Simply-typed $\beta\eta$ equivalence with sums and 0

Gabriel Scherer, Didier Rémy

Gallium – INRIA

February 26, 2016

Plan

1. $\beta\eta$-equivalence, β-normal forms
2. Focused normal forms
3. Simple equivalence algorithm
4. Past work: Obstacles to local approaches
5. Future work: Obstacles to non-focused algorithm

Simply-typed lambda-calculus

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.\, t : A \to B} \qquad \frac{\Gamma \vdash t : A \to B \qquad \Gamma \vdash u : A}{\Gamma \vdash t\ u : B}$$

$$\frac{\begin{array}{c}\Gamma \vdash t_1 : A_1 \\ \Gamma \vdash t_2 : A_2\end{array}}{\Gamma \vdash (t_1, t_2) : A_1 \times A_2} \qquad \frac{\Gamma \vdash t : A_1 \times A_2}{\Gamma \vdash \pi_i\ t : A_i}$$

$$\frac{\Gamma \vdash t : A_i}{\Gamma \vdash \sigma_i\ t : A_1 + A_2} \qquad \frac{\Gamma \vdash t : A_1 + A_2 \qquad \begin{array}{c}\Gamma, x_1 : A_1 \vdash u_1 : C \\ \Gamma, x_2 : A_2 \vdash u_2 : C\end{array}}{\Gamma \vdash \texttt{match}\ t\ \texttt{with} \left| \begin{array}{l}\sigma_1\ x_1 \to u_1 \\ \sigma_2\ x_2 \to u_2\end{array}\right. : C}$$

$$\frac{}{\Gamma \vdash () : 1} \qquad \frac{(x : A) \in \Gamma}{\Gamma \vdash x : A} \qquad \frac{\Gamma \vdash t : 0}{\Gamma \vdash \texttt{absurd}(t) : A}$$

$\beta\eta$-equivalence

$$(\lambda x.\, t)\; u \;\triangleright_\beta\; t[u/x] \qquad\qquad \pi_i\;(t_1, t_2)\;\triangleright_\beta\; t_i$$

$$\texttt{match}\ \sigma_i\ t\ \texttt{with}\ \left|\ \begin{array}{l} \sigma_1\ x_1 \to u_1 \\ \sigma_2\ x_2 \to u_2 \end{array}\right. \;\triangleright_\beta\; u_i[t/x_i]$$

$$\frac{\Gamma \vdash t : A \to B}{t \triangleright_\eta \lambda x.\,(t\ x)} \qquad \frac{\Gamma \vdash t : A_1 \times A_2}{t \triangleright_\eta (\pi_1\ t, \pi_2\ t)} \qquad \frac{\Gamma \vdash t : 1}{t \triangleright_\eta ()}$$

$$\frac{\Gamma \vdash t : A_1 + A_2 \qquad \Gamma \vdash D[t] : C}{D[t] \triangleright_\eta \begin{array}{l}\texttt{match}\ t\ \texttt{with} \\ \left|\ \begin{array}{l} \sigma_1\ x_1 \to D[\sigma_1\ x_1] \\ \sigma_2\ x_2 \to D[\sigma_2\ x_2] \end{array}\right.\end{array}} \qquad \frac{\Gamma \vdash t : 0 \qquad \Gamma \vdash D[t] : C}{D[t] \triangleright_\eta \texttt{absurd}(t)}$$

(Derived equalities:)

$\beta\eta$-equivalence

$$(\lambda x.\, t)\ u \ \triangleright_\beta \ t[u/x] \qquad\qquad \pi_i\ (t_1, t_2)\ \triangleright_\beta\ t_i$$

$$\texttt{match}\ \sigma_i\ t\ \texttt{with}\ \left|\begin{array}{l}\sigma_1\ x_1 \to u_1\\ \sigma_2\ x_2 \to u_2\end{array}\right. \ \triangleright_\beta\ u_i[t/x_i]$$

