A bijective proof of Jackson's formula for the number of factorizations of a cycle

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Abstract

Factorizations of the cyclic permutation $(1\,2\,\ldots\,N)$ into two permutations with respectively n and m cycles, or, equivalently, unicellular bicolored maps with N edges and n white and m black vertices, have been enumerated independantly by Jackson and Adrianov using evaluations of characters of the symmetric group. In this paper we present a bijection between unicellular partitioned bicolored maps and couples made of an ordered bicolored tree and a partial permutation, that allows for a combinatorial derivation of these results.

Our work is closely related to a recent construction of Goulden and Nica for the celebrated Harer-Zagier formula, and indeed we provide a unified presentation of both bijections in terms of Eulerian tours in graphs.

Key words: Symmetric group, factorizations, unicellular bicolored maps, permutations, bicolored trees, Harer-Zagier formula, Eulerian tours.

1 Introduction

From a combinatorial point of view, this paper is concerned with *factorizations* in the group Σ_N of permutations of $[N] = \{1, \ldots, N\}$. More precisely we are interested in counting the number of ways to write the (full cycle) permutation $\gamma_N = (1\,2\,\ldots\,N)$ as a product $\gamma_N = \alpha\beta$ of two permutations α and β of Σ_N that have respectively m and n cycles. In view of the graphical interpretation discussed below, such a factorization is called a *unicellular bicolored map* with n white vertices, m black vertices, and N edges.

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Our main result is a bijective proof of the following theorem.

Theorem 1.1 ([8]) *The number $B(m,n,N)$ of ways to write the (full cycle) permutation $\gamma_N = (1\,2\,\ldots\,N)$ as a product of two permutations of Σ_N with respectively m and n cycles, that is, the number of unicellular bicolored maps with m white vertices, n black vertices, and N edges, satisfies*

$$\sum_{m,n\geq 1} B(m,n,N) y^m z^n = N! \sum_{p,q\geq 1} \binom{N-1}{p-1,q-1}\binom{y}{p}\binom{z}{q}. \tag{1}$$

This statement immediately calls for two remarks. First, since the pairs (α,β) we consider are characterized by the relation $\beta = \alpha^{-1}\gamma_N$, there is exactly one factorization for each permutation α of Σ_N and, accordingly, setting $y = z = 1$ in Formula (1) yields $N!$. Secondly, setting only $z = 1$ and using Vandermonde's convolution formula, Formula (1) reads

$$\sum_{m,n\geq 1} B(m,n,N) y^m = N! \sum_{p\geq 1} \binom{N-1}{p-1}\binom{y}{p} = y(y+1)(y+2)\cdots(y+N-1),$$

in agreement with the fact that the number of permutations of Σ_N with m cycles is an unsigned Stirling number of the first kind [11].

The terminology *unicellular bicolored maps* refers to the fact that pairs of permutations can be used to encode *maps*, or *ribbon graphs*, or to be more explicit *embeddings of graphs in surfaces considered up to homeomorphisms*: detailed description of these encodings as well as examples of their numerous applications in various branches of mathematics and physics can be found in the survey [2] and in the book [9].

In particular Harer and Zagier enumerated unicellular maps with a prescribed number of edges and vertices in order to compute the Euler characteristics of some moduli spaces. Their result can be stated as follows.

Theorem 1.2 ([7]) *The number $A(m,N)$ of ways to write the permutation $\gamma_{2N} = (1\,2\,\ldots\,2N)$ as a product of a fix point free involution (with N cycles of length 2) and a permutation with m cycles, also known as the number of unicellular maps with m vertices and N edges, satisfies*

$$\sum_{m,N} A(m,N) y^m = (2N-1)!! \sum_{p\geq 1} 2^{p-1}\binom{N}{p-1}\binom{y}{p}. \tag{2}$$

At least six proofs of this formula have been proposed in the literature between 1986 and 1999 (see references in [4]). However these proofs rely on computations with characters of the symmetric group or on matrix integrals, and until recently no elementary proof was known. A first combinatorial proof was given by Lass in [10] using the BEST Theorem on the number of Eulerian tours [11]. Finally a bijection directly proving Formula (2) was proposed by Goulden and Nica in [4].

A more refined formula than (2) was in fact already given by Walsh and Lehman in [13] for the number of factorizations (α, β) of γ_{2N} with α a fix point free involution and β a permutation of cycle type μ, for any fixed partition μ of $2N$. Summing over all μ with m parts yields a (complicated) alternative formula for $A(m, N)$. In [5], this refined formula was extended to count the number of factorizations (α, β) of γ_N with α of cycle type λ and β of cycle type μ, for any pair (λ, μ) of partitions of N. Again summing over all λ with m parts and μ with n parts yields a (complicated) formula for $B(m, n, N)$.

However, the much simpler Expression (1) for $B(m, n, N)$ was first obtained by Jackson in the following general form.

Theorem 1.3 ([8]) *The numbers $B(m_1, \ldots, m_k, N)$ of factorizations of γ_N as a product of k permutations with respectively $m_1, \ldots, m_k$ cycles satisfy*

$$\begin{aligned}\sum_{m_1,\ldots,m_k} & B(m_1, \ldots, m_k, N)\, z_1^{m_1} \ldots z_k^{m_k} \\ &= \quad N!\Phi\left\{z_1 \cdots z_k \left((1+z_1)\cdots(1+z_k) - z_1 \cdots z_k\right)^{N-1}\right\}\end{aligned}$$

where Φ is the linear operator on polynomials defined by

$$\Phi(z_1^{\ell_1} \cdots z_k^{\ell_k}) = \binom{z_1}{\ell_1} \cdots \binom{z_k}{\ell_k}.$$

Theorem 1.1 is a mere rephrasing of the case $k = 2$ of this result. The proof of Jackson is based on evaluations of characters of the symmetric group. A quite different expression for $B(m, n, N)$ was independently given by Adrianov. Upon making explicit the Gauss hypergeometric function that appears in his statement, his result can be written as follows.

Theorem 1.4 ([1]) *The number $B(m, n, N)$ satisfies*

$$\sum_{m,n\geq 1} B(m, n, N) y^m z^n = (N-1)! \sum_{k\geq 0} \frac{yz}{k+1} \binom{y+k}{k} \binom{z+k}{k} \binom{y+z}{N-1-2k}.$$

Adrianov adapted the strategy developed for Formula (2) by Zagier, as exposed in [9, Appendix A]. Like Jackson he relies on evalutations of characters of the symmetric group, and it is not clear how to derive direct combinatorial proof from their approach.

In this paper, we propose a new proof of Jackson's Formula (1) for unicellular bicolored maps based on a direct combinatorial construction. Our initial presentation is similar in essence to the construction of Goulden and Nica in [4] for unicolored maps, though some steps are more involved in our case. Namely these authors proved that (unicolored) unicellular maps with a vertex partition are in one-to-one correspondence with couples formed of an ordered tree and a partial pairing. We show that bicolored unicellular maps with white and black vertices separately partitioned are in bijection with couples formed of a bicolored ordered tree and a partial permutation. In order to stress the similarities but also the important differences between our construction and Goulden and Nica's, we first state and prove our result using notational conventions close to theirs.

While our combinatorial approach directly leads to Jackson's formula (with $k = 2$), it does not provides an interpretation of Adrianov's formula. Instead we sketch a proof of the equivalence of these two formulas using Vandermonde convolution formula.

Finally we restate our bijection and Goulden and Nica's in terms of Eulerian tours in certain directed graphs associated to unicellular maps. This point of view reveals a deep connection between these constructions and the BEST Theorem used by Lass in [10] to prove Formula (2).

Outline of the paper

The ingredients of our combinatorial approach are presented in Section 2. The bijection is then described (Section 3) and proved to be one-to-one (Section 4). Next, in Section 5, we show the equivalence of our formula with Adrianov's. Finally Section 6 is concerned with a reformulation of both our and Goulden and Nica's bijections in terms of Eulerian tours on graphs.

2 Unicellular partitioned bicolored maps, trees and partial permutations

2.1 Definition

Let $CC(p, q, N)$ be the set of triples (π_1, π_2, α) such that π_1 and π_2 are partitions of $[N]$ into p and q blocks and α is a permutation of Σ_N such that:

- each block of π_1 is a union of cycles of α,

- and each block of π_2 is a union of cycles of $\beta = \alpha^{-1}\gamma_N$, where $\gamma_N = (1\,2\,\ldots\,N)$.

Any such triple is called a *unicellular partitioned bicolored map* with N edges, and p white and q black blocks.

2.2 The representation of maps as ribbon graphs

We now sketch a graphical representation of unicellular maps using ribbon graphs. This representation is not used in the first description of our bijection (Sections 3 and 4) which is phrased entirely in terms of permutations. However it allows us to make pictures and we believe that it conveys useful insights on the mechanisms of the bijection. Moreover we will use this representation in Section 6 to rephrase our bijection.

