Interactive Oracle Proofs of Proximity to Codes on Graphs

Hugo Delavenne, Tanguy Medevielle, Élina Roussel

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1. Interactive Oracle Proofs of Proximity
2. Flowering protocol

1 Interactive Oracle Proofs of Proximity

2 Flowering protocol

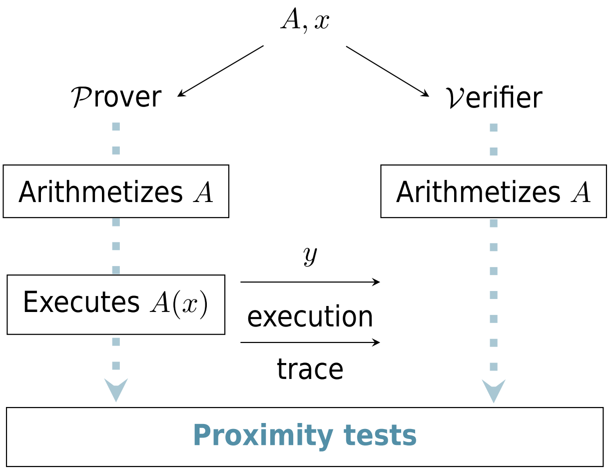
- Interactive Oracle Proof of Proximity
- Fast Reed-Solomon IOPP

SNARK means

- **S**uccinct
- **N**on-interactive
- **AR**gument of
- **K**nowledge

Goal: $\mathcal{P}$ convices $\mathcal{V}$ that

$$y = A(x)$$



Definition **Locally-testable code**

$C \subseteq \mathbb{F}^N$ is (ℓ, δ, s)-locally-testable if there is $\mathcal{V}$ with only ℓ **queries** to u such that

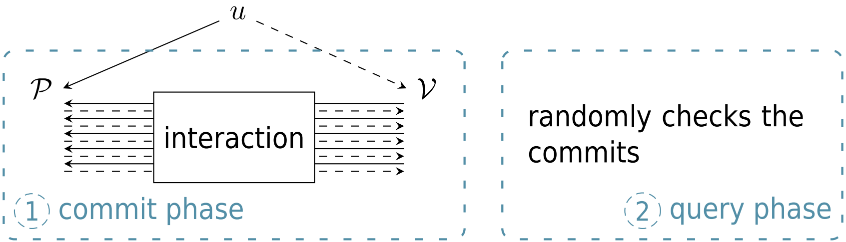
- **Completeness:** if $u \in C$ then $\mathbb{P}(\mathcal{V}^u \text{ accepts}) = 1$
- **Soundness:** if $\Delta(u, C) > \delta$ then $\mathbb{P}(\mathcal{V}^u$ **accepts**$) \leqslant s$

C has **locality** ℓ if C is (ℓ, δ, s)-locally-testable for $\delta, s < 1$.

Example **Reed-Solomon codes are not locally-testable**

RS$[\mathcal{L}, K] := \{f : \mathcal{L} \to \mathbb{F} \mid f \in \mathbb{F}[X]_{\leqslant K-1}\}$ does **not** have locality $\ell \leqslant K$.

Proof.
Any K values correspond to a degree $\leqslant K - 1$ polynomial by interpolation.



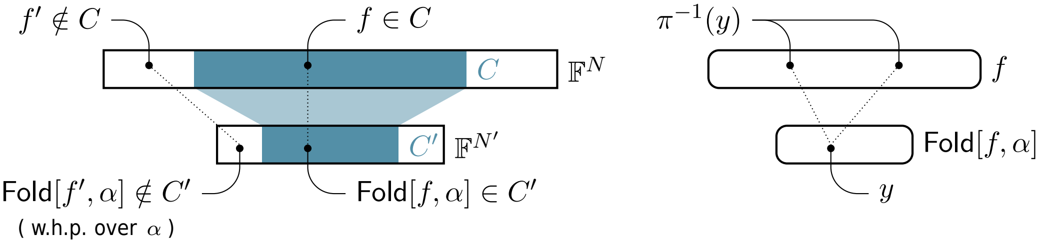
- **Completeness:** if $u \in C$ then $\mathbb{P}(\mathcal{V}^{u,\leftrightarrow\mathcal{P}}$ accepts$) = 1$
- **Soundness:** if $\Delta(u, C) > \delta$ then for any $\mathcal{P}$, $\mathbb{P}(\mathcal{V}^{u,\leftrightarrow\mathcal{P}}$ **accepts**$) \leqslant s$

