 + s_4) + \frac{1}{k^2-1}\,[(k+s_2)(k+s_3) - (k+s_1)(k+s_4)] \\ &= \frac{s_2 s_3 - s_1 s_4}{k^2-1}\,, \end{aligned}$$

In other words,

$$D = \frac{m-n}{w_1 w_2 w_3 w_4}\,\frac{s_2 s_3 - s_1 s_4}{k^2-1}\left(\frac{1}{a} - \frac{1}{b}\right).$$

One now observes that

$$|s_2 s_3| = f(n)g(a)f(m)g(b) = |s_1 s_4|\,.$$

It follows that $D = 0$ depends only on the signs of s_i, $i = 1, \ldots, 4$. In other words, if σ_i is the sign of s_i, then $T_1, \ldots, T_4$ are coplanar if and only if $\sigma_1\sigma_4 = \sigma_2\sigma_3$, that is s_1, s_4 *and* s_2, s_3 *either both have the same signs or both have unequal signs.*

In order to characterize this last condition, one examines the geometrical significance of the sign of σ_i. Note first that one has the explicit computation

$$x_1 = \frac{k y_1 + 1 - k^2}{a} = \frac{k/w_1 + 1 - k^2}{a} = \frac{-(k^2-1)s_1}{a(k+s_1)}\,.$$

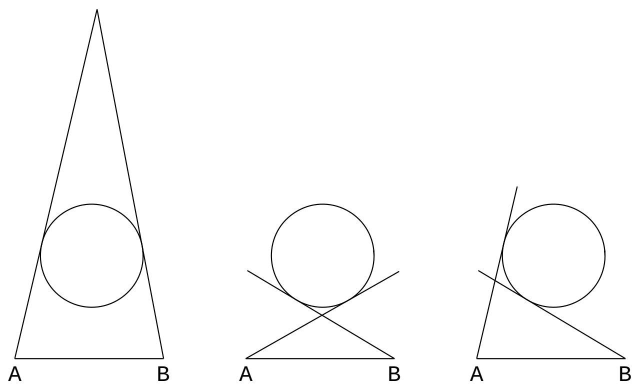
Assuming for the time being that $k > 1$, one has

$$s_1^2 = \frac{a^2}{a^2 + k^2 - 1}\,(1 - m^2(k^2-1)) \leq 1$$

so that $k + s_1 > 0$. It follows that σ_1 equals the sign of x_1 if $a > 0$, and is minus the sign of x_1 is $a < 0$. In other words, if the sphere S is divided into two sides S_1 and S_{-1} according to whether $\mathrm{sign}\,(a) > 0$ or < 0, then σ_1 determines the side of the sphere that T_1 lies in the following way: T_1 lies in $S_{\sigma_1\,\mathrm{sign}(a)}$. The same holds true for $T_2, \ldots, T_4$ and one gets

$$T_1 \in S_{\sigma_1\,\mathrm{sign}(a)}, \quad T_2 \in S_{\sigma_2\,\mathrm{sign}(a)}, \quad T_3 \in S_{\sigma_3\,\mathrm{sign}(b)}, \quad T_4 \in S_{\sigma_4\,\mathrm{sign}(b)}\,. \tag{17}$$

Now consider the two points of tangency T_1, T_3 lying in the plane of A, B. One will say that these are of *Type I* with respect to A, B if T_1 and T_3 lie on different sides of the sphere, as defined above, and of *Type II* with respect to A, B if they lie on the same side of the sphere. Thus, in the diagram, the two figures on the left represent Type I and the figure on the right Type II.



One now translates the condition that $\sigma_1\sigma_4 = \sigma_2\sigma_3$ into this notation.

(i) If $\sigma_1 = \sigma_4$ and $\sigma_2 = \sigma_3$, and $\operatorname{sign}(a) = \operatorname{sign}(b)$, then $T_1, \ldots, T_4$ all lie on S_1 so T_1, T_4 and T_2, T_3 are both of Type II with respect to A, B.

(ii) If $\sigma_1 = \sigma_4$ and $\sigma_2 = \sigma_3$, and $\operatorname{sign}(a) = -\operatorname{sign}(b)$, then T_1, T_4 and T_2, T_3 are both of Type I with respect to A, B.

(iii) If $\sigma_1 = -\sigma_4$ and $\sigma_2 = -\sigma_3$, and $\operatorname{sign}(a) = \operatorname{sign}(b)$, then T_1, T_4 and T_2, T_3 are both of Type I with respect to A, B.

(iv) If $\sigma_1 = -\sigma_4$ and $\sigma_2 = -\sigma_3$, and $\operatorname{sign}(a) = -\operatorname{sign}(b)$, then T_1, T_4 and T_2, T_3 are both of Type II with respect to A, B.

One concludes that $\sigma_1\sigma_4 = \sigma_2\sigma_3$ implies that T_1, T_4 and T_2, T_3 are of the same type with respect to A, B. If $\sigma_1\sigma_4 = -\sigma_2\sigma_3$, then the exact same argument shows that T_1, T_4 and T_2, T_3 cannot be of the same type with respect to A, B. It follows that the condition $\sigma_1\sigma_4 = \sigma_2\sigma_3$ is equivalent to T_1, T_4 and T_2, T_3 being of the same type with respect to A, B. In other words, *the points of tangency are coplanar if and only if T_1, T_4 and T_2, T_3 are of the same type with respect to A, B.*

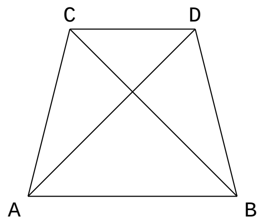
This will be shown in the case at hand. In fact, if the points of tangency lie on the edges of a quadrangle, then they must all be of Type I. In fact, it is easily seen that AT_1 and BT_3 cannot lie on the same side of the sphere when T_1 and T_3 lie on line segments AC and BC. This is obvious from the above diagram and a rigorous proof is left as an exercise.

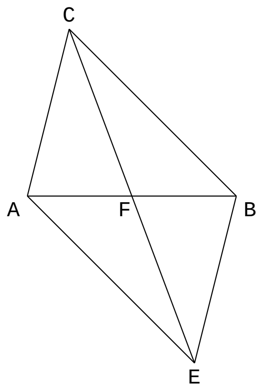
Finally, as can be easily checked, the case $a = 0$ can be proved by continuity from the above argument.

Remark 2.1. The above argument proves the slightly more general result: *Given a quadrangle and a sphere such that the lines extending the edges of the quadrangle are tangent to the sphere, then the points of tangency are coplanar if and only if there are two vertices such that the pairs of points of tangency are of the same type with respect to these vertices.*

Problem 3. *The faces of a triangular pyramid have the same area. Show that they are congruent.*

Solution: A good way to approach this problem is to first characterize the consequence of the statement: *There exists a tetrahedron all of whose sides are congruent to a given triangle if and only if all the angles of the triangle are acute.*





Proof: Let the triangle be ABC, and assume, without loss of generality, that $\angle BAC$ is greater or equal the other angles of the triangle. One place a triangle ABD on the line AB such that C and D lie on the same side of AB and such that ABC is congruent to BAD, in other words, ABD is a mirror image of ABC. As in the above, one rotates ABD about AB to form a triangle ABD'. Clearly, any tetrahedron with all sides congruent to ABC will be formed in this way, so if a solution exists, then it is unique up to rotational symmetry.

(a) Assume that all angles of ABC are acute. It follows that $|CD| < |AB|$. Now let ABE be the triangle ABD rotated 180° about AB, and let F be the intersection of AB and CE. By construction, it follows that triangles ACF and BEF are congruent, so CE bisects AB. By assumption, $\angle BAC \geq \angle ACB$ so $\angle FAC > \angle ACF$. But in the triangle FAC one has, by the law of sines, that $|CF|/\sin \angle FAC$ is equal to $|AF|/\sin \angle ACF$, and one concludes that $|CF| > |AF|$, since $\sin z$ is increasing for $0 < z < 90°$ (recall that $\angle FAC < 90°$).

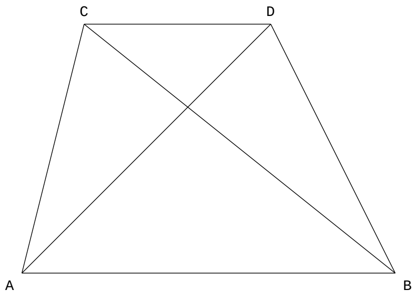
It follows that $|CE| > |AB|$. This implies that there must be a rotation with angle strictly between zero and 180° such that $|CD'| = |AB|$. This value of D' then gives the required tetrahedron.

(b) Assume that at least one angle of ABC is not acute, i.e., $\angle BAC \geq 90°$. Then $|CD| \geq |AB|$, and rotating ABD about AB will only increase the value of $|CD'|$ so that it is strictly greater than $|AB|$. It follows that there can be no solution in this case. □

The original problem uses similar ideas but will require the following technical point:

Lemma 3.1. *Let a, b, c, d be positive real numbers such that a, b are not equal to c, d in some order. Then there is at most one value of x such that there are two triangles with side lengths a, b, x and c, d, x, and with equal areas.*

Assuming this holds, one proves the result by contradiction. One begins as above by trying to construct a tetrahedron all of whose sides have equal areas. Thus, let ABC and ABD be noncongruent triangles with equal areas.



One now forms a tetrahedron by placing ABC and ABD in the same plane with C and D on the same side of AB and then rotating ABD about AB to obtain a new triangle ABD'. Clearly, any tetrahedron with adjacent sides congruent to ABC and ABD will be generated this way.

Let $a = |AC|$, $b = |AD|$, $c = |BC|$, $d = |BD|$. It follows that the other two faces of the tetrahedron are ACD' and BCD' with sides a, b, x and c, d, x, respectively, where $x = |CD'|$. A solution to the problem requires ACD' and BCD' to have equal areas. Since the assumption that ABC is not congruent to ABD implies that a, b are not equal to (c, d) in some order, Lemma 3.1 applies, and there is a unique D' with the required property.

Since triangles ABC and ABD have equal areas, ABC and ABD have equal altitudes with respect to AB, and thus that ACD and BCD must have equal areas as well. This implies that the initial position $D' = D$ gives the only possible solution to ACD' and ABD' having equal areas. Since these lie in the same plane, there is no 3–dimensional solution.

Proof of Lemma 3.1: Let the two triangles be ACD and BCD with notation as above, i.e., $a = |AC|$, $b = AD$, $c = |BC|$, $d = |BD|$, $CD = x$. Let $\alpha = \angle CAD$ and $\beta = \angle CBD$. Then from elementary trigonometry (law of cosines) one has

$$a^2 + b^2 - 2ab\cos\alpha = c^2 + d^2 - 2cd\cos\beta = x^2\,, \tag{18}$$

while equating areas gives

$$ab\sin\alpha = cd\sin\beta\,. \tag{19}$$

Using $\sin^2 z = 1 - \cos^2 z$, one transforms (19) to

$$a^2b^2 - c^2d^2 - a^2b^2\cos^2\alpha + c^2d^2\cos^2\beta = 0\,. \tag{20}$$

Equation (18) implies that

$$a^2b^2\cos^2\alpha = \left(\frac{x^2 - a^2 - b^2}{2}\right)^2, \quad c^2d^2\cos^2\beta = \left(\frac{x^2 - c^2 - d^2}{2}\right)^2.$$

Plugging this into (20) gives

$$a^2b^2 - \left(\frac{x^2 - a^2 - b^2}{2}\right)^2 - c^2d^2 + \left(\frac{x^2 - c^2 - d^2}{2}\right)^2 = 0\,,$$

which leads to

$$x^2(a^2 + b^2 - c^2 - d^2) = \frac{(a^2 - b^2)^2 - (c^2 - d^2)^2}{2}\,. \tag{21}$$

Now if $a^2 + b^2 \neq c^2 + d^2$, then there is at most one positive value of x satisfying (21), and the statement of the Lemma follows. On the other hand, if $a^2 + b^2 = c^2 + d^2$, then (18) implies that $ab\cos\alpha = cd\cos\beta$. Combining this with (19), e.g., by using $\sin^2 z + \cos^2 = 1$, one obtains $ab = cd$. One therefore gets

$$a^2 + 2ab + b^2 = c^2 + 2cd + d^2, \quad a^2 - 2ab + b^2 = c^2 - 2cd + d^2.$$

This implies that $a + b = c + d$ and $a - b = \pm(c - d)$, and one concludes that a, b are equal to c, d in some order, contradicting the hypothesis. □

Problem 4. *The prime decompositions of different integers m and n involve the same primes. The integers $m + 1$ and $n + 1$ also have this property. Is the number of such pairs (m, n) finite or infinite?*

Solution: The number of such pairs is infinite.