A *ribbon graph* is a drawing of a graph in the plane (possibly with edge crossings) such that any vertex v of degree k has a neigborhood homeomorphic to a disk in which the edges incident to v form a star with k branches. Edges are allowed to cross each other outside of these neigborhoods but, as we shall see, such crossings will be irrelevant for our purpose.

The following informal description will be useful to illustrate later constructions. Consider an edge e of a ribbon graph and assume it is oriented. Imagine then a (flat) ant walking on the right hand side of e in the direction given by the orientation. The ant ignores irrelevant edge crossings if it meets any, but when it arrives at a vertex it continues its walk along the next branch of the star without crossing any edge in the neigborhood of the vertex.

A graph is *bicolored* if its vertices are colored into two colors and each edge connects a black vertex to a white one. In this case, the side of the edge which is on the right when going from the white endpoint to the black one is called the *right hand side of the edge* (and the other is the left hand side). In particular an ant walking as above around a bicolored ribbon graph alternatively visits right and left hand sides of edges.

A unicellular bicolored map (α, β) with N edges is then represented as a labelled ribbon graph as follows:

- The ribbon graph has N edges, which carry distinct labels $\{1, \ldots, N\}$.
- Each cycle of α describes the local organization of edges around a white vertex: a cycle $(a_1, \ldots, a_k)$ corresponds to a vertex incident to the edges with label $a_1, \ldots, a_k$ in counterclockwise direction.
- Each cycle of β describes the cyclic organization of the labels of edges around a black vertex.

In terms of the ant walk as described above, the fact that $\alpha\beta = \gamma_N$ implies that starting on the right hand side of the edge with label 1, the ant visits the right hand sides of the edges $1, 2, \ldots, N$ in this order.

Finally the condition on a paritioned bicolored maps (π_1, π_2, α) that the parts π_1 and π_2 be union of cycles of α and β immediately translates as follows for its representation as a labeled ribbon graphs.

- The partition π_1 induces a partition of the white vertices into p subsets.
- The partition π_2 induces a partition of the black vertices into q subsets.

Example 2.1 *Figure (1) gives a ribbon graph representation of the triple* $(\pi_1, \pi_2, \alpha) \in CC(3, 2, 9)$, *defined by* $\alpha = (1)(24)(3)(57)(6)(89)$, $\beta = (1479)(23)(56)(8)$, $\pi_1 = \{\pi_1^{(1)}, \pi_1^{(2)}, \pi_1^{(3)}\}, \pi_2 = \{\pi_2^{(1)}, \pi_2^{(2)}\}$ *with*

$$\begin{array}{lll} \pi_1^{(1)} = \{2,4,6\}, & \pi_1^{(2)} = \{8,9\}, & \pi_1^{(3)} = \{1,3,5,7\}, \\ \pi_2^{(1)} = \{2,3,5,6\}, & \pi_2^{(2)} = \{1,4,7,8,9\} & \end{array}$$

where the numbering of the blocks is arbitrary.
To visualize it better we also assume that each block is associated with some particular shape: $\pi_1^{(1)}$ *with square,* $\pi_1^{(2)}$ *with circle,* $\pi_1^{(3)}$ *with triangle,* $\pi_2^{(1)}$ *with rhombus and* $\pi_2^{(2)}$ *with pentagon. Therefore each vertex of our partitioned map will have a shape corresponding to its block.*

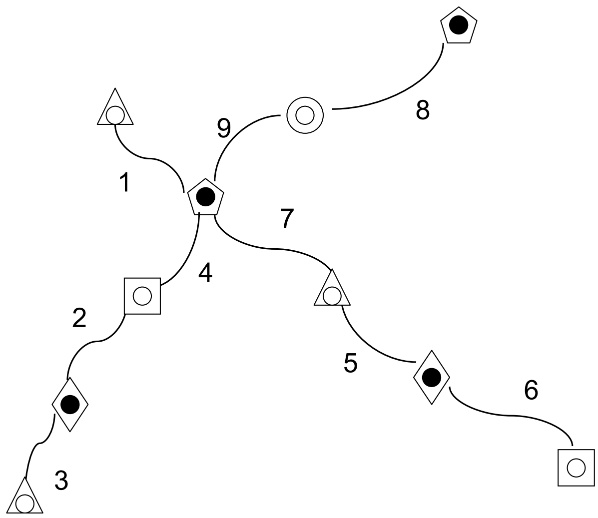


Fig. 1. Example of a partitioned bicolored map

Observe that an ant placed on the right hand side of Edge 1 and starting to walk from its white end point will indeed visit all sides of edges and more precisely will visit the right hand sides in increasing order.

2.3 Connection with (non-partitioned) unicellular bicolored maps

Let $C(p,q,N)$ denote the cardinality $|CC(p,q,N)|$ of the set of partitioned unicellular maps with N edges, p white and q black blocks. Using the Stirling number of the second kind $S(a,b)$ enumerating the partitions of a set of a elements into b non-empty, unordered subsets, we have

$$C(p,q,N) = \sum_{m\geq p, n\geq q} S(m,p)S(n,q)B(m,n,N). \tag{3}$$

In other terms Stirling numbers govern the change of polynomial bases between monomials x^a and falling factorials $(x)_b = \prod_{i=0}^{b-1}(x-i)$, so that $\sum_{b=1}^{a} S(a,b)\,(x)_b = x^a$ (see e.g [12, Theorem??]). The relation above can thus be written

$$\sum_{m,n\geq 1} B(m,n,N) y^m z^n = \sum_{p,q\geq 1} C(p,q,N)\,(y)_p (z)_q, \tag{4}$$

and in view of Formula (1) our aim is to compute the $C(p,q,N)$.

2.4 Trees and partial permutations

Let $BT(p,q)$ denote the set of ordered rooted bicolored trees with p white vertices, q black vertices, and a white root. (Throughout, all bicolored trees are assumed to be ordered and rooted with a white root). The cardinality of $BT(p,q)$ (see e.g. Section 2.7.14 of [3]) is given by

$$|BT(p,q)| = \frac{p+q-1}{pq}\binom{p+q-2}{p-1}^2. \tag{5}$$

We also denote by $PP(X,Y,A)$ the set of partial permutations from any subset of X of cardinality A to any subset of Y (of the same cardinality). The cardinality of this set is given by

$$|PP(X,Y,A)| = \binom{|X|}{A}\binom{|Y|}{A}A!. \tag{6}$$

From now on, for the sake of simplicity, we write $PP(M,N,A)$ for $PP(X,Y,A)$ when $X=[M]$ and $Y=[N]$.

As we have seen, proving Formula (1) boils down in view of Formula (4) to proving

$$C(p,q,N) \stackrel{?}{=} \frac{N!}{p!q!}\binom{N-1}{p-1,q-1}. \tag{7}$$

This formula can be rearranged into

$$\begin{aligned} C(p,q,N) \stackrel{?}{=} & \left[\frac{p+q-1}{pq}\binom{p+q-2}{p-1}^2\right] \\ & \times \left[\binom{N}{N+1-(p+q)}\binom{N-1}{N+1-(p+q)}(N+1-(p+q))!\right], \end{aligned}$$

so that the formula we actually want to prove is

$$|CC(p,q,N)| \stackrel{?}{=} |BT(p,q)| \times |PP(N,N-1,N+1-(p+q))|. \tag{8}$$

Our strategy to prove Theorem 1.1 will thus be to obtain it as a corollary of the following result.

Theorem 2.2 *There is a bijection*

$$\begin{aligned} \Theta_{N,p,q}: \quad CC(p,q,N) & \longrightarrow BT(p,q) \times PP(N,N-1,N+1-(p+q)) \\ (\pi_1,\pi_2,\alpha) & \longmapsto (t,\sigma) \end{aligned} \tag{9}$$

between

- *unicellular partitioned bicolored maps with N edges, p blocks of white vertices and q blocks of black vertices,*
- *and couples formed of a bicolored tree with p white and q black vertices and a partial permutation between subsets of $[N]$ and $[N-1]$ containing $N+1-(p+q)$ elements.*

3 A description of $\Theta_{N,p,q}$

In this section, we construct the announced mapping that associates to a triple $(\pi_1,\pi_2,\alpha) \in CC(p,q,N)$ an ordered bicolored tree t in $BT(p,q)$ and a partial permutation σ in $PP(N,N-1,N+1-(p+q))$.

The ordered bicolored tree. Let $\pi_1^{(1)}, \ldots, \pi_1^{(p)}$ and $\pi_2^{(1)}, \ldots, \pi_2^{(q)}$ be the blocks of the partitions π_1 and π_2 respectively, where the indexing of the blocks is subject only to the condition that $1 \in \pi_1^{(p)}$. Denote by $m_1^{(i)}$ the maximal element of the block $\pi_1^{(i)}$ ($1 \leq i \leq p$) and by $m_2^{(j)}$ the maximal element of $\pi_2^{(j)}$ ($1 \leq j \leq q$). As already said, the index p must be given to the block of partition π_1 containing the element 1, while the indexation of all other blocks is arbitrary.