$$\frac{\Gamma \vdash t : A \to B}{t \triangleright_\eta \lambda x.\,(t\ x)} \qquad \frac{\Gamma \vdash t : A_1 \times A_2}{t \triangleright_\eta (\pi_1\ t, \pi_2\ t)} \qquad \frac{\Gamma \vdash t : 1}{t \triangleright_\eta ()}$$

$$\frac{\Gamma \vdash t : A_1 + A_2 \qquad \Gamma \vdash D[t] : C}{D[t] \triangleright_\eta \begin{array}{l}\texttt{match}\ t\ \texttt{with}\\ \left|\begin{array}{l}\sigma_1\ x_1 \to D[\sigma_1\ x_1]\\ \sigma_2\ x_2 \to D[\sigma_2\ x_2]\end{array}\right.\end{array}} \qquad \frac{\Gamma \vdash t : 0 \qquad \Gamma \vdash D[t] : C}{D[t] \triangleright_\eta \texttt{absurd}(t)}$$

(Derived equalities:)

$$\frac{}{\Gamma \vdash t \approx_\eta u : 1}$$

$\beta\eta$-equivalence

$$(\lambda x.\, t)\; u \;\rhd_\beta\; t[u/x] \qquad\qquad \pi_i\;(t_1, t_2)\;\rhd_\beta\; t_i$$

$$\texttt{match}\;\sigma_i\;t\;\texttt{with}\;\left|\;\begin{array}{l}\sigma_1\;x_1 \to u_1\\ \sigma_2\;x_2 \to u_2\end{array}\right. \;\rhd_\beta\; u_i[t/x_i]$$

$$\frac{\Gamma \vdash t : A \to B}{t \rhd_\eta \lambda x.\,(t\;x)} \qquad \frac{\Gamma \vdash t : A_1 \times A_2}{t \rhd_\eta (\pi_1\;t, \pi_2\;t)} \qquad \frac{\Gamma \vdash t : 1}{t \rhd_\eta ()}$$

$$\frac{\Gamma \vdash t : A_1 + A_2 \qquad \Gamma \vdash D\,[t] : C}{D\,[t] \rhd_\eta \begin{array}{l}\texttt{match}\;t\;\texttt{with}\\ \left|\;\begin{array}{l}\sigma_1\;x_1 \to D\,[\sigma_1\;x_1]\\ \sigma_2\;x_2 \to D\,[\sigma_2\;x_2]\end{array}\right.\end{array}} \qquad \frac{\Gamma \vdash t : 0 \qquad \Gamma \vdash D\,[t] : C}{D\,[t] \rhd_\eta \texttt{absurd}(t)}$$

(Derived equalities:)

$$\frac{}{\Gamma \vdash t \approx_\eta u : 1} \qquad \frac{\Gamma \vdash t : 0}{\Gamma \vdash u_1 \approx_\eta u_2 : A}$$

β-normal forms

Easy in the **negative** fragment $(\to, \times, 1)$. Values and neutrals.

$$v ::= \text{values} \mid \lambda x.\, v \mid (v_1, v_2) \mid n \qquad n ::= \text{neutrals} \mid x \mid \pi_i\ n \mid n\ v$$

Problem: no clear way to add **positives** $(+, 0)$.

β-normal forms

Easy in the **negative** fragment $(\to, \times, 1)$. Values and neutrals.

$$v ::= \text{values} \mid \lambda x.\, v \mid (v_1, v_2) \mid n \qquad n ::= \text{neutrals} \mid x \mid \pi_i\ n \mid n\ v$$

Problem: no clear way to add **positives** $(+, 0)$.

$$v ::= \text{values} \mid \lambda x.\, v \mid (v_1, v_2) \mid n \mid ?\ \sigma_i\ v$$

$$n ::= \text{neutrals} \mid x \mid \pi_i\ n \mid n\ v \mid ?\ \texttt{match}\ n\ \texttt{with} \begin{vmatrix} \sigma_1\ x_1 \to v_1 \\ \sigma_2\ x_2 \to v_2 \end{vmatrix} \mid ?\ \texttt{match}\ n\ \texttt{with} \begin{vmatrix} \sigma_1\ x_1 \to n_1 \\ \sigma_2\ x_2 \to n_2 \end{vmatrix}$$