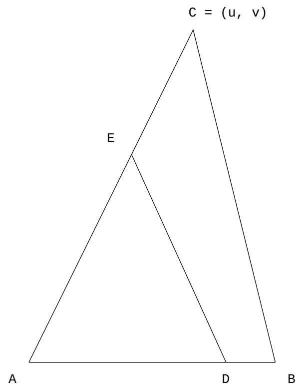
Proof: Let $m = 2^k - 2$, $n = (m + 1)^2 - 1$, for $k = 2, 3, 4, \ldots$. Then $n + 1 = (m + 1)^2$, so $n + 1$ and $m + 1$ have the same prime factors. Moreover, $n = (m + 1)^2 - 1 = m\,(m + 2)$. Since $m + 2$ is a power of 2 and m is already even, it follows that m and n also have the same prime factors.

Remark 4.1. One can ask whether there are infinitely many pairs not of this form. This does not appear to be an easy question and even finding one other pair is non trivial. A computer search revealed that $m = 75 = 3 \cdot 5^2$, $n = 1215 = 3^5 \cdot 5$ also satisfy this condition since $m + 1 = 2^2 \cdot 19$ and $n + 1 = 2^6 \cdot 19$.

In fact, this is a special case of a well known problem of Erdős and Woods in number theory and logic, see the Notes.

Problem 5. *Draw a straight line that halves the area and perimeter of a triangle.*

Solution: Let the triangle be ABC, let $p = a + b + c$ be the perimeter, and, without loss of generality, assume that $b \geq a \geq c$. On AB, let D be such that the length of AD is $t_0 = (p - \sqrt{p^2 - 8bc})/4$, and on AC let E be such that the length of AE is $bc/(2t_0)$. Then the line DE splits the area and perimeter of ABC into two.



Proof: One can think of the triangle ABC as being in the x, y plane with the origin at A, $B = (c, 0)$ and $C = (u, v)$, where $u, v > 0$. Under these assumptions, let D lie on AB such that the length of AD is t and $c/2 \leq t \leq c$. One will construct E on AC such that DE divides the triangle into two equal areas. In fact, let E be such that the length of AE is $bc/(2t)$. Then the area of the triangle ADE is

$$\frac{cv}{2t}\frac{t}{2} = \frac{1}{2}\frac{cv}{2},$$

so E satisfies this property. Note that for $t = c$, one gets $E = C/2$, and for $t = c/2$ one gets $E = C$.

Now the perimeter contribution of AD and AE is $t + bc/(2t)$, so one needs to solve the equation

$$t + \frac{bc}{2t} = \frac{a+b+c}{2},$$

which has solutions $t = (p \pm \sqrt{p^2 - 8bc})/2$. I will show that

$$t_0 = \frac{p - \sqrt{p^2 - 8bc}}{4}$$

satisfies all necessary conditions to give an actual solution, i.e.,

$$p^2 \geq 8bc, \quad t_0 \leq c, \quad t_2 \geq \frac{c}{2}. \tag{22}$$

To prove the first inequality, note that it follows from

$$(-a + b - c)^2 = a^2 + b^2 + c^2 - 2ab - 2bc + 2ac \geq 0,$$

which implies

$$p^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac \geq 8bc.$$

The second inequality in (22) is equivalent to

$$(a+b-3c)^2 \le (a+b+c)^2 - 8bc\,,$$

which reduces to $8ac \ge 0$. Finally, the third inequality in (22) is equivalent to

$$(a+b-c)^2 \ge (a+b+c)^2 - 8bc\,,$$

which reduces to $b \ge a$, which is true by assumption.

Remark 5.1. Since t_0 is a composition of additions, subtractions, divisions, and square roots of the sides of ABC, it follows that DE can be "drawn" with ruler and compass.

Problem 6. *Show that* $(1/\sin^2 x) \le (1/x^2) + 1 - 4/\pi^2$, $0 < x < \pi/2$.

Solution: The question can be rewritten as

$$\frac{1}{x^2} - \frac{1}{\sin^2 x} + 1 - \frac{4}{\pi^2} \ge 0, \quad 0 < x < \pi/2\,. \tag{23}$$

In order to prove this, one begins by showing that

$$\lim_{x\to 0}\left(\frac{1}{\sin^2 x} - \frac{1}{x^2}\right) = \frac{1}{3}\,.$$

This can be done either by expanding into a power series about $x = 0$, or by L'Hôpital's rule, as follows:

$$\frac{1}{\sin^2 x} - \frac{1}{x^2} = \frac{x^2 - \sin^2 x}{x^2 \sin^2 x} \to \frac{0}{0}, \quad x \to 0,$$

so the limit equals the limit of the derivative of the numerator divided by the derivative of the denominator. Iterating this process yields,

$$\frac{2x - \sin 2x}{2x\sin^2 x + x^2 \sin 2x} \to \frac{0}{0}\,, \quad \frac{1-\cos 2x}{\sin^2 x + 2x\sin 2x + x^2\cos 2x} \to \frac{0}{0}\,,$$
$$\frac{2\sin 2x}{3\sin 2x + 6x\cos 2x - 2x^2\sin 2x} \to \frac{0}{0}\,, \quad \frac{2\cos 2x}{6\cos 2x - 8x\sin 2x - x^2\cos 2x} \to \frac{1}{3}\,.$$

Since $1 - 4/\pi^2 > 1/3$, it follows that there is a $\delta > 0$ for which strict inequality holds in (23) for all $0 < x < \delta$.

Next, one rewrites (23) as

$$\frac{\sin x}{\sqrt{1 - a\sin^2 x}} \ge x\,, \tag{24}$$

where $a = 1 - 4/\pi^2$. Clearly, this is an equality for $x = 0$, and a computation shows that it also holds for $x = \pi/2$. To show that the inequality is strict for $0 < x < \pi/2$, one takes the second derivative of

$$f(x) = \frac{\sin x}{\sqrt{1 - a\sin^2 x}}\,,$$

which is easily found:

$$f'(x) = \frac{\cos x}{(1 - a\sin^2 x)^{3/2}}\,, \quad f''(x) = \frac{(a - 1 + 2a\cos^2 x)\,\sin x}{(1 - a\sin^2 x)^{5/2}}\,.$$

Since $a > 1/3$, it follows that $f''(x) > 0$ for $0 < x < x_0$, where x_0 is the unique solution of $f''(x_0) = 0$ in $(0, \pi/2)$ (that x_0 exists and is unique is immediate from the form of $f''(x)$). In other words, $f(x)$ is concave in $(0, x_0)$. Now $f(0) = 0$, and by the first part, it is true that (24) holds for $0 < x < \delta$, so the strict concavity of $f(x)$ implies that $f(x) > x$ for $0 < x \le x_0$.

Since $f''(x)$ only has one zero in $0 < x < \pi/2$ and $f''(\pi/2) = -(\pi/2)^3 < 0$, it follows that $f(x)$ is convex in $(x_0, \pi/2)$. Since $f(x_0) > x_0$ and $f(\pi/2) = \pi/2$, convexity implies that $f(x) > x$ for $x_0 < x < \pi/2$.

Problem 7. *Choose a point on each edge of a tetrahedron. Show that the volume of at least one of the resulting tetrahedrons is* $\le 1/8$ *of the volume of the initial tetrahedron.*

Solution: The most natural interpretation of "resulting tetrahedrons" is the tetrahedra formed by each original corner and the points on edges that are adjacent to the corner, see Remark 7.3.

Lemma 7.1. *If the angles of a vertex of a tetrahedron are fixed, then the volume of the tetrahedron is proportional to the length of the sides adjacent to this vertex.*

Proof: One assumes the well known facts that the area of a triangle is proportional to the base times height, and that the volume of a tetrahedron is proportional to base times height. Let A be the vertex with fixed angles, and B, C, D the other vertices. Let C and D be fixed and let B vary. If one considers ABC to be the base of the tetrahedron, then the height remains fixed as B varies. Similarly, if one considers AB to be the base of ABC, then its height remains fixed. It follows that the volume of the tetrahedron is proportional to AB. By symmetry, this holds for B and C, proving the result.

Alternatively, if one lets α be the angle CBA and β the angle that AD makes with ABC, then the volume of the tetrahedron is simply $\frac{1}{6}|AB| \cdot |AC| \cdot |AD| \sin\alpha \sin\beta$. □

Now let the tetrahedron be T with vertices $A_1, \ldots, A_4$. One then picks a point on each edge so that P_{ij} lies on A_{ij}, $1 \le i, j \le 4$, with the convention that $P_{ij} = P_{ji}$. The resulting tetrahedra are then T_i, $i = 1, \ldots, 4$, where T_i has vertices A_i and P_{ij}, $j \ne i$. Let $v(R)$ the volume of a three dimensional region R, then the problem is to show that one of $v(T_i)/v(T) \le 1/8$.

In order to do this, let $r_{ij} = |A_i P_{ij}|/|A_i A_j|$ be the ratio of the distance of P_{ij} to the corner A_{ij} with respect to the edge $A_i A_j$. Since the angles at the corner A_i of T_i remain fixed as the P_{ij}'s vary, one can apply Lemma 7.1 to get

$$\frac{v(T_i)}{v(T)} = \prod_{j \ne i} r_{ij}\,.$$

Multiplying all these quantities together gives

$$\prod_{i=1}^{4} \frac{v(T_i)}{v(T)} = \prod_{\substack{1 \le i,j \le 4 \\ i \ne j}} r_{ij}\,.$$

Since $r_{ij} = 1 - r_{ji}$, it follows that

$$\prod_{i=1}^{4} \frac{v(T_i)}{v(T)} = \prod_{1 \le i < j \le 4} r_{ij}(1 - r_{ij})\,.$$

Now it is easily shown that $x(1-x) \le 1/4$ for $0 < x < 1$, so it follows that

$$\prod_{i=1}^{4} \frac{v(T_i)}{v(T)} \le \frac{1}{4^6}\,. \tag{25}$$

This implies that not all of the factors on the left of (25) can be $> (1/4^6)^{1/4} = 1/8$. The result follows.

Remark 7.1. This argument generalizes verbatim to n–dimensions. Thus, let S be an n–dimensional simplex, i.e., the set

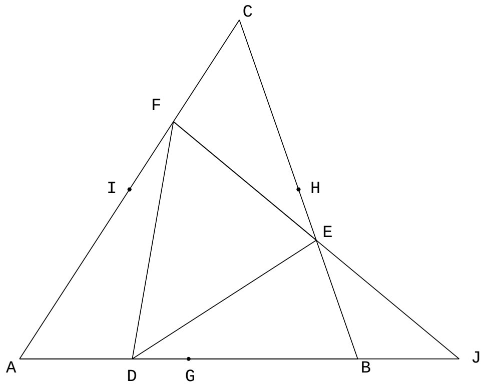
$$S = \{\lambda_1 A_1 + \cdots + \lambda_{n+1} A_{n+1} : \lambda_1, \ldots, \lambda_{n+1} \geq 0,\ \lambda_1 + \cdots + \lambda_{n+1} = 1\},$$

where the vertices $A_1, \ldots, A_{n+1} \in \mathbf{R}^n$ have the property that removing any one results in a linearly independent set. The generalization is: *Pick a point on each edge of an n–dimensional simplex, then one of the simplices obtained by taking an original vertex and the edge points that lie on edges adjacent to it must have volume $\leq 1/2^n$ of the original simplex.* The generalization of this to arbitrary polyhedra is left as a problem for the reader.

Remark 7.2. This problem is frustrating because natural geometric arguments that prove the two dimensional analogue to not seem to generalize well to three dimensions. As an example a simple geometric argument is given for the two dimensional case.