Create a labeled bicolored tree T with p white vertices (labeled $1, \ldots, p$) and q black vertices (labeled $1, \ldots, q$), as follows. The root of T is the white vertex p. For $j = 1, \ldots, q$, the black vertex j is a descendant of the white vertex i if $\beta(m_2^{(j)})$ belongs to the white block $\pi_1^{(i)}$. Similarly, for $i = 1, \ldots, p-1$, the white vertex i is a descendant of the black vertex j if $m_1^{(i)}$ belongs to the black block $\pi_2^{(j)}$. If two black vertices j and k are both descendants of a white vertex i, then j is to the left of k when $\beta(m_2^{(j)}) < \beta(m_2^{(k)})$; if white vertices i, l are both descendants of a black vertex j, then i is to the left of l when $\beta^{-1}(m_1^{(i)}) < \beta^{-1}(m_1^{(l)})$.

Lemma 3.1 *The labeled bicolored tree T is correctly defined.*

PROOF. Let the white vertex i be a descendant of the black vertex j being in its turn a descendant of the white vertex k. Then $m_1^{(i)} \in \pi_2^{(j)}$ and so we have $m_1^{(i)} \leq m_2^{(j)}$. Furthermore, $\beta(m_2^{(j)}) \in \pi_1^{(k)}$ and, as white blocks are unions of disjoined cycles of α, we have $\alpha\beta(m_2^{(j)}) \in \pi_1^{(k)}$, which implies that $\gamma_N(m_2^{(j)}) \leq m_1^{(k)}$. Now let us consider the two possible cases for $m_2^{(j)}$:

(i) If $m_2^{(j)} \neq N$ then $\gamma_N(m_2^{(j)}) = m_2^{(j)} + 1$ and we have $m_1^{(i)} < m_1^{(k)}$.
(ii) If $m_2^{(j)} = N$ then $\gamma_N(m_2^{(j)}) = 1$ and we have $1 \in \pi_1^{(k)}$ which means that $i = p$.

Thus, we can specify a unique path from any white vertex i to the root vertex (a similar reasoning also works for a black vertex) which means that the tree T is well-defined.

□

Remove the labels from T to obtain the bicolored ordered tree t.

Example 3.2 *Let us go back to Example 2.1. We can keep the previous numbering of the blocks since it satisfies the condition $1 \in \pi_1^{(p)}$. Since, $\beta(m_2^{(1)}) = \beta(6) = 5 \in \pi_1^{(3)}$ and $\beta(m_2^{(2)}) = \beta(9) = 1 \in \pi_1^{(3)}$ the black rhombus 1 and the black pentagon 2 are both descendants of the white triangle 3. Moreover, as $\beta(m_2^{(1)}) < \beta(m_2^{(2)})$ vertex 2 is to the left of vertex 1. Since $m_1^{(1)} = 6 \in \pi_2^{(1)}$*

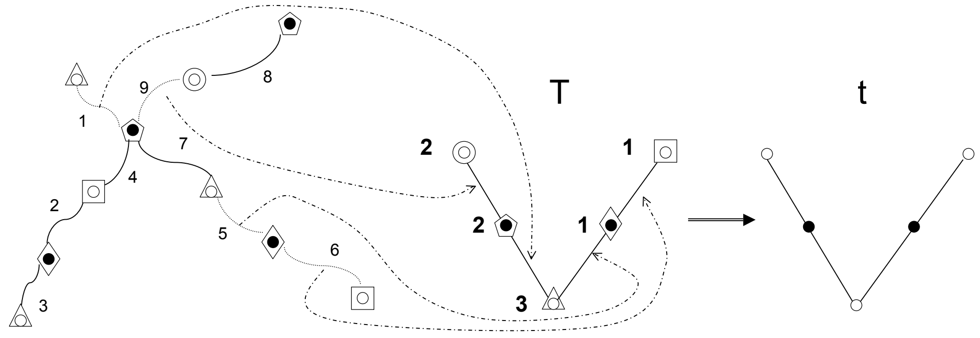


Fig. 2. Construction of the ordered bicolored tree

$m_1^{(2)} = 9 \in \pi_2^{(2)}$ *the white circle* 1 *is a descendant of the black pentagon* 1*, while the white square* 2 *is a descendant of the black rhombus* 2*. In this way we construct first the tree* T *then, removing the labels, get the tree* t *(see Figure 2).*

The partial permutation. The construction of the partial permutation consists of two main steps.

(i) Two relabeling permutations. Consider the *reverse-labeled* bicolored tree t' obtained from t by labeling its vertices as follows: The white vertices are labeled $p, p-1, \ldots, 1$ and the black vertices $q, q-1, \ldots, 1$ in order as they appear when traversing t from bottom-to-top and right-to-left. (Thus the root of t' always has label p.) Then trees T and t' give two, possibly different, labellings of t. Suppose (the (black or white) vertex v of t has label i in T and label j in t'. Then define $\pi_1^j = \pi_1^{(i)}$ if v is white and $\pi_2^j = \pi_2^{(i)}$ if v is black. Repeat this process for all vertices v of t to obtain canonical indexings $\pi_1^1, \ldots, \pi_1^p$ and $\pi_2^1, \ldots, \pi_2^q$ of the blocks of π_1 and π_2, respectively. The reader can easily see that $\pi_1^p = \pi_1^{(p)}$. Let w^i and υ^j be the strings obtained by writing the elements of π_1^i and π_2^j in increasing order. Let $w = w^1 \ldots w^p$ and $\upsilon = \upsilon^1 \ldots \upsilon^q$ be concatenations of $w^1, \ldots, w^p$ and $\upsilon^1, \ldots, \upsilon^q$ respectively. We define $\lambda \in S_N$ by letting w be the first line and $[N]$ the second line in the two-line representation. Similarly, we define $\nu \in S_N$ by letting υ be the first line and $[N]$ the second line in the two-line representation.

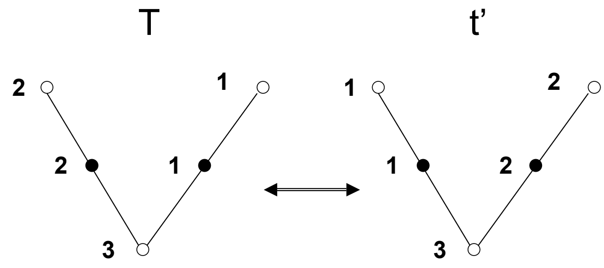


Fig. 3. Relabeling of the blocks

Example 3.3 *Let us continue Example 3.2 by constructing relabeling permutations λ and ν. Figure 3 puts the tree T and the reversed-labeled tree t' side by side to give a natural illustration of the block relabeling:*

$$\pi_1^1 = \pi_1^{(2)}, \pi_1^2 = \pi_1^{(1)}, \pi_1^3 = \pi_1^{(3)},$$
$$\pi_2^1 = \pi_2^{(2)}, \pi_2^2 = \pi_2^{(1)}.$$

The strings w^i and υ^j are given by :

$$w^1 = 89, \quad w^2 = 246, \; w^3 = 1357,$$
$$\upsilon^1 = 14789, \upsilon^2 = 2356.$$

We construct now the relabeling permutations λ and ν.

$$\lambda = \left(\begin{array}{cc|ccc|cccc} 8 & 9 & 2 & 4 & 6 & 1 & 3 & 5 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array}\right) \qquad \nu = \left(\begin{array}{ccccc|cccc} 1 & 4 & 7 & 8 & 9 & 2 & 3 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array}\right)$$

Figures 4 depicts this two new labellings on our example.

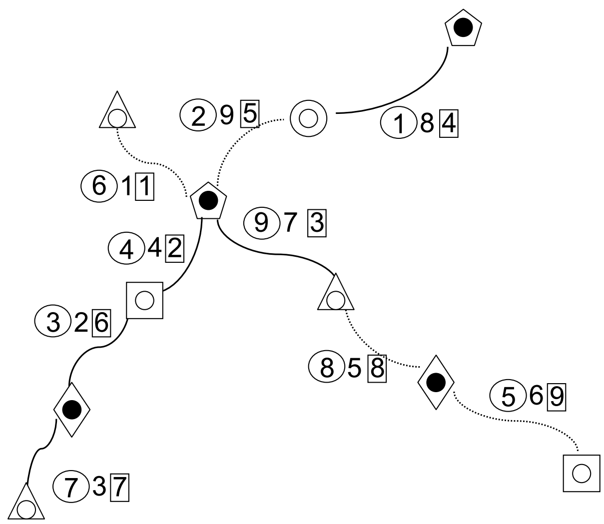


Fig. 4. Relabelings of the partitioned bicolored map

(ii) A partial permutation. We can now introduce a partial permutation that gives an insight into both the connexion between the λ and ν relabeling and the structure of the partitioned bicolored map. Let S be the subset of $[N]$ containing all the edges of the map that were not used to construct the bicolored tree. Namely :

$$S = [N] \setminus \{m_1^1, m_1^2, \ldots, m_1^{p-1}, \beta(m_2^1), \ldots, \beta(m_2^q)\}. \tag{10}$$

We define the partial permutation σ on the set $[N]$ as the partial permutation $\beta^{-1}|_S$ with its domain relabeled according to λ and its image relabeled according to ν:

$$\sigma = \nu \circ \beta^{-1} \circ \lambda^{-1} |_{\lambda(S)} . \tag{11}$$

In Lemmas 3.5 and 3.6 we show that σ is a bijection between two subsets of $N + 1 - (p + q)$ elements and that its image set is included in $[N - 1]$.