β-normal forms

Easy in the **negative** fragment $(\to, \times, 1)$. Values and neutrals.

$$
\begin{aligned}
v ::= &\ \text{values} \\
&\mid \lambda x.\, v \\
&\mid (v_1, v_2) \\
&\mid n
\end{aligned}
\qquad\qquad
\begin{aligned}
n ::= &\ \text{neutrals} \\
&\mid x \\
&\mid \pi_i\ n \\
&\mid n\ v
\end{aligned}
$$

Problem: no clear way to add **positives** $(+, 0)$.

$$
\begin{aligned}
v ::= &\ \text{values} \\
&\mid \lambda x.\, v \\
&\mid (v_1, v_2) \\
&\mid n \\
? &\mid \sigma_i\ v
\end{aligned}
\qquad\qquad
\begin{aligned}
n ::= &\ \text{neutrals} \\
&\mid x \\
&\mid \pi_i\ n \\
&\mid n\ v \\
? &\mid \texttt{match}\ n\ \texttt{with} \left|\begin{array}{l} \sigma_1\ x_1 \to v_1 \\ \sigma_2\ x_2 \to v_2 \end{array}\right. \\
? &\mid \texttt{match}\ n\ \texttt{with} \left|\begin{array}{l} \sigma_1\ x_1 \to n_1 \\ \sigma_2\ x_2 \to n_2 \end{array}\right.
\end{aligned}
$$

(Remark on System L)

Aside: positive and negative types

$$
\begin{array}{lll}
A, B & ::= P \mid N & \text{all types} \\
\\
N, M & ::= & \text{negative types} \\
 & \mid A \to B & \text{function type} \\
 & \mid A \times B & \text{product} \\
 & \mid 1 & \text{unit} \\
\\
P, Q & ::= & \text{positive types} \\
 & \mid A + B & \text{sum} \\
 & \mid 0 & \text{empty} \\
\\
P_{\mathrm{at}}, Q_{\mathrm{at}} & ::= P \mid X & \text{positive or atomic type} \\
N_{\mathrm{at}}, M_{\mathrm{at}} & ::= N \mid X & \text{negative or atomic type}
\end{array}
$$

(white lie on atom polarity)

Normality without loss of generality

Any ($t : A \to B$) may be η-expanded into a λ-abstraction: $\lambda x. (t\ x)$

We can ask all values of type ($A \to B$) to be of the form $\lambda x. t$
Inversion. (Note: no closedness assumption!)

We **cannot** ask all (open) values of type $A + B$ to be of the form $\sigma_i\ t$

$$x : B + A \vdash ? : A + B$$

Focused normal forms: intro

We introduce a system of typed normal forms, split in four judgments.
(Grammar, grammatical categories) $\Longrightarrow$ (Type system, judgments).

"inversion" judgment ($\Gamma \vdash_{\mathsf{inv}} v : A$) for values (w.l.o.g)

"negative neutrals" ($\Gamma \vdash_{\mathsf{ne}} n \Downarrow A$) (destructors)
"positive neutrals" ($\Gamma \vdash_{\mathsf{ne}} p \Uparrow A$) (constructors)

"focused normal forms": ($\Gamma \vdash_{\mathsf{foc}} f : A$) (decisions)

Along the way, I will explain why these restrictions remain **complete**.

Focused normal forms: inversions

Without loss of generality: "inversion" judgment ($\Gamma \vdash_{\mathsf{inv}} v : A$).

$$\frac{\Gamma, x : A \vdash_{\mathsf{inv}} v : B}{\Gamma \vdash_{\mathsf{inv}} \lambda x.\, v : A \to B} \qquad \frac{\Gamma \vdash_{\mathsf{inv}} v_1 : A_1 \qquad \Gamma \vdash_{\mathsf{inv}} v_2 : A_2}{\Gamma \vdash_{\mathsf{inv}} (v_1, v_2) : A_1 \times A_2}$$