Proposition 7.1. *Pick a point on each edge of a triangle. Then one of the triangles formed by joining a vertex of the original triangle to the edge points on the edges adjacent to it has area $\leq 1/4$ of the original triangle.*

Proof: If some vertex, say A, has two edge points, say D, F, at least as close to A as to the other vertices, then the area of ADF is less than $1/4$ of the original triangle. The only other possibility is that each vertex has only one edge point closer to it. Say that D lies on AB such that $AD \leq DB$, that E lies on BC such that $BE \leq EC$, and that F lies on CA such that $CF \leq FA$. Moreover, let G be the midpoint of AB, H the midpoint of BC, and I the midpoint of CA.



One now shows that the area of DEF is $\geq$ to the area of GHI. Since AB and IH are parallel, it follows that FE is either parallel to AB or meets AB at J such that AJF is an acute angle. Now let $x = |GD|$, then in the first case, as x increases, the area of DEF remains constant. In the second case, as x increases, the distance of D to FJ increases, since AJF is acute. Since this equals the distance of D to FE, it follows that as x increases, the area of DEF increases.

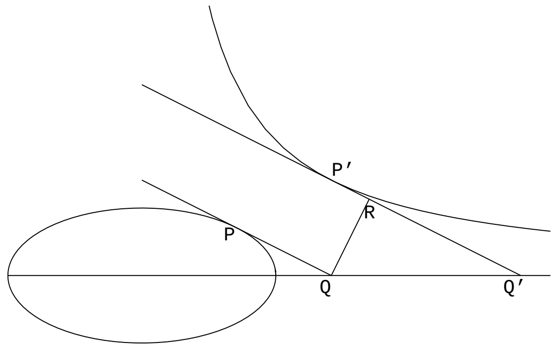
One concludes that the area of DEF is non decreasing in x. Since this argument holds for $|HE|$ and $|IF|$, it follows that the area of DEF is $\leq$ the area of GHI, as claimed. Since the area of GHI is exactly $1/4$ of the area of ABC, the sum of the three remaining triangles is $\leq 3/4$ of ABC, and thus one of the triangles ADF, BDE, CEF must have area lessss than $1/4$ of ABC. $\square$

Generalizing this to three dimensions would entail finding a lower bound on the middle piece of the tetrahedron, i.e., what remains after the four tetrahedra have been removed. However, in three dimensions, the volume of this piece is no longer a monotonic function of the edge points, as was the case in the above argument. Following through with this argument is left as a problem for the reader.

Remark 7.3. If one interprets "resulting tetrahedrons" as meaning any tetrahedron formed by joining one of the edge points to a vertex, then the solution is simple: Pick a vertex A of the tetrahedron. If the points on the three edges from A all lie closer to A than to the other vertex on the edge, then the tetrahedron formed by these edge points and A clearly has volume $\leq 1/8$ of the original tetrahedron. Otherwise, there is an edge point E such that the distance from E to BCD is $\leq 1/2$ the distance from A to BCD. Now clearly, one of the four triangles formed by the edge points at the base has area $\leq 1/4$ of the area of the base. It then follows that the tetrahedron formed by joining this triangle to E has volume $\leq 1/8$ of the original volume.

Problem 8. *We are told that* $a^2 + 4\,b^2 = 4$, $cd = 4$. *Show that* $(a-d)^2 + (b-c)^2 \geq 1.6$.

Solution: The lower bound is found as follows: The form $(a-d)^2+(b-c)^2$ is the square of the Euclidean distance between (a,b) and (d,c), so the question reduces to finding the minimum distance between the curves $x^2 + 4\,y^2 = 4$, and $xy = 4$. The first of these is an ellipse with axes $2, 1$, while the second is a hyperbola. Clearly, this problem is symmetric with respect to the line $x = -y$, so one can restrict oneself to $x \geq -y$, and, as a consequence, to the component of the hyperbola with $c, d > 0$. Thus, in the following, "the hyperbola" will mean the point (x, y) satisfying $xy = 4$ and $x, y > 0$. The main idea is the following simple observation:



Lemma 8.1. *If a convex curve* C *has a tangent line* L *and a concave curve* C' *has a tangent line* L' *such that* L' *is parallel to* L *and neither* C *nor* C' *lie between* L *and* L'*, then the minimum distance between* C *and* C' *is greater than the distance between* L *and* L'.

Proof: Note that if P is a point on C and P' is a point on C', then given the assumptions, PP' must cross L and L', so that $|PP'|$ is greater than the distance between L and L'. $\square$

In order to use this, one must find points on the ellipse and on the hyperbola whose tangents have the same slope. In order to do this, one must first compute the slope of the tangents to these curves.

Thus, consider a point (x, y) on the ellipse. Differentiating $x^2 + 4y^2 = 4$ gives $2x\,dx + 8y\,dy = 0$, so that $dy/dx = -x/(4y)$ is the slope of the tangent line at (x, y). Similarly, the slope of the tangent line at the point (x, y) of the hyperbola is $-4/x^2$.

One now appeals to a "trick" which consists in taking $P = (\sqrt{2}, 1/\sqrt{2})$ and $P' = (2\sqrt{2}, \sqrt{2})$. It turns out that P lies on the ellipse and P' on the hyperbola, and that the slopes of the tangent lines at P and P' are both equal to $-1/2$, as can be checked using the previous paragraph. A further simple computation shows that if L is the tangent line at P and L' the tangent line at Q, then these lines are given by the equations

$$L:\ y=-\frac{x}{2}+\sqrt{2},\quad L':\ y=-\frac{x}{2}+2\sqrt{2}.$$

To compute the distance between L and L', consider their intersections at the x–axis. Thus, L intersects the x–axis at $Q=(2\sqrt{2},0)$, and L' at $Q'=(4\sqrt{2},0)$. Now from Q draw a line perpendicular to L meeting L' at R. Since L' has a slope of $-1/2$ with respect to the x–axis, it follows that the x–axis (pointing to $-\infty$) has a slope of $1/2$ with respect to L' (pointing in the positive y direction). This implies that $|QR|/|RQ'|=1/2$, so that $|QR|/|RQ'|=1/\sqrt{5}$, by the Pythagorean theorem. Since $|QQ'|=2\sqrt{2}$, one gets $|QR|=2\sqrt{2}/\sqrt{5}$.

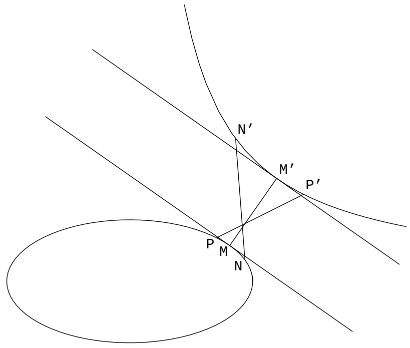
Since QR is perpendicular to L and L', it follows that $|QR|$ equals the distance between L and L'. One concludes that

$$(a-d)^2+(b-c)^2\geq|QR|^2=\frac{8}{5}=1.6.$$

Remark 8.1. *The minimum value of of* $(a-d)^2+(b-c)^2$ *is* $1.77479583276941567010\ldots$.

Proof: Unlike the other material in this section, the proof uses a computer algebra system, as this question does not appear to have a closed form solution.

One proceeds along the lines used in the solution to the problem. The idea is that Lemma 8.1 clearly implies that if M and M' lying on C and C', respectively, are such that MM' is orthogonal to the tangents at M and M', then $|MM'|$ is the minimum distance between C and C'. One therefore finds two such points.



In order to do this, one starts with a given value of α and finds points on the ellipse and the hyperbola whose tangents both have slope $-\alpha$. On the ellipse, let the tangent at $M=(x_0,y_0)$ have slope $-\alpha$ so that $-x_0/(4y_0)=-\alpha$. The condition $x_0^2+4y_0^2=4$ gives

$$x_0=\frac{4\alpha}{\sqrt{4\alpha^2+1}},\quad y_0=\frac{1}{\sqrt{4\alpha^2+1}}.$$

Similarly, let $M'=(x_1,y_1)$ be a point on the hyperbola whose tangent has slope $-\alpha$. Then $-4/x_1^2=-\alpha$, and $x_1y_1=4$ implies that

$$x_1=\frac{2}{\sqrt{\alpha}},\quad y=2\sqrt{\alpha}.$$

In order for MM' to be orthogonal to the tangents one must have $(y_1-y_0)/(x_1-x_0)=1/\alpha$. This gives

$$\frac{2\sqrt{\alpha}-\frac{1}{\sqrt{4\alpha^2+1}}}{\frac{2}{\sqrt{\alpha}}-\frac{4\alpha}{\sqrt{4\alpha^2+1}}}=\frac{2\alpha\sqrt{4\alpha^2+1}-\alpha^{1/2}}{2\sqrt{4\alpha^2+1}-4\alpha^{3/2}}=\frac{1}{\alpha},$$

which simplifies to the equation

(26) $$16\,\alpha^6-28\alpha^4-9\alpha^3+8\alpha^2+4=0\,.$$

This equation was examined using the computer algebra system *Mathematica*. The polynomial on the left hand side of (26) is irreducible over the rationals and the computer algebra system was unable to express the roots using radicals. The approximate roots are

$$-0.979691 \pm 0.34843\, i, \quad -0.04401 \pm 0.493223\, i, \quad 0.699695, \quad 1.34771\,.$$

One can eliminate all but the last two possibilities. Carrying out the above argument using the last root fails, as it ends up giving points N and N' with a negative slope for $N - N'$, see the figure, which means that it cannot be orthogonal to the tangent, since it would then have slope $1/\alpha > 0$. The relevant root is therefore the second to last which, to twenty digits, is

$$\alpha_0 = 0.69969482002339060183\ldots.$$

Following the above argument, one lets

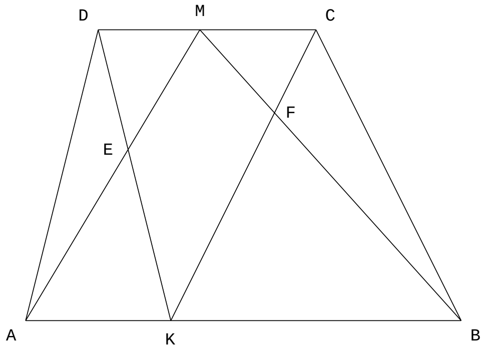
$$M = \left(\frac{4\alpha_0}{\sqrt{4\alpha_0^2+1}}, \frac{1}{\sqrt{4\alpha_0^2+1}}\right) = (1.62722713282531988425\ldots, 0.58140602383297697452\ldots)\,,$$

$$M' = \left(\frac{2}{\sqrt{\alpha_0}}, 2\sqrt{\alpha_0}\right) = (2.39097847459882936932\ldots, 1.67295525346422891327\ldots).$$

One therefore gets the minimum value of $(a-d)^2 + (b-c)^2$ to be $|M - M'|^2 = 1.77479583276941567010\ldots$.

Problem 9. *We are given a point K on the side AB of a trapezoid $ABCD$. Find a point M on the side CD that maximizes the area of the quadrangle which is the intersection of the triangles AMB and CDK.*

Solution: It is not clear from the statement of the problem whether AB is one of the parallel sides of the trapezoid. Since this interpretation seems more natural, I will treat this possibility only and leave the other case to the reader. The answer in this case is: *If AB and CD are parallel, then M is chosen such that $|DM| \cdot |AB| = |AK| \cdot |CD|$.*



Proof: Let t be the area of the trapezoid, h the distance between the parallel lines AB and CD, and let q be the area of the quadrangle in question. The first observation is that the area of AMB plus the area of DKC equals t. To see this, one notes that the areas of AMB and DKC are independent of M and K since AB and CD are parallel. One can therefore take $K = A$ and $M = C$, in which case $ABCD$ is the disjoint union of AMB and DKC.