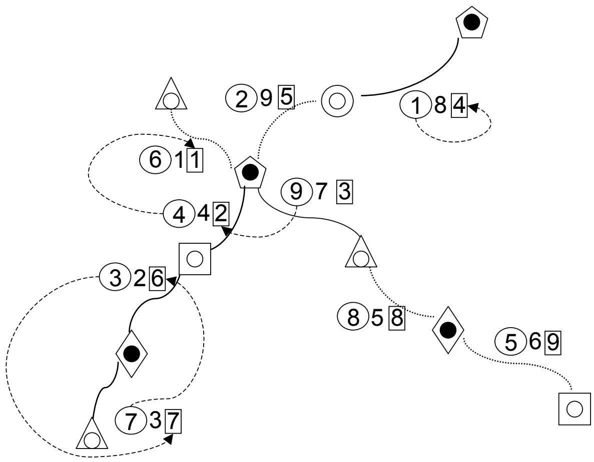


Fig. 5. Connections through σ between λ and ν relabeling

Example 3.4 *In the example previously described the set S is equal to :*

$$S = \{2, 3, 4, 7, 8\}$$

The partial permutation σ is defined by :

$$\sigma = \begin{pmatrix} 1 & 3 & 4 & 7 & 9 \\ 4 & 7 & 1 & 6 & 2 \end{pmatrix}$$

The set of vertices that were not used to construct the tree and their connections to the map through σ can be viewed on Figure 5.

Lemma 3.5 *The set S defined above has cardinality $N + 1 - (p + q)$.*

PROOF. To prove the assertion of this lemma we will show the equivalent statement

$$\{m_1^1, m_1^2, \ldots, m_1^{p-1}\} \cap \{\beta(m_2^1), \ldots, \beta(m_2^q)\} = \emptyset.$$

Assume that there exist $i \in \{1, \ldots, p - 1\}$ and $j \in \{1, \ldots, q\}$, such that

$$\beta(m_2^j) = m_1^i. \tag{12}$$

Then as the blocks of π_2 are stable under β we have $m_1^i \in \pi_2^j$ and $m_1^i \leq m_2^j$. As the blocks of π_1 are stable under α, assumption (12) also implies that $\alpha\beta(m_2^j) = \gamma_N(m_2^j) \in \pi_1^i$. Hence, $\gamma_N(m_2^j) \leq m_1^i$. Combining these two inequalities, we have $\gamma_N(m_2^j) \leq m_2^j$ which occurs only if $m_2^j = N$. In this case, $\gamma_N(m_2^j) = 1$ and $1 \in \pi_1^i$, i.e. $i = p$ which is a contradiction

□

Lemma 3.6 *The element N is not in the image of the permutation σ.*

PROOF. In view of the construction of the relabeling permutation ν, we have $N = \nu(m_2^q)$. But

$$\nu(m_2^q) = \nu \circ \beta^{-1} \circ \lambda^{-1}(\lambda(\beta(m_2^q))).$$

Thus, as $\lambda(\beta(m_2^q))$ does not belong to $\lambda(S)$, N does not belong to the image of the permutation σ.

□

4 A proof that $\Theta_{N,p,q}$ is bijective

4.1 Injectivity

Let (t, σ) be the image of some triple $(\pi_1, \pi_2, \alpha) \in CC(p, q, N)$ by $\Theta_{N,p,q}$. We show in a constructive way that (π_1, π_2, α) is actually uniquely determined by (t, σ).

First we use the tree t and the integers missing in the two lines of σ to find the number of elements in each block of π_1 and π_2, as well as the extension of σ to the whole set $[N]$. By construction of σ, the integers missing in its first line $\lambda(S)$ are

$$\lambda(m_1^1), \ldots, \lambda(m_1^{p-1}), \lambda(\beta(m_2^1)), \ldots, \lambda(\beta(m_2^q)).$$

By definition of λ we have $\lambda(m_1^1) < \ldots < \lambda(m_1^{p-1})$. Moreover, the maximality of m_1^i and the fact that λ is increasing on any block of π_1 implies $\lambda(m_1^{i-1}) < \lambda(s) < \lambda(m_1^i)$ for any $s \in S_i$, where $S_i = \{\beta(m_2^1), \ldots, \beta(m_2^q)\} \cap \pi_1^i$. Thus to recover the exact order of the elements missing in the first line of σ, we need only determine the subsets $\lambda(S_i)$ and the relative order of their elements. To this end, observe that $\lambda(S_i) = \{\lambda(\beta(m_2^{j_1}), \ldots, \beta(m_2^{j_k})\}$ where $j_1, \ldots, j_k$

are all the descendants of white vertex i in t'. Moreover, vertex j_a is to the left of vertex j_b if and only if $\beta(m_2^{j_a}) < \beta(m_2^{j_b})$, which occurs if and only if $\lambda(\beta(m_2^{j_a}) < \lambda(\beta(m_2^{j_b})$ since λ is increasing on π_1^i. Thus the order of $\lambda(S_i)$ is naturally induced by the left-right order of the descendants of the white vertex i in t'.

Consider the set $\nu \circ \beta^{-1}(S)$ in the second line of σ. The missing elements are

$$\nu(m_2^1), \ldots, \nu(m_2^q), \nu(\beta^{-1}(m_1^1)), \ldots, \nu(\beta^{-1}(m_1^{p-1})).$$

Similarly to the first line of σ, we use the structure of t', the relation between ν and t', as well as the fact that $\nu(\beta^{-1}(m_1^{i_1})) \leq \nu(\beta^{-1}(m_1^{i_2}))$ if i_1 and i_2 are descendants of the same black vertex with i_1 on the left of i_2 to order these elements. Once the order on both of the sets of missing elements is established, the missing integers can be uniquely identified. Hence, the extension $\bar{\sigma} = \nu \circ \beta^{-1} \circ \lambda^{-1}$ of the partial permutation σ to the whole set $[N]$ is uniquely determined since

$$\forall i \in [p-1], \quad \bar{\sigma}(\lambda(m_1^i)) = \nu(\beta^{-1}(m_1^i)), \tag{13}$$
$$\text{and } \forall j \in [q], \quad \bar{\sigma}(\lambda(\beta(m_2^j))) = \nu(m_2^j). \tag{14}$$

Now, the knowledge of $\lambda(m_1^1), \ldots, \lambda(m_1^{p-1})$ and $\nu(m_2^1), \ldots, \nu(m_2^q)$ allows us to determine the number of elements in each of the blocks of partitions $\lambda(\pi_1) = \lambda(\pi_1^1), \ldots, \lambda(\pi_1^p)$ and $\nu(\pi_2) = \nu(\pi_2^1), \ldots, \nu(\pi_2^q)$. Indeed, the blocks of the above partitions are intervals:

$$\lambda(\pi_1^i) = \begin{cases} [\lambda(m_1^1)], & \text{for } i = 1, \\ [\lambda(m_1^i)] \setminus [\lambda(m_1^{i-1})], & \text{for } 2 \leq i \leq p-1, \\ [N] \setminus [\lambda(m_1^{p-1})], & \text{for } i = p, \end{cases}$$
$$\nu(\pi_2^i) = \begin{cases} [\nu(m_2^1)], & \text{for } i = 1, \\ [\nu(m_2^i)] \setminus [\nu(m_2^{i-1})] & \text{for } 2 \leq i \leq q. \end{cases}$$

Hence, $\lambda(\pi_1)$ and $\nu(\pi_2)$ are uniquely determined by (t, σ). But, since π_2 is stable under β, we can use $\bar{\sigma}$ to recover $\lambda(\pi_2)$. Indeed

$$\bar{\sigma}^{-1}(\nu(\pi_2)) = \lambda \circ \beta \circ \nu^{-1}(\nu(\pi_2)) = \lambda \circ \beta(\pi_2) = \lambda(\pi_2). \tag{15}$$

In particular, since $\bar{\sigma}$ and $\nu(\pi_2)$ are uniquely determined, so is $\lambda(\pi_2)$.

Example 4.1 *We give here an illustration of the first steps of the injectivity proof. Let us suppose that we are given the parameters $N = 10, p = 3, q = 2$, the following partial permutation*

$$\sigma = \begin{pmatrix} 3 & 4 & 5 & 6 & 8 & 10 \\ 4 & 6 & 3 & 1 & 8 & 7 \end{pmatrix}$$

and the bicolored ordered tree shown in Figure 6.