Focused normal forms: inversions

Without loss of generality: "inversion" judgment ($\Gamma \vdash_{\text{inv}} v : A$).

$$\frac{\Gamma, x : A \vdash_{\text{inv}} v : B}{\Gamma \vdash_{\text{inv}} \lambda x.\, v : A \to B} \qquad \frac{\Gamma \vdash_{\text{inv}} v_1 : A_1 \qquad \Gamma \vdash_{\text{inv}} v_2 : A_2}{\Gamma \vdash_{\text{inv}} (v_1, v_2) : A_1 \times A_2}$$

$$\frac{\Gamma, x : A_1 \vdash_{\text{inv}} v_1 : C \qquad \Gamma, x : A_2 \vdash_{\text{inv}} v_2 : C}{\Gamma, x : A_1 + A_2 \vdash_{\text{inv}} \texttt{match}\ x\ \texttt{with}\ \mid \sigma_1\, x \to v_1 \mid \sigma_2\, x \to v_2 : C}$$

Focused normal forms: inversions

Without loss of generality: "inversion" judgment ($\Gamma \vdash_{\text{inv}} v : A$).

$$\frac{\Gamma, x : A \vdash_{\text{inv}} v : B}{\Gamma \vdash_{\text{inv}} \lambda x.\, v : A \to B} \qquad \frac{\Gamma \vdash_{\text{inv}} v_1 : A_1 \qquad \Gamma \vdash_{\text{inv}} v_2 : A_2}{\Gamma \vdash_{\text{inv}} (v_1, v_2) : A_1 \times A_2}$$

$$\frac{\Gamma, x : A_1 \vdash_{\text{inv}} v_1 : C \qquad \Gamma, x : A_2 \vdash_{\text{inv}} v_2 : C}{\Gamma, x : A_1 + A_2 \vdash_{\text{inv}} \texttt{match}\ x\ \texttt{with}\ \mid \sigma_1\ x \to v_1 \mid \sigma_2\ x \to v_2 : C}$$

$$D[x] \rhd_\eta \texttt{match}\ x\ \texttt{with} \left| \begin{array}{l} \sigma_1\ x_1 \to D[\sigma_1\ x_1] \\ \sigma_2\ x_2 \to D[\sigma_2\ x_2] \end{array} \right.$$

Focused normal forms: inversions

Without loss of generality: "inversion" judgment ($\Gamma \vdash_{\text{inv}} v : A$).

$$\frac{\Gamma, x : A \vdash_{\text{inv}} v : B}{\Gamma \vdash_{\text{inv}} \lambda x.\, v : A \to B} \qquad \frac{\Gamma \vdash_{\text{inv}} v_1 : A_1 \qquad \Gamma \vdash_{\text{inv}} v_2 : A_2}{\Gamma \vdash_{\text{inv}} (v_1, v_2) : A_1 \times A_2}$$

$$\frac{\Gamma, x : A_1 \vdash_{\text{inv}} v_1 : C \qquad \Gamma, x : A_2 \vdash_{\text{inv}} v_2 : C}{\Gamma, x : A_1 + A_2 \vdash_{\text{inv}} \texttt{match}\ x\ \texttt{with}\ \mid \sigma_1\, x \to v_1 \mid \sigma_2\, x \to v_2 : C}$$

$$\frac{}{\Gamma \vdash_{\text{inv}} () : 1} \qquad \frac{}{\Gamma, x : 0 \vdash_{\text{inv}} \texttt{absurd}(x) : A}$$

Focused normal forms: inversions

Without loss of generality: "inversion" judgment ($\Gamma \vdash_{\mathsf{inv}} v : A$).