One then uses

$$\text{area}\,(AMK \cup DKC) = \text{area}\,(AMK) + \text{area}\,(DKC) - q\,,$$

to get $q = t - \text{area}\,(AMK \cup DKC)$. Now let E be the intersection of DK and AM, and F the intersection of MB and KC. One has

$$\text{area}\,(AMK \cup DKC) = \text{area}\,(AEK) + \text{area}\,(DEM) + \text{area}\,(KFB) + \text{area}\,(MFC) + q\,,$$

so

$$q = \frac{t}{2} - \frac{\text{area}\,(AEK) + \text{area}\,(DEM) + \text{area}\,(KFB) + \text{area}\,(MFC)}{2}\,.$$

But since AB is parallel to DC, it follows that triangle AEK is similar to triangle MED and triangle KFB is similar to CFM. Now let h_1 be the altitude of DEM, i.e., the distance from E to DM, and h_2 be the altitude of AEK, i.e., the distance from E to AK. One then has $h_2 = h_1|AK|/|DM|$. Moreover, since AB and CD are parallel, one also has $h_1 + h_2 = h$. One concludes that

$$h_1 = h\,\frac{|DM|}{|DM| + |AK|}\,.$$

This then implies that

$$\text{area}\,(AEK) + \text{area}\,(DEM) = \frac{h}{2}\,\frac{|AK|^2 + |DM|^2}{|AK| + |DM|}\,.$$

One similarly gets

$$\text{area}\,(KFB) + \text{area}\,(MFC) = \frac{h}{2}\,\frac{|KB|^2 + |MC|^2}{|KB| + |MC|}\,.$$

Let $x = |DM|$, so that $|MC| = |DC| - x$, then one wants to maximize

$$\frac{t}{2} - \frac{h}{4}\left(\frac{x^2 + |AK|^2}{x + |AK|} + \frac{(|DC| - x)^2 + |KB|^2}{|DC| - x + |KB|}\right)\,,$$

which is equivalent to minimizing

$$\begin{aligned} f(x) &= \frac{x^2 + |AK|^2}{x + |AK|} + \frac{(|DC| - x)^2 + |KB|^2}{|DC| - x + |KB|} = x - |AK| + \frac{2|AK|^2}{x + |AK|} - |KB| - x + \frac{2|KB|^2}{|DC| - x + |KB|} \\ &= -|AB| + \frac{2|AK|^2}{x + |AK|} + \frac{2|KB|^2}{|DC| - x + |KB|}\,. \end{aligned}$$

Solving for $f'(x) = 0$ gives

$$\frac{|AK|^2}{(x + |AK|)^2} = \frac{|KB|^2}{(|DC| - x + |KB|)^2}\,,$$

and since all quantities are positive, one can take positive square roots to obtain $|AK| \cdot |CD| = x\,|AB|$, which is exactly the expression claimed above.

To complete the proof, one checks that this gives a minimum of $f(x)$. But this follows from the fact that

$$f''(x) = \frac{4|AK|^2}{(x + |AK|)^3} + \frac{4|KB|^2}{(|DC| - x + |KB|)^3} > 0\,.$$

Problem 10. *Can one cut a three–faced angle by a plane so that the intersection is an equilateral triangle?*

Solution: In general, one cannot cut a three–faced angle by a plane so that the intersection is an equilateral triangle.

Proof: One can take "three–faced" angle to mean the set of points $\{xU + yV + zW\,:\; x, y, z \geq 0\}$, where U, V, and W are unit vectors that do not all lie in a plane. The problem is therefore to find $x, y, z > 0$

such that $\|zW - xU\| = \|zW - yV\| = \|xU - yV\|$. Clearly, one can assume that $z = 1$, so the problem is equivalent to finding $x, y > 0$ such that

$$(27)(W - xU) \cdot (W - xU) = (W - yV) \cdot (W - yV), \quad (W - xU) \cdot (W - xU) = (xU - yV) \cdot (xU - yV)\,.$$

A counter example is given by $U = (1, 0, 0)$, $V = (0, 1, 0)$, $W = (0, 1/\sqrt{2}, 1/\sqrt{2})$. Thus, assume that there exist x, y that satisfy (27) for this choice of U, V, W. Then the left hand equation of (27) gives

$$x^2 + 1 = \left(y - \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}\,,$$

so that

$$y = \frac{1}{\sqrt{2}} \pm \sqrt{x^2 + \frac{1}{2}}\,.$$

Since one must have $x, y > 0$, the only solution is

$$y = \frac{1}{\sqrt{2}} + \sqrt{x^2 + \frac{1}{2}}\,. \tag{28}$$

The right hand equation of (27) implies that $x^2 + y^2 = x^2 + 1$. However, (28) implies that $y \geq \sqrt{2} > 1$, so there is a contradiction, and this proves the result.

Remark 10.1. One can give some general conditions on the existence of a solution (a complete characterization is left to the reader). In particular, it can be shown that there is a solution if either (a) $U \cdot V, U \cdot W, V \cdot W < 1/2$, or (b) $U \cdot V, U \cdot W, V \cdot W > 1/2$.

Proof: The idea is to solve the left hand equation of (27) by finding a solution $y = f(x)$, $x \geq 0$, such that $y = f(x) > 0$ for $x > 0$, and $f(x)$ is continuous. One then defines

$$g(x) = (W - xU) \cdot (W - xU), \quad h(x) = (xU - f(x)V) \cdot (xU - f(x)V)\,,$$

so that a solution exists when $g(x) = h(x)$. In order to do this, one finds a value $x_1 > 0$ such that $g(0) - h(0)$ and $g(x_1) - h(x_1)$ have opposite sign. Continuity will then imply that there is an $x_0 > 0$ for which $g(x_0) = h(x_0)$.

To prove (a), assume, without loss of generality, that $V \cdot W \geq U \cdot W$. One now solves the left hand side of (27) to get

$$y = V \cdot W + \sqrt{(x - (U \cdot W))^2 + (V \cdot W)^2 - (U \cdot W)^2}\,.$$

Let $f(x)$ the right hand side of this equation. Note that $f(x) > 0$ for $x > 0$, since by assumption, $U \cdot W < 0$ when $V \cdot W < 0$.

Now if $V \cdot W \leq 0$, then $f(0) = 0$, and otherwise $f(0) = 2(V \cdot W)$. Furthermore, $g(0) = 1$, while $h(0) = 0$ if $V \cdot W = 0$, and $h(0) = 4(V \cdot W)^2$ if $V \cdot W > 0$. By assumption $4(V \cdot W)^2 < 1$, so in either case, one has $h(0) < g(0)$.

Now, it is clear that $f(x)/x \to 1$ as $x \to \infty$, so

$$g(x) = x^2 + 1 - 2x(U \cdot W) \sim x^2\,,$$

while

$$h(x) = x^2 + [f(x)]^2 - 2xf(x)(U \cdot V) \sim (2 - 2\,(U \cdot V))\,x^2 = (1 + \varepsilon)\,x^2, \quad \varepsilon > 0\,,$$

since it was assumed that $U \cdot V < 1/2$. It follows that there is an $x_1 > 0$ for which $g(x_1) < h(x_1)$ and the proof follows as outlined above.

(b) Again, assume that $W \cdot U \leq W \cdot V$. In this case, one solves the right hand side of (27) but this time one takes the solution

$$y = V \cdot W - \sqrt{(x - (U \cdot W))^2 + (V \cdot W)^2 - (U \cdot W)^2}\,.$$

Let $f(x)$ be the right hand side of this equation. It follows that $f(0) = 0$, $f(2(U \cdot W)) = 0$, and $f(x) > 0$ for $2(U \cdot W) > x > 0$. Now $g(0) = 1$ and $h(0) = 0$, so $g(0) > h(0)$. On the other hand, $g(2(U \cdot W)) = 1$ while $h(2(U \cdot W)) = 4(U \cdot W)^2 > 1$, so that $h(2(U \cdot W)) > g(2(U \cdot W))$, and the result follows by continuity.

Problem 11. *Let H_1, H_2, H_3, H_4, be the altitudes of a triangular pyramid. Let O be an interior point of the pyramid and let h_1, h_2, h_3, h_4 be the perpendiculars from O to the faces. Show that*

$$H_1^4 + H_2^4 + H_3^4 + H_4^4 \geq 1024\, h_1 \cdot h_2 \cdot h_3 \cdot h_4 . \tag{29}$$

Solution: Let $ABCD$ be the tetrahedron and let its faces be F_1, F_2, F_3, F_4 with areas f_1, f_2, f_3, f_4, respectively. Let H_i be the altitude to F_i, and if P is an interior point of the tetrahedron, let $h_i = h_i(P)$ be the distance of P to F_i.

Recall that the volume of a tetrahedron is $\frac{1}{3}$base $\times$ height (knowledge of the exact constant $\frac{1}{3}$ is not important here). Thus, letting V be the volume of the tetrahedron, one has $H_i = 3V/f_i$, $i = 1, \ldots, 4$. Moreover, P divides $ABCD$ into 4 non–overlapping tetrahedra $PABC$, $PABD$, $PACD$, and $PBCD$. These tetrahedra have volumes $h_i f_i/3$ in some order, so one also gets the identity $h_1 f_1 + h_2 f_2 + h_3 f_3 + h_4 f_4 = 3V$.

Now both sides of (29) are homogeneous of degree 4, so without loss of generality one can normalize the tetrahedron to have volume 1/3. It follows that

$$H_i = \frac{1}{f_i},\ i = 1, \ldots, 4, \quad h_1 f_1 + h_2 f_2 + h_3 f_3 + h_4 f_4 = 1. \tag{30}$$

One next finds an upper bound for $h_1 h_2 h_3 h_4$ by maximizing $\alpha(y_1, y_2, y_3, y_4) = y_1 y_2 y_3 y_4$ given the constraints $y_1 f_1 + y_2 f_2 + y_3 f_3 + y_4 f_4 = 1$ and $y_i \geq 0$ (whether this maximum is attained by an actual interior point of the tetrahedron is left as a problem for the reader).

One observes that there will be a maximum with $y_i > 0$ since $\alpha(y_1, y_2, y_3, y_4)$ vanishes if any of the y_i's is zero. This implies that the maximum will be a local maximum, and one applies the following principle: Let S be a smooth surface of dimension of $n-1$ in Euclidean n–space and γ a real valued smooth function on S. Then at a local maximum s_0 of γ, the vector $\nabla\gamma = \left(\frac{\partial \gamma}{\partial x_1}, \ldots, \frac{\partial \gamma}{\partial x_n}\right)$ is a multiple of the normal to S at s_0. To see why this should be true, recall that $\nabla\gamma$ points in the direction of maximum growth of γ, so if s_0 is a local maximum, then moving away from s_0 along S, i.e., locally orthogonally to the normal vector at s_0, should never increase γ.

One now lets $y_i = 1/(4f_i) + x_i$, $i = 1, \ldots, 4$, and notes that maximizing α is equivalent to maximizing $\beta = \log\alpha$. This reduces the problem to maximizing

$$\beta(x_1, x_2, x_3, x_4) = \log\left[\left(\frac{1}{4f_1} + x_1\right)\left(\frac{1}{4f_2} + x_2\right)\left(\frac{1}{4f_3} + x_3\right)\left(\frac{1}{4f_4} + x_4\right)\right],$$

given that

$$x_1 f_1 + x_2 f_2 + x_3 f_3 + x_4 f_4 = 0\,. \tag{31}$$

One easily computes

$$\nabla\beta(x_1,x_2,x_3,x_4) = \left(\frac{1}{\frac{1}{4f_1}+x_1}, \frac{1}{\frac{1}{4f_2}+x_2}, \frac{1}{\frac{1}{4f_3}+x_3}, \frac{1}{\frac{1}{4f_4}+x_4}\right),$$

while (31) defines a plane with normal vector $\mathbf{n} = (f_1, f_2, f_3, f_4)$. Equating $\nabla\beta = t\,\mathbf{n}$, for $t \neq 0$, results in

$$x_i = \frac{1}{f_i}\left(\frac{1}{t} - \frac{1}{4}\right), \quad i = 1, \ldots, 4.$$

Applying (31) shows that in fact $x_1 = x_2 = x_3 = x_4 = 0$. It follows that the maximum of α occurs at

$$y_1 = \frac{1}{4f_1}, \quad y_2 = \frac{1}{4f_2}, \quad y_3 = \frac{1}{4f_3}, \quad y_4 = \frac{1}{4f_4},$$

and thus the maximal value of α is $1/(2^8\, f_1 f_2 f_3 f_4)$. This implies that

$$h_1 h_2 h_3 h_4 \leq \frac{1}{2^8\, f_1 f_2 f_3 f_4}.$$

On the other hand, (30) implies that

$$H_1^4 + H_2^4 + H_3^4 + H_4^4 = \frac{1}{f_1^4} + \frac{1}{f_2^4} + \frac{1}{f_3^4} + \frac{1}{f_3^4}.$$

The final result will therefore follow from the inequality

$$z_1 z_2 z_3 z_4 \leq \frac{z_1^4 + z_2^4 + z_3^4 + z_4^4}{4}. \tag{32}$$

To prove this, one starts with $(a-b)^2 \geq 0$ which implies

$$ab \leq \frac{a^2+b^2}{2}. \tag{33}$$

One therefore has

$$z_1 z_2 z_3 z_4 \leq \left(\frac{z_1^2+z_2^2}{2}\right)\left(\frac{z_3^2+z_4^2}{2}\right) \leq \frac{1}{4}\,\frac{(z_1^2+z_2^2)^2+(z_3^2+z_4^2)^2}{2} = \frac{1}{4}\,\frac{z_1^4+2z_1^2z_2^2+z_2^4+z_3^4+2z_3^2z_4^2+z_4^4}{2},$$

which leads to (32) upon applying (33) to $z_1^2 z_2^2$ and $z_3^2 z_4^2$.