[BCS16] Eli Ben-Sasson, Alessandro Chiesa, and Nicholas Spooner. Interactive Oracle Proofs.
In *Theory of Cryptography: 14th International Conference, TCC 2016-B, Beijing, China, October 31-November 3, 2016, Proceedings, Part II 14*, pages 31–60. Springer, 2016

Definition **Folding operator**

Let $C \subseteq \mathbb{F}^N$, $C' \subseteq \mathbb{F}^{N'}$. **Fold** : $\mathbb{F}^N \times \mathbb{F} \to \mathbb{F}^{N'}$ must satisfy for $f \in \mathbb{F}^N$

- **Completeness:** If $f \in C$ then $\mathrm{Fold}[f, \alpha] \in C'$
- **Soundness:** If $\Delta(f, C) > \delta$ then $\Delta(\mathrm{Fold}[f, \alpha], C') > \delta - \varepsilon$ w.h.p. over α
- **Locality:** $\mathcal{V}$ computes $\mathrm{Fold}[f, \alpha](y)$ with only queries to $f(\pi^{-1}(y))$



Folding polynomials

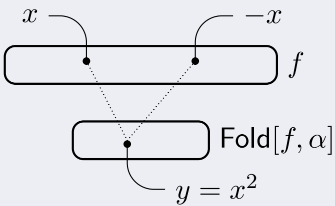
Idea: Test even and odd parts $f(X) =: f_{\text{even}}(X^2) + X f_{\text{odd}}(X^2)$

Definition **Folding**

Let $\mathcal{L}_1 := \{x^2 \mid x, -x \in \mathcal{L}_0\}$.

Define $\mathbf{Fold}[f, \alpha] : \mathcal{L}_1 \to \mathbb{F}$

$$\begin{aligned}\text{Fold}[f,\alpha](x^2) &:= f_{\text{even}}(x^2) + \alpha f_{\text{odd}}(x^2)\\ &= \frac{f(x)+f(-x)}{2} + \alpha \frac{f(x)-f(-x)}{2x}\end{aligned}$$



Field restriction: repeated foldings require $\approx |\mathcal{L}_0|$th roots of unity

e.g. $\mathbb{F}_p$ with $p = 2^{64} - 2^{32} + 1$ has 2^{32}th roots of unity

[BBHR18] Eli Ben-Sasson, Iddo Bentov, Yinon Horesh, and Michael Riabzev. Fast Reed-Solomon Interactive Oracle Proofs of Proximity. In *45th international colloquium on automata, languages, and programming (ICALP 2018)*. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2018