$$\frac{\Gamma, x : A \vdash_{\mathsf{inv}} v : B}{\Gamma \vdash_{\mathsf{inv}} \lambda x.\, v : A \to B} \qquad \frac{\Gamma \vdash_{\mathsf{inv}} v_1 : A_1 \qquad \Gamma \vdash_{\mathsf{inv}} v_2 : A_2}{\Gamma \vdash_{\mathsf{inv}} (v_1, v_2) : A_1 \times A_2}$$

$$\frac{\Gamma, x : A_1 \vdash_{\mathsf{inv}} v_1 : C \qquad \Gamma, x : A_2 \vdash_{\mathsf{inv}} v_2 : C}{\Gamma, x : A_1 + A_2 \vdash_{\mathsf{inv}} \mathtt{match}\ x\ \mathtt{with}\ \mid \sigma_1\, x \to v_1 \mid \sigma_2\, x \to v_2 : C}$$

$$\frac{}{\Gamma \vdash_{\mathsf{inv}} () : 1} \qquad \frac{}{\Gamma, x : 0 \vdash_{\mathsf{inv}} \mathtt{absurd}(x) : A}$$

$$\frac{\Gamma \text{ negative or atomic} \qquad \Gamma \vdash_{\mathsf{foc}} v : A \qquad A \text{ positive or atomic}}{\Gamma \vdash_{\mathsf{inv}} v : A}$$

Focused normal forms: inversions

Without loss of generality: "inversion" judgment ($\Gamma \vdash_{inv} v : A$).

$$\frac{\Gamma, x : A \vdash_{inv} v : B}{\Gamma \vdash_{inv} \lambda x.\, v : A \to B} \qquad \frac{\Gamma \vdash_{inv} v_1 : A_1 \qquad \Gamma \vdash_{inv} v_2 : A_2}{\Gamma \vdash_{inv} (v_1, v_2) : A_1 \times A_2}$$

$$\frac{\Gamma, x : A_1 \vdash_{inv} v_1 : C \qquad \Gamma, x : A_2 \vdash_{inv} v_2 : C}{\Gamma, x : A_1 + A_2 \vdash_{inv} \texttt{match}\ x\ \texttt{with}\ \mid \sigma_1\, x \to v_1 \mid \sigma_2\, x \to v_2 : C}$$

$$\frac{}{\Gamma \vdash_{inv} () : 1} \qquad \frac{}{\Gamma, x : 0 \vdash_{inv} \texttt{absurd}(x) : A}$$

$$\frac{\Gamma \text{ negative or atomic} \qquad \Gamma \vdash_{foc} v : A \qquad A \text{ positive or atomic}}{\Gamma \vdash_{inv} v : A}$$

(discuss ordering)

Focused normal forms: neutrals

Destructors, constructors: $(\Gamma \vdash_{\mathsf{ne}} n \Downarrow A)$, $(\Gamma \vdash_{\mathsf{ne}} p \Uparrow A)$.

$$\frac{(x : N) \in \Gamma}{\Gamma \vdash_{\mathsf{ne}} x \Downarrow N}$$

Focused normal forms: neutrals

Destructors, constructors: $(\Gamma \vdash_{\mathsf{ne}} n \Downarrow A)$, $(\Gamma \vdash_{\mathsf{ne}} p \Uparrow A)$.

$$\frac{(x : N) \in \Gamma}{\Gamma \vdash_{\mathsf{ne}} x \Downarrow N} \qquad \frac{\Gamma \vdash_{\mathsf{ne}} n \Downarrow A_1 \times A_2}{\Gamma \vdash_{\mathsf{ne}} \pi_i\ n \Downarrow A_i}$$

Focused normal forms: neutrals

Destructors, constructors: $(\Gamma \vdash_{\mathsf{ne}} n \Downarrow A)$, $(\Gamma \vdash_{\mathsf{ne}} p \Uparrow A)$.

$$\frac{(x : N) \in \Gamma}{\Gamma \vdash_{\mathsf{ne}} x \Downarrow N} \qquad \frac{\Gamma \vdash_{\mathsf{ne}} n \Downarrow A_1 \times A_2}{\Gamma \vdash_{\mathsf{ne}} \pi_i \; n \Downarrow A_i} \qquad \frac{\Gamma \vdash_{\mathsf{ne}} n \Downarrow A \to B \qquad \Gamma \vdash_{\mathsf{ne}} p \Uparrow A}{\Gamma \vdash_{\mathsf{ne}} n \; p \Downarrow B}$$