Problem 12. *Solve the system of equations $y(x+y)^2 = 9$, $y(x^3-y^3) = 7$.*

Solution: The only real solution is $x = 2$, $y = 1$.

Proof: Clearly this is a solution. To show that this is the only one, let $x = ty$, then the system becomes

$$y^3(t+1)^2 = 9, \quad y^4(t^3-1) = 7. \tag{34}$$

Taking the first equation to the 4th power and cubing the second and dividing yields

$$\frac{(t+1)^8}{(t^3-1)^3} = \frac{9^4}{7^3},$$

which reduces to finding the roots of

$$f(t) = 9^4\,(t^3-1)^3 - 7^3\,(t+1)^8.$$

Any real positive root t_0 of $f(t)$ will yield a solution x_0, y_0 of (34) by letting $y_0 = (9^4/(t+1)^8)^{1/12}$ and $x_0 = t_0 y_0$. Conversely, the above shows that every solution of (34) yields a positive real root of $f(t)$.

Clearly, $t = 2$ is a root of $f(t)$, and this corresponds to the solution $x = 2$, $y = 1$. By the previous argument, one only has to show that $f(t)$ has no other positive real root. This can be done by directly computing

$$\frac{f(t)}{t-2} = 6561t^8 + 12779t^7 + 22814t^6 + 16341t^5 + 13474t^4 + 2938t^3 + 6351t^2 + 3098t + 3452\,,$$

and noting that all the coefficients are positive so there is no other positive real root. This computation can be done in a straightforward way by expanding

$$f(t) = 6561t^9 - 343t^8 - 2744t^7 - 29287t^6 - 19208t^5 - 24010t^4 + 475t^3 - 9604t^2 - 2744t - 6904,$$

and then doing a long division by $t - 2$. Such a computation was achieved in full during a train ride from IHES to Paris. Moreover, the fact that division by $t - 2$ must leave a zero remainder provides an internal check for the computation.

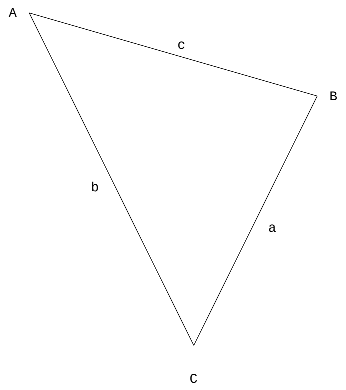
Problem 13. *Show that if a, b, c are the sides of a triangle and A, B, C are its angles, then*

$$\frac{a+b-2c}{\sin(C/2)} + \frac{b+c-2a}{\sin(A/2)} + \frac{a+c-2b}{\sin(B/2)} \geq 0\,.$$

Solution: By collecting terms, one can rewrite the expression as

$$(a-b)\left(\frac{1}{\sin(B/2)} - \frac{1}{\sin(A/2)}\right) + (a-c)\left(\frac{1}{\sin(C/2)} - \frac{1}{\sin(A/2)}\right) + (b-c)\left(\frac{1}{\sin(C/2)} - \frac{1}{\sin(B/2)}\right)\,.$$

One now observes that each summand on the right is nonnegative.



In fact, consider a triangle ABC with sides $a = BC$, $b = AC$, $c = AB$. Then $b \geq a$ if and only if $\angle B \geq \angle A$, and since $A, B < 180°$, if and only if $\sin(B/2) \geq \sin(A/2)$. It follows that

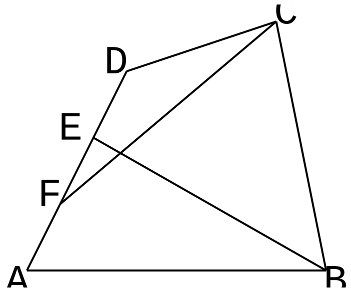
$$(a-b)\left(\frac{1}{\sin(B/2)} - \frac{1}{\sin(A/2)}\right) \geq 0\,.$$

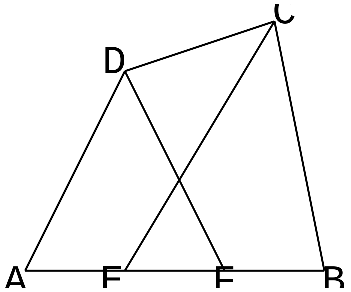
Since this is true for any two sides and corresponding vertices, it holds for the other terms and the result follows.

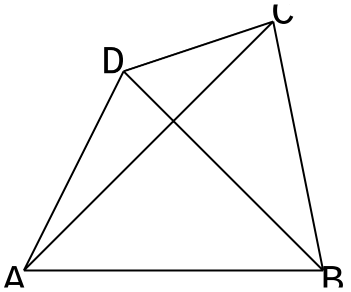
Problem 14. *In how many ways can one represent a quadrangle as the union of two triangles?*

Solution: If the quadrangle is convex then there are exactly two ways, and if it not convex then there are an infinite number of representations.

Proof: First consider a convex quadrangle $ABCD$, and assume that $ABCD$ is the union of two triangles T_1 and T_2.







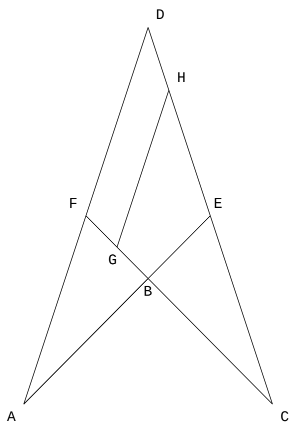
First one shows that one of the triangles must contain three vertices. For if this were not the case, then each of the triangles would contain exactly two vertices leading to two cases.

In the first case, each triangle contains two vertices on the same edge of the quadrangle. Without loss of generality, assume that T_1 contains AB and T_2 contains CD. Since a triangle is a closed convex set, and DA is not contained completely in T_1, there is a point E in the interior of AD for which $AE \subset T_1$ but $ED - \{E\}$ is not $\subset T_1$. This condition implies that E is a vertex of T_1. Since a triangle is convex and the quadrangle is convex none of the sides AE, BE, or AB of T_1 can be extended outside of $ABCD$, so it follows that $T_1 = ABE$. Similarly, there is an F in the interior of AD such that $T_2 = CDF$. However, this shows that $T_1 \cup T_2$ does not include the interior of BC, so this case is not possible.

In the second case, each triangle contains opposite vertices of the quadrangle. Without loss of generality, assume that T_1 contains A and C and T_2 contains B and D. By assumption, AB is not completely contained in T_1 or T_2, so as above, there is a point E in the interior of AB which is a vertex of T_1 and $T_1 = AEC$. Similarily, there is a point F in the interior of AB such that $T_2 = BDF$. This again implies that $T_1 \cup T_2$ does not contain the interior of BC, so this case is not possible either.

It follows that one of the triangles contains three vertices. Without loss of generality, assume that this is T_1 and that the vertices are A, B, C. Since the quadrangle is convex, none of the edges AB, AC, or BC can be extended and still remain in T_1 so $T_1 = ABC$. It follows that $BDA \subset T_2$, and since none of the edges of BDA can be extended, one has that $BDA = T_2$.

Thus, each choice of three vertices of $ABCD$ yields a partition into two triangles. There are 4 such choices, but two of these are equal by symmetry, so there are two choices: $ABC \cup BDA$ and $ABD \cup BCD$.



Consider the case where the quadrangle is not convex. Let the quadrangle be $ABCD$, where the angle B is greater than 180°. Now extend AB so that it cuts CD at E and extend CB so that it cuts AD at F. Also, let G be any point on BF and H be any point on ED. For any such choice, the quadrangle is the union of the triangles AED and CGH.

Remark 14.1. It seems clear that the examiners only meant *non–overlapping* triangles (note that the two triangles can *never* be disjoint). The "trap" was to only consider convex quadrangles (which some students actually fell into [20]), but the examiners were in fact trapped by failing to consider the most general case.

In the non–convex case, there are three ways to represent the quadrangle as a union of non–overlapping triangles which, using the above notation, are $ABD \cup CBD$, $AED \cup CBE$, and $ABF \cup CFD$. The argument is similar to the convex case and is left to the reader.

Problem 15. *Show that the sum of the numbers* $1/(n^3+3n^2+2n)$ *for* n *from 1 to 1000 is* $< 1/4$.

Solution: The factorization $n^3+3n^2+2n = n(n+1)(n+2)$ leads to the partial fraction expansion

$$\frac{1}{n^3+3n^2+2n} = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right) + \frac{1}{n+1} - \frac{1}{n+2}\,.$$

Now let $N > 3$, e.g., $N = 1000$, then

$$\sum_{n=1}^{N} \frac{1}{n^3+3n^2+2n} = \frac{1}{2}\sum_{n=1}^{N}\left(\frac{1}{n} - \frac{1}{n+2}\right) - \sum_{n=1}^{N}\left(\frac{1}{n+1} - \frac{1}{n+2}\right)\,.$$

Each sum reduces by telescopic summation and this gives

$$\frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{N+1}-\frac{1}{N+2}\right) - \left(\frac{1}{2}-\frac{1}{N+2}\right) = \frac{1}{4} + \frac{1}{2(N+2)} - \frac{1}{2(N+1)} = \frac{1}{4} - \frac{1}{2(N+1)(N+2)} < \frac{1}{4}\,.$$

Problem 16. *Solve the equation* $x^4 - 14x^3 + 66x^2 - 115x + 66.25 = 0$.