$$f_0 : \mathcal{L}_0 \to \mathbb{F}$$

$\mathcal{P}$ $\mathcal{V}$

$$\alpha_0 \xleftarrow{\$} \mathbb{F}$$

$$f_1 : \mathcal{L}_1 \to \mathbb{F}$$

$$f_1 = \mathsf{Fold}[f_0, \alpha_0]$$

$$\vdots$$

$$\alpha_{r-1} \xleftarrow{\$} \mathbb{F}$$

$$f_r : \mathcal{L}_r \to \mathbb{F}$$

$$f_r = \mathsf{Fold}[f_{r-1}, \alpha_{r-1}]$$

1 commit phase

$$x_0 \xleftarrow{\$} \mathcal{L}_0,\ x_1 := x_0^2$$

$$f_1(x_1) \stackrel{?}{=} \mathsf{Fold}[f_0, \alpha_0](x_1)$$

$$\vdots$$

$$x_r := x_{r-1}^2$$

$$f_r(x_r) \stackrel{?}{=} \mathsf{Fold}[f_{r-1}, \alpha_{r-1}](x_r)$$

repeat m times

$$f_r \stackrel{?}{\in} \mathsf{RS}[\mathcal{L}_r, K/2^r]$$

2 query phase

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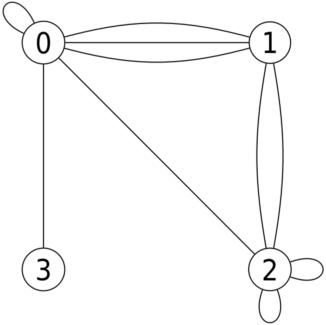
1 Interactive Oracle Proofs of Proximity

2 Flowering protocol

- Codes on graphs
- Folding graphs
- Flowering protocol
- Graphs considered

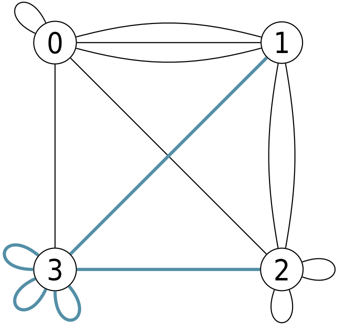
$\Gamma = (V, E)$ is a n-RIM:

- **Multigraph:** multiple edges and loops



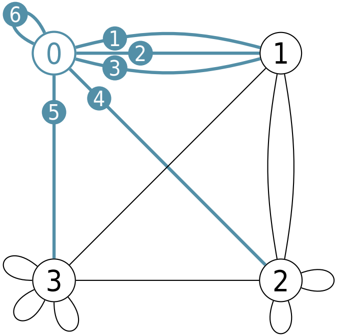
$\Gamma = (V, E)$ is a n-RIM:

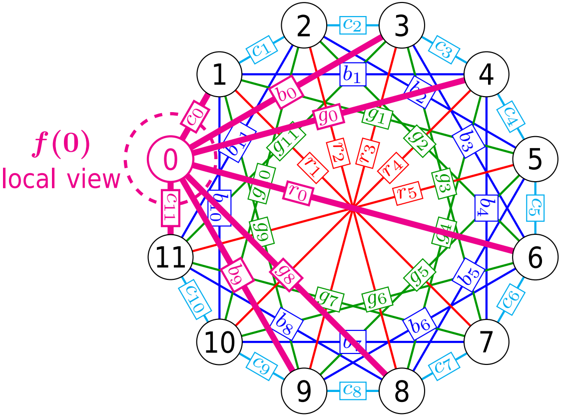
- ▷ **Multigraph:** multiple edges and loops
- ▶ **Regular:** same number n of edges



$\Gamma = (V, E)$ is a n-RIM:

- **Multigraph:** multiple edges and loops
- **Regular:** same number n of edges
- **Indexed:** edge is $(v, \ell) \in V \times [n]$
  Write $E(v, \ell) \in V$ the neighbor of v by ℓ





Word $f : V \times [n] \to \mathbb{F}$ on a graph Γ

Definition **Code $\mathcal{C}[\Gamma, C_0]$**

Given Γ a n-RIM and $C_0 \subseteq \mathbb{F}^n$,

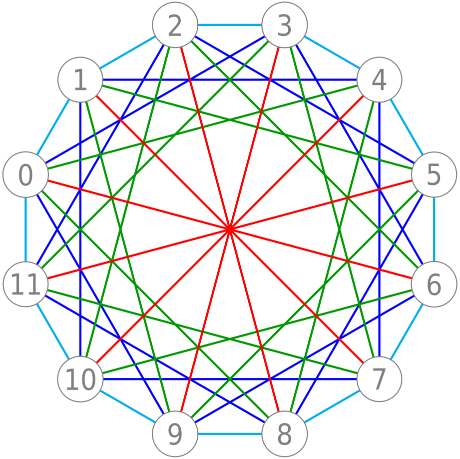
$$f \in \mathcal{C}[\Gamma, C_0] \iff \forall v,\ f(v) \in C_0.$$

We'll only use $C_0 = \mathsf{RS}[n, k]$.