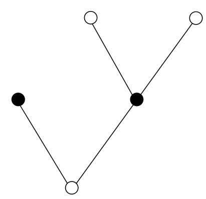


Fig. 6. A bicolored tree

Consider the set $\lambda(S)$ *in the first line of* σ*. The missing elements of* $\lambda(S)$ *are clearly* $1, 2, 7,$ *and* 9*, so we must have*

$$\{\lambda(m_1^1), \lambda(m_1^2), \lambda(\beta(m_2^1), \lambda(\beta(m_2^2)\} = \{1, 2, 7, 9\}.$$

¿From the reverse-labeling tree we conclude that

$$\lambda(m_1^1) = 1, \lambda(m_1^2) = 2, \lambda(\beta(m_2^1)) = 7, \lambda(\beta(m_2^2)) = 9.$$

Consider the set $\nu \circ \beta^{-1}(S)$ *in the second line of* σ*. The missing elements are*

$$\nu(m_2^1), \nu(m_2^2), \nu(\beta^{-1}(m_1^1)), \nu(\beta^{-1}(m_1^2)).$$

We have

$$\nu(m_2^1) = 2, \nu(\beta^{-1}(m_1^1)) = 5, \nu(\beta^{-1}(m_2^2)) = 9, \nu(m_2^2) = 10.$$

Now we can extend σ *to the permutation* $\bar{\sigma}$ *on the set* $[N]$*:*

$$\bar{\sigma} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 5 & 9 & 4 & 6 & 3 & 1 & 2 & 8 & 10 & 7 \end{pmatrix}.$$

Note, that as $\lambda(m_1^1) = 1, \lambda(m_1^2) = 2, \nu(m_2^1) = 2, \nu(m_2^2) = 10,$ *we also can identify the images of white blocks by* λ *and images of black blocks by* ν*:*

$$\lambda(\pi_1^1) = \{1\}, \quad \lambda(\pi_1^2) = \{2\}, \lambda(\pi_1^3) = \{3, 4, 5, 6, 7, 8, 9, 10\},$$
$$\nu(\pi_2^1) = \{1, 2\}, \nu(\pi_2^2) = \{3, 4, 5, 6, 7, 8, 9, 10\}.$$

Using (15) we obtain the relabeling of partition π_2:

$$\begin{aligned}
&\lambda(\pi_2^1) = \{6,7\},\\
&\lambda(\pi_2^2) = \{1,2,3,4,5,8,9,10\}.
\end{aligned} \tag{16}$$

Now let us show that λ and ν are uniquely determined as well. As $1 \in \pi_1^p$ and λ is an increasing function on each block of π_1, $\lambda(1)$ is necessarily the least element of $\lambda(\pi_1^p)$. Let then $\lambda(\pi_2^k)$ be the block of $\lambda(\pi_2)$ such that $\lambda(1) \in \lambda(\pi_2^k)$. As ν is an increasing function on each block of π_2, necessarily $\nu(1)$ is the least element of $\nu(\pi_2^k)$.

Now assume that for a given i in $[N-1]$, $\lambda(1), \ldots, \lambda(i)$ and $\nu(1), \ldots, \nu(i)$ have been determined. As π_1 is stable under α, necessarily $\beta(i)$ and $i+1 = \gamma_N(i) = \alpha \circ \beta(i)$ belong to the same block of π_1. Hence $\lambda(i+1)$ and $\lambda(\beta(i))$ belong to the same block of $\lambda(\pi_1)$. But

$$\lambda(\beta(i)) = \lambda \circ \beta \circ \nu^{-1}(\nu(i)) = \bar{\sigma}^{-1}(\nu(i)). \tag{17}$$

As a consequence, $\lambda(i+1)$ and $\bar{\sigma}^{-1}(\nu(i))$ belong to the same block of $\lambda(\pi_1)$. Finally, as λ is an increasing function on each block of π_1, $\lambda(i+1)$ is necessarily the least element of the block of $\lambda(\pi_1)$ containing $\bar{\sigma}^{-1}(\nu(i))$ that has not been used yet to identify $\lambda(1), \ldots, \lambda(i)$.

Let $\lambda(\pi_2^l)$ be the block of $\lambda(\pi_2)$ containing $\lambda(i+1)$. Since ν is an increasing function on each block of π_2, $\nu(i+1)$ is uniquely determined as being the least element of the block $\nu(\pi_2^l)$ that has not already been used to identify $\nu(1), \ldots, \nu(i)$. By iterating the above procedure for all the integers in $[N-1]$ we see that λ and ν are uniquely determined.

To end this proof, we remark that

$$\begin{aligned}
&\pi_1 = \lambda^{-1}(\lambda(\pi_1)),\\
&\pi_2 = \nu^{-1}(\nu(\pi_2)),\\
\text{and}\quad &\alpha = \gamma_N \circ \beta^{-1} = \gamma_N \circ \nu^{-1} \circ \bar{\sigma} \circ \lambda.
\end{aligned}$$

As a result, at most one triple (π_1, π_2, α) can be associated with (t, σ) through $\Theta_{N,p,q}$. Moreover, if such a triple exists, it can be computed using the description of $\Theta_{N,p,q}^{-1}$ given by the above proof.

Example 4.2 *We apply the iterative reconstruction of λ and ν to the previous example. A table of three lines and N columns will be used to sum up the available information on λ and ν in each step of the reconstruction: the first line is given by $[N]$, while the second and third lines represent the relabellings* *by* λ *and* ν*. We initialize the procedure by putting* $3 = \min(\lambda(\pi_1^3))$ *at the first position of the line for* λ*:*

$$
\begin{array}{cl}
\gamma_N & : 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10 \\
\lambda & : 3 * * * * * * * * \ * \\
\nu & : * * * * * * * * * \ *
\end{array}
$$

Now, looking at (16) we establish that the element 3 *belongs to* $\lambda(\pi_2^2)$*. As the least element of* $\nu(\pi_2^2)$ *is* 3*, we put* 3 *in the first position of the third line of our table:*

$$
\begin{array}{cl}
\gamma_N & : 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10 \\
\lambda & : 3 * * * * * * * * \ * \\
\nu & : 3 * * * * * * * * \ *
\end{array}
$$

To establish the next white block we take the image of the last discovered element $\nu(1)$ *by* $\bar{\sigma}^{-1}$*:*

$$\bar{\sigma}^{-1}(\nu(1)) = \bar{\sigma}^{-1}(3) = 5.$$

Thus $\bar{\sigma}^{-1}(\nu(1))$ *belongs to* $\lambda(\pi_1^3)$*. We then deduce that* $\lambda(2)$ *is the least element of* $\lambda(\pi_1^3)$ *which has not been met yet, i.e* 4*. We write* 4 *at the second position of the line for* λ*:*

$$
\begin{array}{cl}
\gamma_N & : 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10 \\
\lambda & : 3\ 4 * * * * * * * \ * \\
\nu & : 3 * * * * * * * * \ *
\end{array}
$$

We iterate the process until λ *and* ν *are fully reconstructed :*

$$
\begin{array}{cl}
\gamma_N & : 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ \ 9\ \ 10 \\
\lambda & : 3\ 4\ 5\ 1\ 6\ 7\ 8\ 9\ 10\ \ 2 \\
\nu & : 3\ 4\ 5\ 6\ 1\ 2\ 7\ 8\ \ 9\ \ 10
\end{array}
$$

Once λ *and* ν *are known, we have reconstructed the partitioned map:*

$$
\begin{aligned}
\pi_1 &= \{\{4\}\{10\}\{1,2,3,5,6,7,8,9\}\}\,, \\
\pi_2 &= \{\{5,6\}\{1,2,3,4,7,8,9,10\}\}\,, \\
\alpha &= (13256798)(4)(10).
\end{aligned}
$$

The bicolored map represented by this permutation is the unicellular map of genus 2 shown in Figure 7.

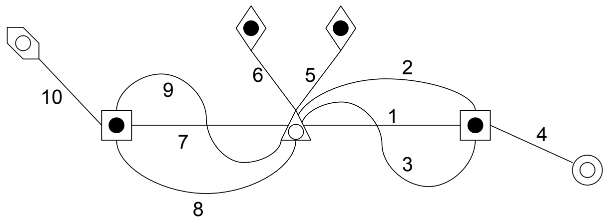


Fig. 7. The partitioned bicolored map once reconstructed

4.2 Surjectivity

Let us now proceed by showing that $\Theta_{N,p,q}$ is a surjection. Clearly, up to the reconstruction of λ and ν the first steps of the procedure described in the previous section can be applied to any couple (t, σ) belonging to $BT(p,q) \times PP(N, N-1, N-1-(p+q))$. Namely we can define the extension $\bar{\sigma}$ of σ to the whole set $[N]$ as well as the partitions $\lambda(\pi_1)$, $\lambda(\pi_2)$ and $\nu(\pi_2)$. Then we use the next lemma to show that the reconstruction of λ and ν can also always be successfully completed.