Solution: The roots of $x^4 - 14x + 66x^2 - 115x + 66.25$ are

$$\frac{7+i}{2} + \sqrt{4+2i}\,,\quad \frac{7+i}{2} - \sqrt{4+2i}\,,\quad \frac{7-i}{2} + \sqrt{4-2i}\,,\quad \frac{7-i}{2} - \sqrt{4-2i}\,,\quad \text{where } i = \sqrt{-1}.$$

Proof: Let $f(x) = x^4 - 14x + 66x^2 - 115x + 66.25$. Letting $x = y/2$ reduces $f(x) = 0$ to $g(y) = 0$, where

$$g(y) = y^4 - 28y^3 + 264y^2 - 920y + 1060\,.$$

One removes the cubic term by letting $y = z + 7$ so that $g(y) = 0$ is reduced to $h(z) = 0$, where

$$h(z) = z^4 - 30z^2 + 32z + 353\,.$$

Now $h(z) \equiv z^4 - z - 1 \pmod 3$, which is easily seen to be irreducible modulo 3. It follows that $f(x)$ is irreducible over the rationals, so there is no very easy solution to this problem. However, since this is an examination problem (one conjectures that students were not expected to be familiar with the solution to the general quartic), there might still be an "easy" solution. In particular, one could hope that $f(x)$ factors over a quadratic extension of the rational numbers. With this in mind, one writes

$$z^4 - 30z^2 + 32z + 353 = (z^2 + a\sqrt{D}\,z + b + c\sqrt{D})\,(z^2 - a\sqrt{D}\,z + b - c\sqrt{D})\,, \tag{35}$$

where a, b, c, D are integers (more generally, b and c could be half–integers) and D is squarefree. Equating terms in (35) one gets the conditions

$$\text{(I) } 2b - a^2D = -30, \quad \text{(II) } -2acD = 32\,, \quad \text{(III) } b^2 - c^2D = 353\,.$$

From (II) one concludes that D must be one of $-2, 2, -1$. If $D = -2$, then (III) has the solution $b = \pm 15$, $c = \pm 8$. But then, (I) would imply that a is divisible by 15, which is inconsistent with (II). If $D = 2$, then (III) has the solution $b = \pm 19$, $c = \pm 2$ (other solutions can be easily excluded). But then $a = \pm 4$ which is inconsistent with (I). Finally, if $D = -1$, then (III) has the solution $b = \pm 17$, $c = \pm 8$, and (II) implies that $a = \pm 2$. Trying out all the possible sign combinations, one eventually finds that $a = -2$, $c = -8$, $b = -17$ solves (I), (II), and (III). One therefore has the factorization

$$z^4 - 30z^2 + 32z + 353 = (z^2 - 2iz - 17 - 8i)\,(z^2 + 2iz - 17 + 8i). \tag{36}$$

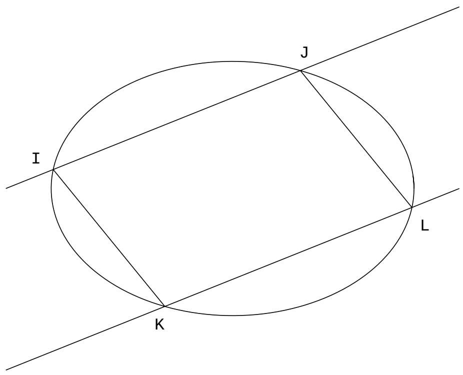
Applying the quadratic formula to each term yields roots $z = i \pm 2\sqrt{4 + 2i}$ for the left factor of (36) and $z = -i \pm 2\sqrt{4 - 2i}$ for the right factor of (36). The final answer follows on substituting $x = (z + 7)/2$.

Problem 17. *Can a cube be inscribed in a cone so that 7 vertices of the cube lie on the surface of the cone?*

Solution: It is not possible to inscribe a cube in a cone so that 7 vertices of the cube lie on the cone.

Proof: If this were possible, then there would be a face $ABCD$ with all vertices on the cone, and the parallel face $EFGH$ would have at least 3 vertices on the cone. Now the face $ABCD$ lies on a plane which cuts the cone in a conic section, i.e., either in a hyperbola, parabola, ellipse, or two intersecting lines. Only an ellipse can circumscribe a square at 4 points, therefore, the intersection is an ellipse, say E_1. Since $EFGH$ is parallel to $ABCD$, it lies on a plane which also intersect the cone at an ellipse, say E_2.

Now if $ABCD$ is symmetric with respect to E_1, i.e., its sides are parallel to the major or minor axes of E_1. Since $ABCD$, $EFGH$ are parallel and E_1 and E_2 are parallel, it follows that $EFGH$ is symmetric with respect to E_2. This implies that the vertices of $EFGH$ can meet E_2 at either 0, 2, or 4 points. This implies that $EFGH$ meets E_2 at 4 points. However, it is clear that there is a unique square that is inscribe symmetrically in an ellipse. Since E_1 and E_2 are parallel, they are similar, i.e., the ratio of their axes is the same, so the fact that they inscribe the same square implies that they are equal. This is clearly impossible, as different parallel sections of a cone must be different. It should also be noted that this also implies that E_1 cannot be a circle. The only other possiblity is that $ABCD$ is not symmetric with respect to E_1. However, this cannot happen as every inscribed square in an ellipse must have its sides parallel to the major or minor axes.



In order to prove this, without loss of generality, consider an ellipse $x^2 + y^2/a^2 = 1$, $a > 0$, and a line $y = mx + b$, $m \neq 0$. If these intersect at (x, y), then $(a^2 + m^2)x^2 - 2mbx + b^2 - a^2 = 0$, so the intersection points are

$$I = \left(\frac{-mb - a\sqrt{a^2 + m^2 - b^2}}{a^2 + m^2}, \frac{ba^2 - am\sqrt{a^2 + m^2 - b^2}}{a^2 + m^2} \right),$$

$$J = \left(\frac{-mb + a\sqrt{a^2 + m^2 - b^2}}{a^2 + m^2}, \frac{ba^2 + am\sqrt{a^2 + m^2 - b^2}}{a^2 + m^2} \right).$$

The length of the chord IJ is then

$$\frac{2a\sqrt{(1 + m^2)(a^2 + m^2 - b^2)}}{a^2 + m^2}.$$

It follows that given a and m, the only way to get two equal chords is to take $y = mx + b$ and $y = mx - b$ (this corresponds to the symmetry of the ellipse). The solutions corresponding to $-b$ are

$$K = \left(\frac{mb - a\sqrt{a^2 + m^2 - b^2}}{a^2 + m^2}, \frac{-ba^2 - am\sqrt{a^2 + m^2 - b^2}}{a^2 + m^2} \right),$$

$$L = \left(\frac{mb + a\sqrt{a^2 + m^2 - b^2}}{a^2 + m^2}, \frac{-ba^2 + am\sqrt{a^2 + m^2 - b^2}}{a^2 + m^2} \right).$$

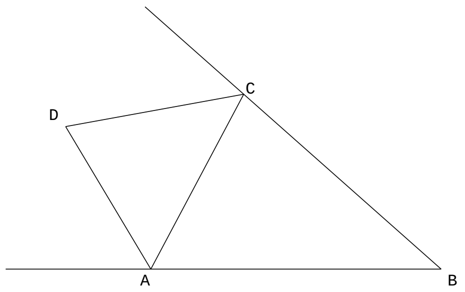
If these 4 points are to lie on a square then the angle KLJ must be $90°$, in other words, the slope of LJ must be $-1/m$ since the slope of KL is m. Since

$$L - J = \left(\frac{2mb}{a^2 + m^2}, \frac{-2ba^2}{a^2 + m^2} \right),$$

this slope is $-a^2/m$. It follows that $a^2 = 1$, which implies that the ellipse is a circle. However, this contradicts the assumption that the ellipse circumscribes the square asymmetrically, and completes the proof.

Problem 18. *The angle bisectors of the exterior angles A and C of a triangle ABC intersect at a point of its circumscribed circle. Given the sides AB and BC, find the radius of the circle.* [From [19]: "The condition is incorrect: this doesn't happen."]

Solution: As indicated by A. Shen, the statement is incorrect. In fact, the following is true: *In a triangle ABC, the angle bisectors of the exterior angles of A and C cannot meet on the circumscribed circle of ABC.*



Proof: Let the exterior angle bisectors of A and C meet at the point D. If D were to lie on the circumscribed circle of ABC, then $ABCD$ would be a cyclic quadrilateral. One appeals to the fact that in a cyclic quadrilateral the sum of opposite angle is $180°$ (this result seems well–known and the easy proof is left to the reader). One now observes that the angle BAD is equal to $(180° - A)/2 + A = 90° + A/2$ and similarly the angle BCD is $90° + C/2$. It follows that the sum of the angles BAD and BCD is $180° + (A+C)/2 > 180°$, which is a contradiction.

Problem 19. *A regular tetrahedron $ABCD$ with edge a is inscribed in a cone with a vertex angle of $90°$ in such a way that AB is on a generator of the cone. Find the distance from the vertex of the cone to the straight line CD.*

Solution: The statement of the problem is incorrect as the tetrahedron cannot be inscribed in the cone. Inscribing a tetrahedron in the cone means that all its vertices lying on the cone and that, apart from its vertices, it lies entirely inside a connected component of $\mathbf{R}^3$ minus the cone. As will be seen below, this is not possible. If one takes "inscribe" to mean only that the tetrahedron has all its vertices on the (double) cone, then the answer is $\frac{\sqrt{34}\,a}{8}$. However, this interpretation would imply that a cube could be inscribed in a cone, contradicting the result of Problem **17**.

Proof: Without loss of generality, one can take the cone to be given by the equation $x^2 + y^2 = z^2$ and the generator to be the line $x = 0$, $y = z$. Moreover, one can take the tetrahedron to have side length $= 1$, so that for a tetrahedron of side a, the answer will be a times the answer for this case. Since A and B lie on the generator, one can assume, without loss of generality, that $A = (0, t, t)$, $B = (0, t + 1/\sqrt{2}, t + 1/\sqrt{2})$. Letting

$$C = \left(-\frac{1}{2}, t - \frac{1}{2} + \frac{1}{2\sqrt{2}}, t + \frac{1}{2} + \frac{1}{2\sqrt{2}}\right), \quad D = \left(\frac{1}{2}, t - \frac{1}{2} + \frac{1}{2\sqrt{2}}, t + \frac{1}{2} + \frac{1}{2\sqrt{2}}\right),$$

it is easily checked that $ABCD$ is a regular tetrahedron. In order that C and D lie on the cone, one must solve

$$\frac{1}{4} + \left(t - \frac{1}{2} + \frac{1}{2\sqrt{2}}\right)^2 = \left(t + \frac{1}{2} + \frac{1}{2\sqrt{2}}\right)^2,$$

which has the unique solution $t_0 = 1/8 - 1/\sqrt{8}$. It follows that the tetrahedron must have vertices

$$A = \left(0, \frac{1}{8} - \frac{1}{\sqrt{8}}, \frac{1}{8} - \frac{1}{\sqrt{8}}\right), \quad B = \left(0, \frac{1}{8} + \frac{1}{\sqrt{8}}, \frac{1}{8} + \frac{1}{\sqrt{8}}\right), \quad C = \left(-\frac{1}{2}, -\frac{3}{8}, \frac{5}{8}\right) \quad D = \left(\frac{1}{2}, -\frac{3}{8}, \frac{5}{8}\right).$$

Since $1/8 < 1/\sqrt{8}$, it follows that the interior of AB and the interior of CD lie on two different components connected components of $\mathbf{R}^3$ minus the cone, so that $ABCD$ is not strictly inscribed in the double cone. In any case, the midpoint of CD is $(0, -3/8, 5/8)$ so that the distance from the vertex $(0, 0, 0)$ to CD is $\sqrt{34}/8$.

One must also consider the possibility of inscribing the tetrahedron asymmetrically by rotating it about $y = z$. However, the intersection of the cone with the possible rotations of the C and D about $y = z$ form a circle which lies in a plane orthogonal to the generator of the cone. Since the cone has vertex angle $90°$, the intersection of this plane with the cone is a parabola. A circle and a parabola can intersect at two points at most, so this implies that any intersection point must be symmetric, and there are no other solutions.

Remark 19.1. The following shows how one was led to the original construction of C and D. One begins with a simple result about regular tetrahedra.

Lemma 19.1. *Let $ABCD$ be a regular tetrahedron and φ be the angle between AB and $AC + AD$, i.e., the angle that a side from the base to the summit makes with the base, then $\varphi = \arccos 1/\sqrt{3}$.*

Proof: This is clearly equivalent to showing that for a regular tetrahedron of side 1, the distance of the summit to the base is $\sqrt{2/3}$. Thus, let $A = (0, \sqrt{3}/2, 0)$, $B = (0, y, z)$, $C = (-1/2, 0, 0)$, $D = (1/2, 0, 0)$. One has $\|C - B\| = 1$ and $\|A - B\| = 1$, so that

$$y^2 + z^2 = \frac{3}{4}, \quad \left(x - \frac{\sqrt{3}}{2}\right)^2 + z^2 = 1 .$$

Clearly $B = (0, 1/(2\sqrt{3}), \sqrt{2/3})$ solves these equations, and this proves the Lemma 19.1. $\square$

In order to inscribe the tetrahedron in the cone, one translates it by $(0, -\sqrt{3}/2, 0)$ so that it has vertices $A = (0, 0, 0)$, $B = (0, -1/\sqrt{3}, \sqrt{2/3})$, $C = (-1/2, -\sqrt{3}/2, 0)$, $D = (1/2, -\sqrt{3}/2, 0)$. In this position, B has angle $\pi - \varphi$ in the y–z plane and one wants it to have angle $\pi/4$, so one rotates the tetrahedron $\psi = \varphi - 3\pi/4$ degrees with respect to the x axis, then slides it up the cone by adding $(0, k, k)$. Let $R_{x,\psi}$ be the rotation, then one is solving for k such that $R_{x,\psi}(C) + (0, k, k)$ lies on $x^2 + y^2 = z^2$.