Example

For $v = 0$, being a codeword requires $(c_0, b_0, g_0, r_0, g_8, b_9, c_{11}) \in \mathsf{RS}[7, k]$.

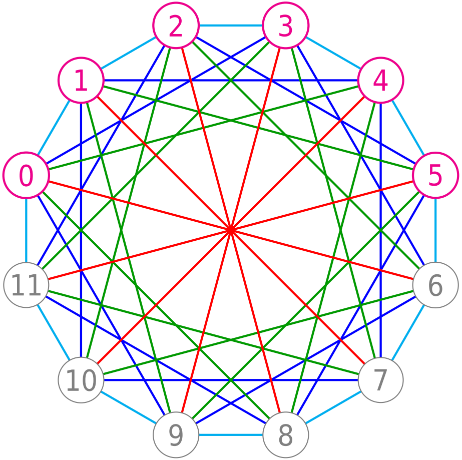
Cut-graph $\Gamma' = \mathrm{Cut}[\Gamma, V']$:



[DMR25] Hugo Delavenne, Tanguy Medevielle, and Élina Roussel. Interactive Oracle Proofs of Proximity to Codes on Graphs, 2025.
This work

Cut-graph $\Gamma' = \mathsf{Cut}[\Gamma, V']$:

- Choose vertices $V' \subseteq V$



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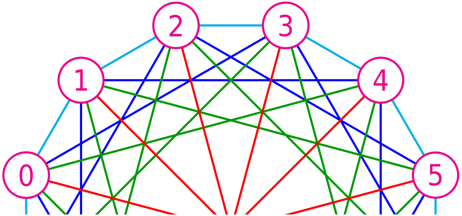
Cut-graph $\Gamma' = \mathsf{Cut}[\Gamma, V']$:

- Choose vertices $V' \subseteq V$
- Cut the rest



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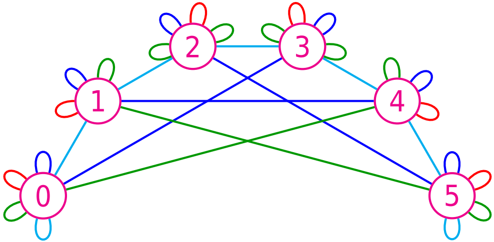
This work

Cut-graph $\Gamma' = \text{Cut}[\Gamma, V']$:

- ▷ Choose vertices $V' \subseteq V$
- ▷ Cut the rest
- ▶ Turn cut edges into petals

$$E_{V'}(v, \ell) = \begin{cases} E(v, \ell) & \text{if } E(v, \ell) \in V' \\ v & \text{otherwise} \end{cases}$$

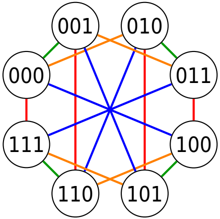


[DMR25] Hugo Delavenne, Tanguy Medevielle, and Élina Roussel. Interactive Oracle Proofs of Proximity to Codes on Graphs, 2025.
This work

Definition RIM isomorphism

A bijection $\varphi : V' \to V''$ is an **isomorphism** $\Gamma' \to \Gamma''$ if

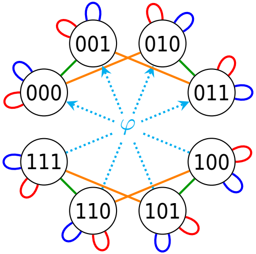
$$\forall (v', \ell) \in V' \times [n], \quad \varphi(E'(v', \ell)) = E''(\varphi(v'), \ell).$$



Definition Flowering cut

With $V'' = V \setminus V'$, if Cut$[\Gamma, V'] \sim$ Cut$[\Gamma, V'']$, the cut is **flowering**.