Lemma 4.3 *Given any couple $(t, \sigma) \in BT(p,q) \times PP(N, N-1, N-1-(p+q))$, the iterative procedure for the reconstruction of λ and ν can always be performed and gives valid output in any case.*

PROOF. First, we notice there are only two ways the procedure can be prevented. Either for a given i in $[N-1]$, $\bar{\sigma}^{-1}(\nu(i))$ belongs to a block of $\lambda(\pi_1)$ that has all its elements already used for the construction of $\lambda(1), \ldots, \lambda(i)$ so that we cannot define $\lambda(i+1)$; or $\lambda(i+1)$ belongs to a block of $\lambda(\pi_2)$ such that the corresponding block of $\nu(\pi_2)$ has all its elements already used for the construction of $\nu(1), \ldots, \nu(i)$ and we are not able to define $\nu(i+1)$. We show by induction that neither of these situations can occur.

Assume that we have already successfully iterated the procedure up to $i \leq N-1$. Also assume that we cannot define $\lambda(i+1)$ for the reason stated above. Let $\lambda(\pi_1^k)$ be the block containing $\bar{\sigma}^{-1}(\nu(i))$. Then:

(i) If $\lambda(\pi_1^k)$ does not contain $\lambda(1)$, then $|\pi_1^k| + 1$ different integers, including $\nu(i)$, that have been used for the construction of ν have their image under $\bar{\sigma}^{-1}$ in $\lambda(\pi_1^k)$. This contradicts the fact that $\bar{\sigma}^{-1}$ is a bijection.
(ii) If $\lambda(1)$ belongs to $\lambda(\pi_1^k)$ (thus $k = p$), we only know that $|\pi_1^k|$ different integers have their image by $\bar{\sigma}^{-1}$ in $\lambda(\pi_1^p)$. However, according to our definition of $\bar{\sigma}$, $\lambda(\pi_1)$ and $\lambda(\pi_2)$, if the white vertex in t corresponding to a given block

π_1^a is the descendant of the black vertex associated with π_2^b then

$$\lambda(m_1^a) \in \lambda(\pi_2^b). \tag{18}$$

In other words we cannot have used all the elements of $\nu(\pi_2^b)$ for the construction of ν until the maximum element of $\lambda(\pi_1^a)$ (and therefore all the elements of $\lambda(\pi_1^a)$) has been used for the construction of λ. In a similar way, if the black vertex associated to π_2^c is the direct descendant of the white one corresponding to π_1^d, we have

$$\bar{\sigma}^{-1}(\nu(m_2^c)) \in \lambda(\pi_1^d). \tag{19}$$

Thus all the elements of $\nu(\pi_2^c)$ must be used for the reconstruction of ν before all the elements of $\lambda(\pi_1^d)$ are used for the reconstruction of λ. To summarize, all the elements of a block associated to a vertex x (either black or white) are not used for the construction of λ and ν until all the elements of the blocks associated with vertices that are descendant of x are used for the same construction. As π_1^p is associated with the root of t, if all the elements of $\lambda(\pi_1^p)$ have already been used for the construction of λ and ν, it means that all the elements of all the other blocks of $\lambda(\pi_1)$ and $\nu(\pi_2)$ have been already used as well. The reconstruction is hence completed and $i = N$. That is a contradiction with our assumption $i \leq N - 1$.

Once $\lambda(i+1)$ is found, we notice that $\nu(i+1)$ can always be defined. Indeed, if $\lambda(i+1)$ belongs to a block $\lambda(\pi_2^l)$ such that all the elements of $\nu(\pi_2^l)$ have been already used to construct ν, it would mean that $|\pi_2^l|+1$ different integers belong to $\lambda(\pi_2^l)$, which is a contradiction. Our induction is completed by the obvious remark that $\lambda(1)$ and $\nu(1)$ can always be defined.

□

For the final step of this proof we need to show that once λ and ν are constructed the permutation α defined by

$$\alpha = \gamma_N \circ \nu^{-1} \circ \bar{\sigma} \circ \lambda, \tag{20}$$

satisfies the two following conditions:

$$\alpha(\pi_1) = \pi_1 \tag{21}$$

and

$$\alpha^{-1}\gamma_N(\pi_2) = \pi_2. \tag{22}$$

Condition (22) comes from the fact that we have defined

$$\lambda(\pi_2) = \bar{\sigma}^{-1}(\pi_2), \tag{23}$$

so that,

$$\pi_2 = \lambda^{-1} \circ \bar{\sigma}^{-1} \circ \nu \circ \gamma_N^{-1} \circ \gamma_N(\pi_2) \tag{24}$$

and as a consequence

$$\pi_2 = \alpha^{-1} \circ \gamma_N(\pi_2). \tag{25}$$

Condition (21) can be shown using the fact that for all i in $[N]$, $\lambda(i)$ and $\bar{\sigma}^{-1} \circ \nu \circ \gamma_N^{-1}(i)$ belong to the same block of $\lambda(\pi_1)$. Hence, $\lambda^{-1} \circ \bar{\sigma}^{-1} \circ \nu \circ \gamma_N^{-1}(i)$ and i belong to the same block of π_1. Finally, the blocks of π_1 are stable under α^{-1} and therefore also stable under α.

This concludes the proof that $\Theta_{N,p,q}$ is a bijection.

5 The equivalence between Formula (1) and Adrianov's

In this section we show that Adrianov's statement in [1] is equivalent to Formula (1). Upon making explicit the expansion of the Gauss hypergeometric function used by Adrianov in his statement, his formula reads:

$$\sum_{m,n\geq 1} \frac{B(m,n,N)}{(N-1)!}\, y^{n-1} z^{m-1} = \sum_{k\geq 0} \frac{\binom{y+k}{k}\binom{z+k}{k}\binom{y+z}{N-1-2k}}{k+1} \tag{26}$$

Lemma 5.1 below allows us to rewrite this as

$$\sum_{m,n\geq 1} \frac{B(m,n,N)}{(N-1)!} y^{n-1} z^{m-1} = \sum_{\substack{l,l',k\geq 0\\ 2k+l+l'=N-1}} \frac{\binom{k+l}{k}\binom{k+l'}{k}\binom{y+k}{k+l}\binom{z+k}{k+l'}}{k+1} \tag{27}$$

Now multiplying both sides by yz and expanding the last two binominal coefficients with Vandermonde's convolution formula yields

$$\sum_{m,n\geq 1} \frac{B(m,n,N)}{(N-1)!} y^n z^m = \sum_{\substack{p,q\geq 1,\, l,l',k\geq 0\\ 2k+l+l'=N-1}} \frac{\binom{k+l}{k}\binom{k+1'}{k}\binom{k+1}{p-l}\binom{k+1}{q-l'}}{(k+1)(p-1)!(q-1)!} (y)_{(p)}(z)_{(q)} \tag{28}$$

In view of Formula (4) Adrianov's statement is thus equivalent to saying that

$$\frac{C(p,q,N)}{(N-1)!} = \sum_{\substack{l,l',k\geq 0\\ 2k+l+l'=N-1}} \frac{\binom{k+l}{k}\binom{k+l'}{k}\binom{k+1}{p-l}\binom{k+1}{q-l'}}{(k+1)(p-1)!(q-1)!} \tag{29}$$

or, upon rearranging binomial coefficients,

$$\frac{C(p,q,N)}{(N-1)!} = \sum_{\substack{k,l,l'\geq 0\\ 2k+l+l'=N-1}} \frac{(k+1)\binom{p}{l}\binom{q}{l'}\binom{k+l}{p-1}\binom{k+l'}{q-1}}{p!q!}. \tag{30}$$

which, in view of Lemma 5.2 below, is equivalent to the Formula (7) we have proved combinatorially.

Lemma 5.1

$$\binom{y+k}{k}\binom{z+k}{k}\binom{y+z}{N-1-2k} = \sum_{\substack{l,l'\geq 0\\ l+l'=N-1-2k}} \binom{k+l}{k}\binom{k+l'}{k}\binom{y+k}{k+l}\binom{z+k}{k+l'}.$$

Proof. The binomial coefficients on the right hand side of the above equation can be interpreted as multiple derivatives of polynomial functions. Namely,

$$\binom{y+k}{k}\binom{z+k}{k}\binom{y+z}{N-1-2k} = \left.\frac{\left\{\left[(1+u)^{y+k}\right]^{(k)}\left[(1+u)^{z+k}\right]^{(k)}\right\}^{(N-1-2k)}}{k!^2(N-1-2k)!}\right|_{[u=0]}$$

Expanding the formula for the $(N-1-2k)$th derivative of a product gives

$$\binom{y+k}{k}\binom{z+k}{k}\binom{y+z}{N-1-2k} = \sum_{\substack{l,l'\geq 0\\ l+l'=N-1-2k}} \binom{N-1-2k}{l}\left.\frac{\left[(1+u)^{y+k}\right]^{(k+l)}\left[(1+u)^{z+k}\right]^{(k+l')}}{k!^2(N-1-2k)!}\right|_{[u=0]}.$$

and the formula follows upon expanding and rearranging factorials. $\square$

Lemma 5.2

$$\sum_{\substack{k,l,l'\geq 0\\ 2k+l+l'=N-1}} (k+1)\binom{p}{l}\binom{q}{l'}\binom{k+l}{p-1}\binom{k+l'}{q-1} = N\binom{N-1}{p-1,q-1}$$

Proof. As pointed out by an anonymous referee, this formula can be proved through a direct combinatorial interpretation. However since we were unable to use this interpretation to devise a combinatorial proof of Adrianov's formula, we content with a simple proof based again on Vandermonde's convolution.