It only remains to compute $R_{x,\psi}(C)$, which follows from

$$R_{x,\psi}(C) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} + \frac{1}{2\sqrt{2}} \\ \frac{1}{2} + \frac{1}{2\sqrt{2}} \end{pmatrix} .$$

using

$$\cos\psi = \cos\frac{3\pi}{4}\cos\varphi + \sin\frac{3\pi}{4}\sin\varphi = -\frac{1}{\sqrt{6}} + \frac{1}{\sqrt{3}}, \quad \sin\psi = \cos\frac{3\pi}{4}\sin\varphi - \sin\frac{3\pi}{4}\cos\varphi = -\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{6}} .$$

Problem 20. *Let $\log(a, b)$ denote the logarithm of b to base a. Compare the numbers $\log(3, 4) \cdot \log(3, 6) \cdot \ldots \cdot \log(3, 80)$ and $2\log(3, 3) \cdot \log(3, 5) \cdot \ldots \cdot \log(3, 79)$*

Solution: $\log(3, 4) \cdot \log(3, 6) \cdot \ldots \cdot \log(3, 80) > 2\log(3, 3) \cdot \log(3, 5) \cdot \ldots \cdot \log(3, 79)$.

Proof: Since there are the same number of $\log(3, \cdot)$ terms on each side, the base 3 in the logarithm can be cancelled out and the above is equivalent to

$$\log 4 \cdot \log 6 \cdot \ldots \cdot \log 80 > 2\log 3 \cdot \log 5 \cdot \ldots \cdot \log 79 .$$

Taking logarithms of both sides, leads to the equivalent statement

$$\sum_{k=2}^{40} \log\log(2k) > \log 2 + \sum_{k=2}^{40} \log\log(2k - 1) . \tag{37}$$

The proof of this will rely on two basic facts: that $\log\log x$ is concave for $x \geq 3$ and that

$$\int_2^{41} \frac{dx}{(2x-1)\log(2x-1)} = \log 2\,. \tag{38}$$

To see that $\log\log x$ is concave for $x \geq 3$, note that

$$\frac{d^2}{dx^2}\log\log x = \frac{1}{x^2}\left(1 - \frac{1}{\log x}\right) > 0, \quad x > e\,.$$

In order to prove (38), note that

$$\int_2^{41} \frac{dx}{(2x-1)\log(2x-1)} = \left.\frac{\log\log(2x-1)}{2}\right|_2^{41} = \frac{\log\log 81 - \log\log 3}{2} = \log 2\,.$$

From the concavity of $\log\log x$ one has for $k \geq 2$,

$$\log\log(2k) > \frac{\log\log(2k+1) + \log\log(2k-1)}{2}\,.$$

Subtracting $\log\log(2k-1)$ from each side gives

$$\log\log(2k) - \log\log(2k-1) > \frac{\log\log(2k+1) - \log\log(2k-1)}{2}\,.$$

But the right hand side of this is exactly equal to

$$\frac{1}{2}\log\log(2x-1)\Big|_k^{k+1} = \int_k^{k+1} \frac{dx}{(2x-1)\log(2x-1)}\,.$$

It follows that

$$\sum_{k=2}^{40}[\log\log(2k) - \log\log(2k-1)] > \int_2^{41} \frac{dx}{(2x-1)\log(2x-1)} = \log 2\,,$$

by (38). This last inequality is exactly (37) and the result follows.

Problem 21. *A circle is inscribed in a face of a cube of side a. Another circle is circumscribed about a neighboring face of the cube. Find the least distance between points of the circles.*

Solution: The minimum distance is $a/\sqrt{20 + 8\sqrt{6}}$.

Proof: It is sufficient to treat the case of $a = 2$, as the solution is linear in a. One can thus consider the cube to have vertices at $(\pm1, \pm1, \pm1)$ and that the inscribed circle is given by $(\cos t, \sin t, 1)$, $0 \leq t < 2\pi$, and that the circumscribed circle is given by $(1, \sqrt{2}\sin u, \sqrt{2}\cos u)$, $0 \leq u < 2\pi$. The minimum distance will therefore be the minimum of $\sqrt{(\cos t - 1)^2 + (\sin t - \sqrt{2}\sin u)^2 + (1 - \sqrt{2}\cos u)^2}$. One therefore minimizes

$$(\cos t - 1)^2 + (\sin t - \sqrt{2}\sin u)^2 + (1 - \sqrt{2}\cos u)^2 = 5 - 2\cos t - 2\sqrt{2}\sin t\sin u - 2\sqrt{2}\cos u\,. \tag{39}$$

This is equivalent to maximizing

$$\cos t + \sqrt{2}\sin t \sin u + \sqrt{2}\cos u\,. \tag{40}$$

One does this by first considering u to be constant, and maximizing over t, and then maximizing over u. This requires the following

Lemma 21.1. *Let α and β be real numbers, then*

$$\max_{t\in[0,2\pi)}(\alpha\cos t + \beta\sin t) = \sqrt{\alpha^2 + \beta^2}\,.$$

Proof: Let $\gamma = \sqrt{\alpha^2+\beta^2}$, then there is a φ such that $\alpha/\gamma = \sin\varphi$ and $\beta/\gamma = \cos\varphi$. It follows that $\alpha\cos t + \beta\sin t = \gamma\sin(t+\varphi)$, which immediately implies the result. □

Continuing with the proof, Lemma 21.1 shows that the maximum of (40) is $\sqrt{1+2\sin^2 u} + \sqrt{2}\cos u$, which can be rewritten as $\sqrt{3-2\cos^2 u} + \sqrt{2}\cos u$. Since $\cos u$ takes on all values in $[-1,1]$, maximizing this last form over u is equivalent to maximizing

$$\sqrt{3-2x^2} + \sqrt{2}\,x, \quad x \in [-1,1]. \tag{41}$$

One checks for critical points by taking the derivative and setting it equal to zero. This gives

$$\sqrt{2} - \frac{2x}{3-2x^2} = 0\,,$$

so that $x = \sqrt{3}/2$, and the resulting value in (41) is $\sqrt{6}$. Since there is only one critical point in $[-1,1]$ the only other possible maxima are at $x = \pm 1$, but these give $1 \pm \sqrt{2}$ which are both smaller than $\sqrt{6}$.

It follows that the maximum value of (40) is $\sqrt{6}$ and plugging this back into (39) gives the minimum value

$$5 - 2\sqrt{6} = \frac{1}{5+2\sqrt{6}}\,.$$

The result then follows by substitution.

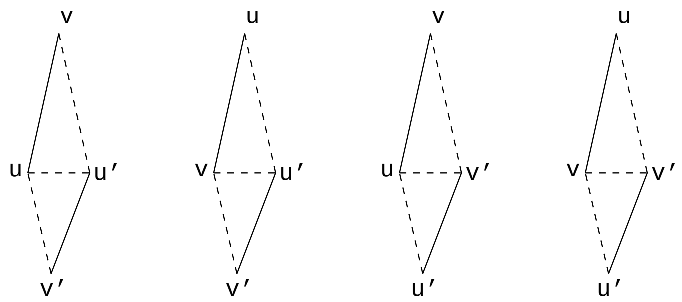
Remark 21.1. A. Shen [20] notes that there is an elegant solution to the problem: Consider two spheres with center at the center of the cube with each containing one of the circles mentioned in the problem. Clearly, the distance between the circles cannot be less than the distances between the spheres. On the other hand, it is easy to see that there is a ray from the center that intersects both circles. It follows that this distance is minimal.

Problem 22. *Given k segments in a plane, show that the number of triangles all of whose sides belong to the given set of segments is less than $C\,k^{3/2}$, for some constant $C > 0$.*

Solution: One has to interpret this as asking for triangles whose edges exactly belong to the set of given segments, see Section 4.

The problem is equivalent to bounding the number of triples $\{a,b\}$, $\{b,c\}$, $\{c,a\}$, where $\{a,b\}$, $\{b,c\}$, $\{c,a\}$ correspond to the endpoints of 3 distinct segments. Under this formulation it becomes clear that the fact that the e_i are line segments is unimportant and that the problem rests on the fact that each e_i joins its 2 endpoints. In other words, one is really considering a (combinatorial) graph V with vertices the endpoints of the segments and edges the e_i's. The problem can therefore be restated as: *Let V be a graph, then the number of triangles in the graph is $\leq C\,k^{3/2}$, where k is the number of edges in the graph.* Note that a triangle in a graph is a simply a set of 3 vertices that is completely connected. The main idea is the following

Lemma 22.1. *Given a graph with k edges, the number of undordered pairs of distinct triangles which have a common edge is $\leq 2\,k^2$.*



Proof: Let $e_1, \dots, e_k$ be the edges. To each unordered pair of edges (e, e'), where $e = \{u, v\}$ and $e' = \{u', v'\}$, one associates the 4 pairs of triangles

$$\{uvu', u'v'u\}, \quad \{vuu', u'v'v\}, \quad \{uvv', v'u'u\}, \quad \{vuv', v'u'v\}\,.$$

Thus each unordered pair of edges gives rise to at most 4 pairs of triangles with a common edge. Moreover, it is clear that any pair of triangles with a common edge will be generated in this way. It follows that the number of pairs of triangles with a common edge is ≤ 4 times the number of unordered pairs of edges. Since the number of unordered pairs of edges is $k(k-1)/2 \le k^2/2$, the result follows. $\square$

Now given a graph V, let T be the total number of triangles, and for each edge e, let t_e be the number of triangles containing e. For each edge e, the number of pairs of triangles having e as a common edge is $t_e\,(t_e - 1)/2$. Since distinct triangles cannot have more than one edge in common, the estimate of Lemma 22.1 implies

$$\sum_e \frac{t_e\,(t_e-1)}{2} \le 2\,k^2\,.$$

A simple computation shows that $t \le t(t-1)/2$ for $t \ge 3$, so

$$\sum_e t_e^2 \le 4\,k^2 + \sum_e t_e \le 4\,k^2 + \sum_e \frac{t_e(t_e-1)}{2} + 2\sum_e 1 \le 6\,k^2 + 2k \le 7\,k^2\,, \tag{42}$$

since $\sum_e 1 = k$ is the number of edges (it is assumed that $k \ge 3$, otherwise there are no triangles). Now

$$\left(\sum_e t_e\right)^2 \le 2 \sum_{t_e} \sum_{t_{e'} \le t_e} t_e\, t_{e'} \le 2k \sum_{t_e} t_e^2 \le 14\,k^3\,, \tag{43}$$

by (42). One concludes, by noting that each triangle contains exactly 3 edges, so that

$$\sum_e t_e = 3T\,.$$

Plugging this into (43) one obtains

$$T \le \frac{\sqrt{14}}{3}\,k^{3/2}\,,$$

which gives the result with $C = \sqrt{14}/3$.

Remark 22.2. Ofer Gabber has noted that the above method can be improved to give the optimal constant $C = \sqrt{2}/3$. This can be done using almost exactly the same techniques as follows (Ofer Gabber used a different approach). As before, one begins with

Lemma 22.2. *Given a graph with k edges, the number of ordered pairs of triangles which have a common edge is $\le 2\,k(k-1)$.*

Proof: One considers *ordered* pairs of distinct edges. Thus, let $(uv, u'v')$ be an ordered pair of edges. First assume that none of u, v equals u', v'. Then, as in the above, one can make at most 4 ordered pairs of triangles with a common edge, where one always lets the triangle containing uv be the first component of the pair. However, any such pairs of triangles, if they exist, will be counted twice in total. For example, if the triangles uvu' and $u'v'u$ exist, then $u'v$ and uv' are edges, so that the ordered pair $(uvu', u'v'u)$ will also be counted by $(u'v, uv')$.

Next assume that one of u, v equals one of u', v', without loss of generality, say $u' = u$. Then the possible pairs of triangles one can construct are $\{uvv', uvv'\}$, i.e., the two triangles are equal, and pairs $\{uvw, v'uw\}$, where w is another vertex unequal to u, v, v'. This last possibility will already have been counted twice above by the ordered pairs $(uv, v'w)$ and $(vw, v'u)$ with distinct vertices, so can be left out of this count. In the first case, the pair of $\{uvv', uvv'\}$, if it exists, will be counted 6 times: once by each of (uv, uv'), (uv', uv), (vu, vv'), (vv', vu), $(vv', v'u)$, $(v'u, vv')$.