The cut-word Cut$[f, V']$ is the restriction $f_{|V' \times [n]}$.



Definition Folding

For $\alpha \in \mathbb{F}$, $(v', \ell) \in V' \times [n]$,

$$\text{Fold}[f, \alpha](v', \ell) := \text{Cut}[f, V'](v', \ell) + \alpha \text{Cut}[f, V''](\varphi^{-1}(v'), \ell).$$

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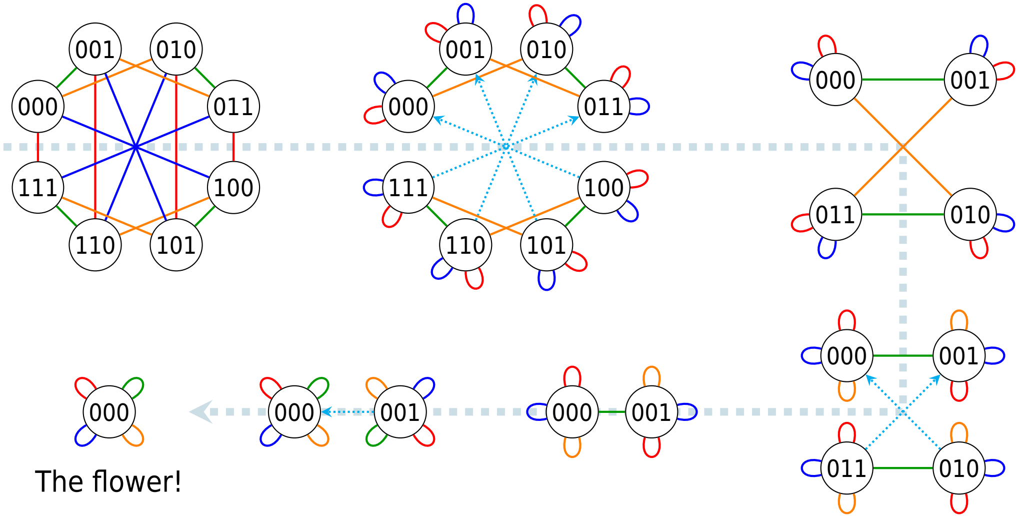
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The flower!

$f_0 : V_0 \times [n] \to \mathbb{F}$

$\mathcal{P}$ $\mathcal{V}$

$\alpha_0 \xleftarrow{\$} \mathbb{F}$

$f_1 : V_1 \times [n] \to \mathbb{F}$

$f_1 = \mathsf{Fold}[f_0, \alpha_0]$

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$\alpha_{r-1} \xleftarrow{\$} \mathbb{F}$

$f_r : V_r \times [n] \to \mathbb{F}$

$f_r = \mathsf{Fold}[f_{r-1}, \alpha_{r-1}]$

1 commit phase

$v_0 \xleftarrow{\$} V_0, \ v_1 := \pi_1(v_0)$

$f_1(v_1) \stackrel{?}{=} \mathsf{Fold}[f_0, \alpha_0](v_1)$

$\vdots$

$v_r := \pi_r(v_{r-1})$

$f_r(v_r) \stackrel{?}{=} \mathsf{Fold}[f_{r-1}, \alpha_{r-1}](v_r)$

$f_r \stackrel{?}{\in} \mathsf{RS}[n, k]$

repeat m times

2 query phase

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Let $C = \mathsf{RS}[N, K]$ or $\mathcal{C}[\Gamma, \mathsf{RS}[n, k]]$.