Consider the left hand side of the equation from the statement of the lemma and set $j = k+l$, so that $k+l' = N-1-j$ and $l' = N-1-l-2(j-l) = N-1-2j+l$: it rewrites as

$$\sum_{j\geq l\geq 0} (j-l+1)\binom{p}{l}\binom{q}{N-1-2j+l}\binom{j}{p-1}\binom{N-1-j}{q-1}$$

Consider the summand for $l > j \geq 0$. If $j > N-q$ the last binomial is 0. Otherwise $N-q-j \geq 0$ and $l-j > 0$ so that $N-2j+l > q$ and the second binomial is 0 or $(j-l+1)=0$. In any case, these terms do not contribute so that the summation can be reset on all $j,l \geq 0$ instead of $j \geq l \geq 0$.

Then, using Vandermonde's convolution, the terms $\binom{p}{l}\binom{q}{K+l}$ and $l\binom{p}{l}\binom{q}{K+l}$ can be summed over l to $\binom{p+q}{p+K}$ and $p\binom{p+q-1}{p+K}$ respectively (with $K = N-1-2j$). Hence the sum simplifies to

$$\sum_{j\geq 0}\left((j+1)\binom{p+q}{N-1-2j+p} - p\binom{p+q-1}{N-1-2j+p}\right)\binom{j}{p-1}\binom{N-1-j}{q-1}$$

or, taking out the factors depending only on p and q and rearranging binomials,

$$\frac{(p+q-1)!}{(p-1)!(q-1)!}\sum_{j\geq 0}\left((p+q)\binom{j+1}{N-j-q} - p\binom{j}{N-j-q}\right)\frac{\binom{N-j}{j-p+1}}{N-j}.$$

Using a generalized Vandermonde convolution formula [6, Formula 5.62] to sum $\binom{j+1}{N-j-q}\binom{N-j}{j-p+1}\frac{1}{N-j}$ and $\binom{j}{N-j-q}\binom{N-j}{j-p+1}\frac{1}{N-j}$ over j into $\binom{N+1}{p+q}\frac{1}{N+1-p}$ and $\binom{N}{p+q-1}\frac{1}{N+1-p}$ respectively, the result is obtained.

$\square$

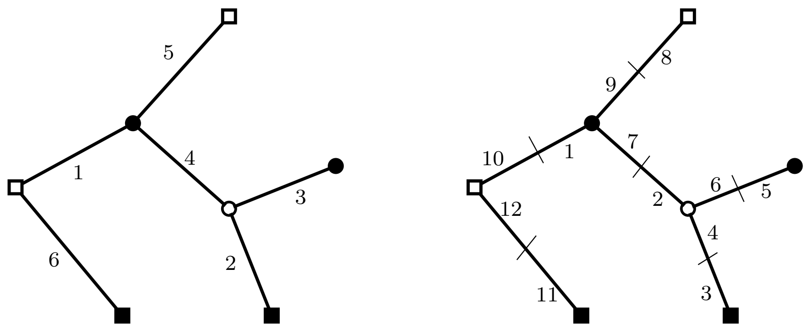
6 Eulerian tours and the graphical interpretation of the bijection

In this section we rephrase Goulden and Nica's bijection as well as ours in terms of Eulerian tours on directed graphs.

In order to state Goulden and Nica's bijection we need to give the combinatorial definition of unicellular (not necessarily bicolored) maps. Recall that γ_{2N} denotes the permutation $\gamma_{2N} = (1\,2\,\ldots\,2N)$. A *unicellular map* M with N edges is formed of a fix point free involution α and a permutation β of Σ_{2N} such that $\alpha\beta = \gamma_{2N}$.

As such, unicellular maps with N edges appear to be a special kind of unicellular bicolored maps with $2N$ edges (with the extra requirement that α be a fix-point free involution). For our purpose it will be more useful to observe that, as their name suggests, unicellular bicolored maps with N edges are in turn a special kind of unicellular maps with N edges. This is easy to understand if we represent unicellular maps as (half-edge-)labeled ribbon graphs as follows:

- The elements of $[2N]$ index the *half-edges* of M.
- The involution α indicates opposite half-edges: two half-edges i and j in $[2N]$ form an edge if $\alpha(i) = j$.



$\alpha = (1\,6)(2\,3\,4)(5)$ $\qquad$ $\bar{\alpha} = (1\,10)(2\,7)(3\,4)(5\,6)(8\,9)(11\,12)$

$\beta = (1\,4\,5)(2)(3)(6)$ $\qquad$ $\bar{\beta} = (1\,7\,9)(2\,4\,6)(3)(5)(8)(10\,11)(12)$

Fig. 8. A unicellular bicolored map and the associated unicellular map.

- The cycles of $\beta = \alpha\gamma_{2N}$ describe the cyclic organization of half-edges around each vertex: if $\mathcal{B} = (b_1, \ldots, b_k)$ is a cycle of β then for all $i = 1, \ldots, k$, we let $v(b_i)$ represent the vertex to which b_i is incident.

Again the condition that $\alpha\beta = \gamma_N$ implies that a walking ant starting on the right hand side of the half-edge with label 1 will visit all right hand sides of half-edges in increasing order.

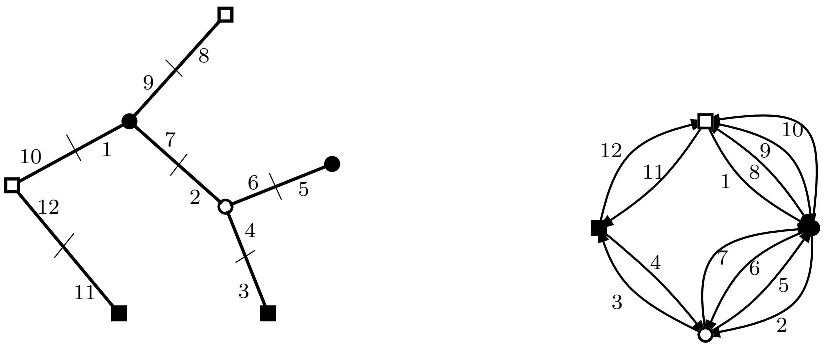
Then, given a unicellular bicolored map represented as a ribbon graph, it can be viewed as a unicellular map by simply forgetting the vertex colors and relabelling the half-edge in increasing order along a visit of the ant. Formally, given a unicellular bicolored map (α, β) with $\alpha\beta = \gamma_N$, the associated unicellular map $(\bar{\alpha}, \bar{\beta})$ is given by the relations $\bar{\alpha}(2i-1) = 2\beta^{-1}(i)$, $\bar{\alpha}(2i) = 2\beta(i)-1$ and $\bar{\beta} = \bar{\alpha}\gamma_{2N}$. This correspondence is illustrated by Figure 8.

A *directed graph* G is formed of a set of vertices V and a set of arcs A with two mappings, namely or : $A \to V$ giving the origin of each arc, and ex : $A \to V$ giving its extremity. Two arcs a_1 and a_2 are opposite if $\mathrm{ex}(a_1) = \mathrm{or}(a_2)$ and $\mathrm{or}(a_1) = \mathrm{ex}(a_2)$. A *pairing of opposite arcs* is a fix-point free involution ϕ on the set of arcs such that $\mathrm{or}(a) = \mathrm{ex}(\phi(a))$ for all a. A *tour* in G is a sequence $(a_1, \ldots, a_k)$ of arcs such that $\mathrm{ex}(a_i) = \mathrm{or}(a_{i+1})$ for all i and $\mathrm{ex}(a_k) = \mathrm{or}(a_1)$. Equivalently a tour $\tau = (a_1, \ldots, a_k)$ can be given by its origin $\mathrm{or}(a_1) = \mathrm{ex}(a_k)$, and a successor function, $\tau(a_i) = a_{i+1}$. An *Eulerian tour* on G is a tour that visits exactly once each arc of G.

There is a very close relation between unicellular partitioned maps and Eulerian tour, as illustrated by Figure 9 and the following lemma.