Now let M be the number of ordered pairs of edges with no common vertex and N be the number of distinct ordered pairs of edges with a common vertex. Once again, one let t_e be the number of triangles containing the edge e. The above shows that

$$2\sum_e t_e(t_e - 1) \leq 4M, \quad \frac{6}{3}\sum_e t_e \leq N\,.$$

The second inequality follows from the above argument and the fact that $\sum_e t_e$ counts each triangle 3 times. It follows that

$$\sum_e t_e^2 \leq 2M + 2\,N \leq 2k(k-1)\,, \tag{44}$$

since $M + N = k(k-1)$ is the number of ways of choosing distinct pairs of edges. □

One applies the improved inequality (better known as the Cauchy–Schwarz inequality)

$$\left(\sum_i a_i b_i\right)^2 \leq \left(\sum_i a_i^2\right)\left(\sum_i b_i^2\right)\,, \tag{45}$$

and (44) to get

$$\left(\sum_e t_e\right)^2 \leq 2k^2(k-1)\,,$$

and the result follows as before.

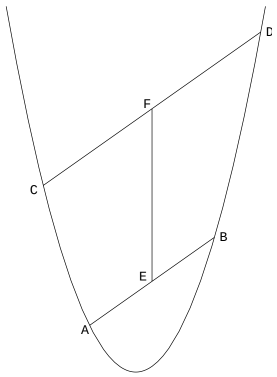
The fact that the value $C = \sqrt{2}/3$ is optimal is proved by considering the complete graph with n vertices which has $\varepsilon_n = n(n-1)/2$ edges and $\tau_n = n(n-1)(n-2)/6$ triangles, so that $\tau_n/\varepsilon_n^{3/2} \to \sqrt{2}/3$, as $n \to \infty$.

Remark 22.2. The argument directly generalizes to show that for each n, there is a constant C_n such that the number of n–gons all of whose sides belong to the segments is $\leq C_n\, k^{n/2}$. In fact, Lemma 22.1 already proves this for $n = 4$.

Remark 22.3. Igor Rivin has proved all the above results using algebraic methods, i.e., using the spectral theory of the adjacency matrix [18]. Moreover, this paper proves the analogous optimal results for higher length cycles.

Problem 23. *Use ruler and compasses to construct, from the parabola $y = x^2$, the coordinate axes.*

Solution: Let $A = (a, a^2)$ and $B = (b, b^2)$ be two points on the parabola. One can draw the line segment joining AB with ruler and compass. This line has slope $(b^2 - a^2)/(b - a) = a + b$. Let $C = (c, c^2)$ be a third point on the parabola unequal to A or B. One can then use ruler and compass to draw a line L through C parallel to AB and say that this line meets the parabola at $D = (d, d^2)$. Since CD has the same slope as AB, it follows that $a + b = c + d$. One can then use ruler and compass to construct E, the midpoint of AB and F, the midpoint of CD. It follows that $E = ((a + b)/2, (a^2 + b^2)/2)$ and $F = ((c + d)/2, (c^2 + d^2)/2)$. Using ruler and compass one constructs the line segment EF.



Since $a + b = c + d$, it follows that EF is parallel to the y axis. Using ruler and compass, one constructs a line L' perpendicular to EF through E. Let L' meet the parabola G and H. Using ruler and compass, one construct I, the midpoint of G, and then draws the line L'' through G which is parallel to EF. It follows that L'' is the y axis. Let L'' intersect the parabola at J. Using ruler and compass, one constructs the line L''' through J which is perpendicular to L''. Then L''' is the x-axis.

Problem 24. *Find all a such that for all $x < 0$ we have the inequality $ax^2 - 2x > 3a - 1$.*

Proof: The condition is that $0 \le a \le 1/3$.

Proof: By letting $x \mapsto -x$, the condition is equivalent to characterizing a for which

$$ax^2 + 2x > 3a - 1, \quad x > 0, \tag{46}$$

holds. If $0 \le a \le 1/3$, then $3a - 1 \le 0$ so the right side of (46) is non–positive, while the left side is positive, so the inequality holds. On the other hand, if $a > 1/3$, then $3a - 1 > 0$, so there is a small positive value of x for which (46) fails. Thus, if $1 > a > 1/3$, then one can take $x = (3a - 1)/3$, since

$$ax^2 + 2x < 3x < 3a - 1\,.$$

If $a \ge 1$, then one can take $x = 1/(3\sqrt{a})$ since

$$ax^2 + 2x = \frac{1}{9} + \frac{2}{3\sqrt{a}} < 1 < 3a - 1\,.$$

Finally, if $a < 0$, then $ax^2 + 2x \to -\infty$ as $x \to \infty$, i.e., (46) fails for all sufficiently large x.

Problem 25. *Let A, B, C be the angles and a, b, c the sides of a triangle. Show that*

$$60^\circ \le \frac{aA + bB + cC}{a + b + c} \le 90^\circ\,.$$

Solution: Since $A + B + C = 180°$, the statement can be rewritten as

$$\frac{A+B+C}{3} \leq \frac{aA+bB+cC}{a+b+c} \leq \frac{A+B+C}{2} \,. \tag{47}$$

To prove the right hand inequality, one multiplies by $2(a+b+c)$ to get the equivalent statement

$$Ab + Ac + Ba + Bc + Ca + Cb - Aa - Bb - Cc \geq 0 \,.$$

Collecting terms, this can be rewritten as

$$A(b + c - a) + B(a + c - b) + C(a + b - c) \geq 0 \,.$$

One now observes that each summand is positive. This follows from the triangle inequality which implies that $b + c > a$, $a + c > b$, and $a + b > c$. One can therefore conclude that the inequality on the right of (47) is strict.

To prove the left hand inequality, one multiplies by $3(a+b+c)$ to get the equivalent statement

$$2aA + 2bB + 2cC - aB - aC - bA - bC - cA - cB \geq 0 \,.$$

Collecting terms, this becomes

$$(A - B)(a - b) + (A - C)(a - c) + (B - C)(b - c) \geq 0 \,.$$

One now observes that, as in Problem 13, each of these terms is non–negative. For example, $a \geq b$ if and only if $A \geq B$, so $(A - B)(a - b) \geq 0$, and similarly for the other terms.

4 Notes

Problem 1. This problems appears to be a standard result in elementary geometry, "the butterfly theorem" [4, Problem 10.13.33] (French edition only), [6, p. 409] [9] [10] [14]. However, this result was not included in any standard geometry textbooks used by the candidates [20]. In [10], it is stated that a proof was given in 1815 by W.G. Horner (of Horner's method for polynomials) and that the shortest proof depends on projective geometry [9, p. 78]. Marcel Berger [4, Berger2] has stated that the butterfly theorem is a good example of a deceptive result. In particular, it is a statement about circles and lengths which lead one to look for a metric proof. However, as is seen above such arguments are quite awkward, whereas the correct point of view is projective. A projective generalization is given in [9] and [6]. Using the notation of the problem, this can be stated as: *Let AB be a chord of a conic section and let MN, ST be chords whose intersection does not lie on AB. If MN and ST both intersect AB at K and SN intersects AB at Q and MT intersects AB at Q, then K has the same harmonic conjugate with respect to P and Q and with respect to A and B.*

The proofs of Pavol Severa and David Ruelle both seem to be candidates "for the book" [1].

The first solution found by the author proceeded by converting it into a purely algebraic framework (the same is true for problem **2**. This has the advantage of almost guaranteeing a solution, even if one has missed the "idea" of the intended solution (this is confirmed by the fact that this in fact worked). Moreover, the "conceptual" solution of problem **1** used intermediate results, e.g., Lemma 1.2, which seemed to be as subtle as the original statement, whereas the algebraic proof was fairly direct.

However, algebraic methods have the disadvantage that they require much algebraic computation in which any slight error destroys any possibility of obtaining the solution. Morever, in order to keep the computations at a manageable level, one must be somewhat clever in setting up the algebraic formulation, as well as deciding how to proceed with the computation, e.g., see the solution to problem **2**. On the other hand, these considerations vanish almost completely if one allows oneself the use of a computer algebra system. Using such a system, the answer follows almost immediately from the algebraic formulation. One can argue that such proofs are more in the nature of verifications, in particular, they may not reveal how the result was orginally discovered. These issues are discussed in [11] [23].

Problem 3. This question appears as problem 10.13.11 in [4, Vol. 1].

Problem 4. This question is a special case of a problem of Erdős and Woods [12] [22]. Thus, for an integer $k \geq 2$, one considers m, n for which $m+i, n+i$ have the same prime divisors for $i = 0, \ldots, k-1$. The problem in question is $k = 2$ for which all the known examples are given in the above solution. It is conjectured that there exists a $k > 2$ such that if $m+i, n+i$ have the same prime divisors for $i = 0, \ldots, k-1$ then $m = n$. This conjecture has applications to logic [22]. This question and its generalizations has been studied by Balasubramanian, Langevin, Shorey, and Waldschmidt [2] [3].

Problem 5. In the formulation of [19], the word "perimeter" is given as "circumference."

Problem 6. In [19], the condition $0 < x < \pi/2$ is omitted which renders the condition invalid. To see this, note that the left hand side of (23) equals zero when $x = \pi/2$, but the derivative of the left hand side at $x = \pi/2$ is $-16/\pi^3 < 0$, so there is a small $\varepsilon > 0$ for which the left hand side of (23) is negative in $(\pi/2, \pi/2 + \varepsilon)$.

Problem 8. In [19], the condition is incorrectly given as $a^2 + b^2 = 4$, $cd = 4$, and the corresponding statement is false. In fact, the minimum value of $(a-d)^2+(b-c)^2$ is $4(3-2\sqrt{2})$ which is smaller than 1.6 and is actually smaller than 1, since

$$4(3-2\sqrt{2}) = \frac{4}{3+2\sqrt{2}} < 1\,.$$

The minimum value in this case is found as in the solution of Problem **8** given above: One is computing the distance between the curves $x^2+y^2=4$, and $xy=4$. The first of these is a circle of radius 2, while the second is a hyperbola.

Let L be the line $x+y=\sqrt{2}$, then this is clearly a tangent to the circle since it meets the circle at the point $(\sqrt{2},\sqrt{2})$ and is perpendicular to the radius. Likewise, the line L' given by $x+y=4$ is tangent to $xy=4$ since it meets it at $(2,2)$ and the slope of $y=4/x$ at $x=2$ is $-4/(2^2)=-1$.

Lemma 8.1 implies that the minimum distance between the two curves is $\geq$ the distance between L and L'. However, the line joining $(\sqrt{2},\sqrt{2})$ and $(2,2)$ has slope 1 so is perpendicular to L and L', thus the distance between these two points will be the actual minimum distance between the curves. One then computes the minimum distance to be

$$\sqrt{2(2-\sqrt{2})^2} = \sqrt{2}\,(2-\sqrt{2}) = 2\sqrt{2}-2\,,$$

so that the minimum of $(a-d)^2+(b-c)^2$ is

$$(2\sqrt{2}-2)^2 = 4\,(3-2\,\sqrt{2})\,,$$

as claimed.

Problem 14. In [19], the examiners were incorrectly given as Ugol'nikov and Kibkalo.

Problem 15. In [19], the examiners were incorrectly given as Ugol'nikov and Kibkalo.

Problem 21. In [19, p. 7], this is given as an example of a "murderous" problem, as it was the most difficult problem of the second round of the All–Union Olympiad in 1985. It was solved by 6 participants, partly solved by 3, and not solved by 91.

Problem 22. In [19], the original formulation was: Given k segments in the plane, give an upper bound for the number of triangles all of whose sides belong to the given set of segments. [Numerical data were given, but in essence one was asked to prove the estimate $O(k^{15})$.]

This formulation has the typographical error $O(k^{15})$ for $O(k^{1.5})$.

Parting thought: The following is my own suggestion for the type of problem considered in this paper and I leave it as an exercise for the reader: *Consider two triangles whose perimeters add up to a constant. What is the minimum value of the sum of the squares of the lengths of the edges of their symmetric difference (points which belong to exactly one triangle)?*

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