Proposition **FRI & Flowering complexities [BBHR18, DMR25]**

FRI and **Flowering** with m repetitions have complexities:

Protocol	FRI	Flowering
Prover	$< 8N$	$< 3N$
Verifier	$< 2m \log K$	$< 4m \log^2 N$
Query	$2m \log K$	$< 2m \log^2 N$
Round	$\log K$	$< \log N$

[BBHR18] Eli Ben-Sasson, Iddo Bentov, Yinon Horesh, and Michael Riabzev. Fast Reed-Solomon Interactive Oracle Proofs of Proximity. **In *45th international colloquium on automata, languages, and programming (ICALP 2018)*. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2018**

[DMR25] Hugo Delavenne, Tanguy Medevielle, and Élina Roussel. Interactive Oracle Proofs of Proximity to Codes on Graphs, 2025. **This work**

Let $C = \mathsf{RS}[N, K]$ or $\mathcal{C}[\Gamma, \mathsf{RS}[n, k]]$.

Theorem **FRI & Flowering completeness and soundness [BCI+23, DMR25]**

- **Completeness:** If $f \in C$ then $\mathcal{V}$ accepts with probability 1.
- **Soundness:** If $\Delta(f, C) > \delta$ then for any $\mathcal{P}$, $\mathcal{V}$ accepts with probability

$$\leqslant \frac{10^7 N^{3.5} \log K}{K^{1.5} |\mathbb{F}|} + \left(1 - \min\left(\delta, 1 - 1.05\sqrt{K/N}\right)\right)^m \qquad \textbf{(FRI)}$$

$$\leqslant \frac{N \log N}{|\mathbb{F}|} + (1 - \delta + \log N/N)^m . \qquad \textbf{(Flowering)}$$

[BCI+23] Eli Ben-Sasson, Dan Carmon, Yuval Ishai, Swastik Kopparty, and Shubhangi Saraf. Proximity Gaps for Reed–Solomon Codes. ***J. ACM*, 70(5), October 2023**

[DMR25] Hugo Delavenne, Tanguy Medevielle, and Élina Roussel. Interactive Oracle Proofs of Proximity to Codes on Graphs, 2025. **This work**

Cayley graphs

Definition **Cayley graph Cay(G, S)**

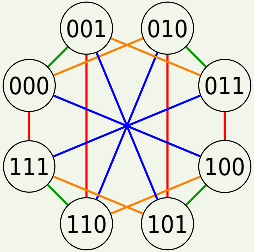
Fix $(G, \cdot)$. Let $S \subseteq G$ symmetric generating. Define $\Gamma = (G, E)$ where $E(g, s) = g{\cdot}s$.

We take

- $G = (\mathbb{F}_2^r, +)$
- $S \subseteq G$ of size n
- $\Gamma = \text{Cay}[G, S]$
- $C = \mathcal{C}[\Gamma, \text{RS}[n, k]]$

Example

With $G = (\mathbb{F}_2^3, +)$ and $S = \{100, 010, 001, 111\}$,



[Cay78] Arthur Cayley. Desiderata and Suggestions: No. 2. The Theory of Groups: Graphical Representation. ***American Journal of Mathematics***, **1(2):174–176, 1878**

Using S the columns of a parity check matrix of a $[n, n-r, d]_2$ binary code

Proposition **Parameters of the code**

- $N = n2^{r-1}$
- $\mathrm{rate}\, C \geqslant \frac{2k}{n} - 1$
- $\frac{1}{2}\delta \leqslant \Delta(C) \leqslant \delta$, with $\delta = \mathbf{2^{d-r-1}} \left(1 - \frac{k-1}{n}\right)$

If $d \ll r$ (i.e. far from MDS), δ is **terrible** ($O(1/N)$).

[DMR25] Hugo Delavenne, Tanguy Medevielle, and Élina Roussel. Interactive Oracle Proofs of Proximity to Codes on Graphs, 2025.
This work

Competing parameters with FRI

- Better soundness
- Complexity to improve
- No field restriction

New cuts, new graphs (work in progress)

- 2 disjoint cuts V', V'' $\rightarrow$ m covering cuts $V_1, V_2, ..., V_m$
- Flowering any Cayley graph $\rightarrow$ expander codes, constant minimal distance

Thank you!