Lemma 6.1 *There is a bijection between*

- *unicellular maps with N edges and with vertices partitioned into k blocks,*
- *and triples (G, τ, ϕ) where G is a directed graph with $2N$ arcs and k vertices,*



$\alpha = (1\,10)(2\,7)(3\,4)(5\,6)(8\,9)(11\,12)$ $\qquad$ $\phi = (1\,10)(2\,7)(3\,4)(5\,6)(8\,9)(11\,12)$

$\beta = (1\,7\,9)(2\,4\,6)(3)(5)(8)(10\,11)(12)$

Fig. 9. A unicellular map and the corresponding triple (G, τ, ϕ).

τ is an Eulerian tour on G, and ϕ is a pairing of opposite arcs.

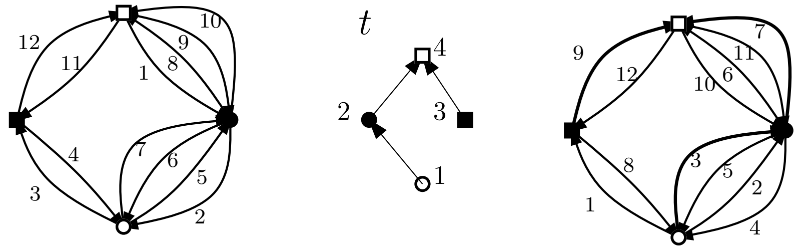
Moreover the image of the set of unicellular partitioned bicolored maps is the set of triples (G, τ, ϕ) such that G is a bicolored graph.

Proof. Let M be a partitioned map, described by a pair (α, π), with α a fix-point free involution on $[2N]$, $\beta = \alpha\gamma_{2N}$, and $\pi = (\pi^{(1)}, \ldots, \pi^{(k)})$ a partition of the vertices of M. The image of M under the bijection is the triple (G, τ, ϕ) obtained as follows:

- The graph G has one vertex $\hat{\pi}^{(i)}$ for each block $\pi^{(i)}$ of vertices of M.
- The graph G has an arc $\hat{a}$ with $\mathrm{or}(\hat{a}) = \hat{\pi}^{(i)}$ and $\mathrm{ex}(\hat{a}) = \hat{\pi}^{(j)}$ for each half-edge a of M with $v(a) \in \pi^{(j)}$ and $v(\alpha(a)) \in \pi^{(i)}$ (observe that arcs and half-edges point in opposite directions).
- The pairing ϕ associates $\hat{a}$ and $\hat{b}$ if $b = \alpha(a)$.
- The tour is $\tau = (\hat{1}, \hat{2}, \ldots, \hat{2N})$. In other terms, the tour corresponds to the ant walk around M. In particular $\hat{b} = \tau(\hat{a})$ if $b = \gamma_{2N}(a)$.

Conversely consider a triple (G, τ, ϕ) where G is a graph with $2N$ arcs and k vertices, $\tau = (a_1, \ldots, a_{2N})$ is an Eulerian tour on G and ϕ is a pairing of opposite arcs of G. A partitioned map $M = (\alpha, \pi)$ is constructed as follows:

- The half-edge set is $\{1, \ldots, 2N\}$ with each half-arc i corresponding to the arc a_i of the tour.
- The fix-point free involution α corresponds to the pairing ϕ: $\alpha(i) = j$ if $\phi(a_i) = a_j$.
- The vertices of M are described by cycles of $\beta = \alpha\gamma_{2N}$.
- For all i, the two half-edges i and $j = \beta(i)$ correspond to arcs a_i and a_j that have the same extremity in G. Indeed $j = \alpha\gamma_{2N}(i) = \alpha(i + 1)$, so that a_{i+1} and a_j are opposite, and $\mathrm{ex}(a_i) = \mathrm{or}(a_{i+1}) = \mathrm{ex}(a_j)$.
- The partition π has a part π_v for each vertex v of G: π_v contains all the vertices of M that are described by cycles of half-edges corresponding to



$\phi = (1\,10)(2\,7)(3\,4)(5\,6)(8\,9)(11\,12)$ $\qquad$ $\psi = (1\,8)(4\,5)(6\,11)$

Fig. 10. The unicolor construction.

arcs ending in v.

These two constructions are clearly inverse one of the other and they transport bipartition. $\square$

Lemma 6.1 allows us to give a graphical description of Goulden and Nica's bijection and of ours in terms of Eulerian tours. This description recycles an ingredient of the BEST theorem, namely the *last passage tree* associated to an Eulerian tour $\tau = (a_1, \ldots, a_{2N})$ on a directed graph G. This is a tree constructed as follows:

- The vertices of the last passage tree are the vertices of G, and its root is the origin of a_1.
- To each non root vertex v is associated a last passage arc $\ell(v)$ of G: this arc is the last arc with origin v in the tour τ. Then the father of the vertex v in the last passage tree is defined to be the vertex $\mathrm{ex}(\ell(v))$ at which the arc $\ell(v)$ is pointing.

More precisely, to a triple (G, τ, ϕ) we shall associate a *last passage ordered tree* t, which is an ordered version of the last passage tree in which the siblings of a vertex v are ordered according to the order in which the opposites under ϕ of last visiting arcs are encountered on the tour τ:

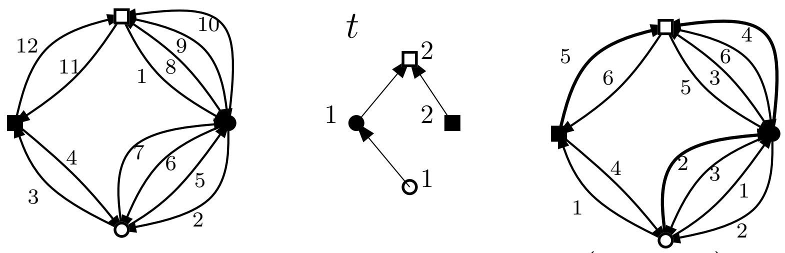
- Given two siblings u and u' of v, $u < u'$ if $\phi(\ell(u))$ appears before $\phi(\ell(u'))$ in the tour τ.

Observe that if G is bicolored then so is the tree t.

The last passage ordered tree is the ordered tree constructed by both Goulden and Nica's and our bijections. The rest of the two constructions slightly differ.

Proposition 6.2 *Triples (G, τ, ϕ) as in Lemma 6.1 are in bijection with pairs (t, ψ) where t is an ordered tree with k vertices and ψ is a involution on $\{1, \ldots, 2N\}$ with $2k$ fix points.*

Proof. The tree t is the last passage ordered tree associated with τ. Let us



$\phi = (1\,10)(2\,7)(3\,4)(5\,6)(8\,9)(11\,12)$ $\qquad$ $\xi = \begin{pmatrix} 1 & 3 & 6 \\ 4 & 2 & 3 \end{pmatrix}$

Fig. 11. The bicolored construction.

order the vertices of G by left-to-right breadth first search in t. Then define a canonical numbering of arcs of G as the order in t on their origin refined by their order of appearance in the tour τ. The involution ψ associates the canonical label i of an arc a and the canonical label j of an arc b if a and b are opposite arcs of G (that is, $b = \phi(a)$) and neither a nor b is a last passage arc.

This construction, illustrated by Figure 10, is exactly the bijection defined by Goulden and Nica in [4]. $\square$

The analogous result for bicolored maps reads as follows.

Proposition 6.3 *Triples (G, τ, ϕ) as in Lemma 6.1 with G bicolored with p white and q black blocks are in bijection with pairs (t, ψ) where t is an ordered bicolored tree with p white and q black vertices and ξ is a partial permutation in $PP(N, N-1, N+1-(p+q))$.*

Proof. The construction is similar to the previous one: the tree is the last passage ordered tree associated with τ, and its black and white vertices are ordered separately by left-to-right reverse breadth first search. Then arcs with a black origin are canonically numbered using the order on black vertices refined by their order of appearance in the tour, while arcs with a white origin are canonically numbered using the order on white vertices, again refined by their order of appearance in the tour. Finally the partial permutation ξ maps the canonical label i of an arc a with white origin onto the label j of an arc b with black origin if $b = \phi(a)$, and neither a nor b is a last passage arc.

This construction, illustrated by Figure 11, is exactly the bijection of the previous pages. (In particular one can check that the ordering of siblings of vertices in t is consistent between the two descriptions.) $\square$

In this reformulation it appears that the two constructions satisfy the following relation. To any unicellular bicolored partitioned map with N edges is associated an ordered tree τ, a partial involution ψ (upon forgetting the colors and applying Goulden and Nica's construction), and a partial permutation ξ

(applying our construction): then ψ maps even elements of $\{1,\dots,2N\}$ onto odd ones, and, for all $i \in \{1,\dots,N\}$ such that the arc $2i-1$ and its opposite are not last passage arcs, $2\xi(i) = \psi(2i-1)$.

It should be observed also that the use of breadth first search is irrelevant for the construction: any canonical order on the vertices of t such that each vertex is visited after its siblings would do the job. On the contrary, the order between siblings (with respect to the order of the opposite to the last passage arcs in the tour) is crucial to recover properly the missing elements.

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