Focused normal forms: neutrals

Destructors, constructors: $(\Gamma \vdash_{\text{ne}} n \Downarrow A)$, $(\Gamma \vdash_{\text{ne}} p \Uparrow A)$.

$$\frac{(x : N) \in \Gamma}{\Gamma \vdash_{\text{ne}} x \Downarrow N} \qquad \frac{\Gamma \vdash_{\text{ne}} n \Downarrow A_1 \times A_2}{\Gamma \vdash_{\text{ne}} \pi_i \; n \Downarrow A_i} \qquad \frac{\Gamma \vdash_{\text{ne}} n \Downarrow A \to B \qquad \Gamma \vdash_{\text{ne}} p \Uparrow A}{\Gamma \vdash_{\text{ne}} n \; p \Downarrow B}$$

$$\frac{\Gamma \vdash_{\text{ne}} p \Uparrow A}{\Gamma \vdash_{\text{ne}} \sigma_i \; p \Uparrow A_1 + A_2}$$

Focused normal forms: neutrals

Destructors, constructors: $(\Gamma \vdash_{\mathsf{ne}} n \Downarrow A)$, $(\Gamma \vdash_{\mathsf{ne}} p \Uparrow A)$.

$$\frac{(x : N) \in \Gamma}{\Gamma \vdash_{\mathsf{ne}} x \Downarrow N} \qquad \frac{\Gamma \vdash_{\mathsf{ne}} n \Downarrow A_1 \times A_2}{\Gamma \vdash_{\mathsf{ne}} \pi_i\ n \Downarrow A_i} \qquad \frac{\Gamma \vdash_{\mathsf{ne}} n \Downarrow A \to B \qquad \Gamma \vdash_{\mathsf{ne}} p \Uparrow A}{\Gamma \vdash_{\mathsf{ne}} n\ p \Downarrow B}$$

$$\frac{\Gamma \vdash_{\mathsf{ne}} p \Uparrow A}{\Gamma \vdash_{\mathsf{ne}} \sigma_i\ p \Uparrow A_1 + A_2} \qquad \frac{(x : X^+) \in \Gamma}{\Gamma \vdash_{\mathsf{ne}} x \Uparrow X^+}$$

Focused normal forms: neutrals

Destructors, constructors: $(\Gamma \vdash_{\text{ne}} n \Downarrow A)$, $(\Gamma \vdash_{\text{ne}} p \Uparrow A)$.

$$\frac{(x : N) \in \Gamma}{\Gamma \vdash_{\text{ne}} x \Downarrow N} \qquad \frac{\Gamma \vdash_{\text{ne}} n \Downarrow A_1 \times A_2}{\Gamma \vdash_{\text{ne}} \pi_i\ n \Downarrow A_i} \qquad \frac{\Gamma \vdash_{\text{ne}} n \Downarrow A \to B \qquad \Gamma \vdash_{\text{ne}} p \Uparrow A}{\Gamma \vdash_{\text{ne}} n\ p \Downarrow B}$$

$$\frac{\Gamma \vdash_{\text{ne}} p \Uparrow A}{\Gamma \vdash_{\text{ne}} \sigma_i\ p \Uparrow A_1 + A_2} \qquad \frac{(x : X^+) \in \Gamma}{\Gamma \vdash_{\text{ne}} x \Uparrow X^+} \qquad \frac{\Gamma \vdash_{\text{inv}} v : N \qquad N \text{ negative}}{\Gamma \vdash_{\text{ne}} v \Uparrow N}$$

Focused normal forms: decisions

"focused" judgment ($\Gamma \vdash_{\mathsf{foc}} t : A$).

Either kind of neutrals:

Focused normal forms: decisions

"focused" judgment ($\Gamma \vdash_{\mathsf{foc}} t : A$).

Either kind of neutrals:

$$\frac{\Gamma \vdash_{\mathsf{ne}} p \Uparrow P}{\Gamma \vdash_{\mathsf{foc}} p : P} \qquad \frac{\Gamma \vdash_{\mathsf{ne}} n \Downarrow X^{-}}{\Gamma \vdash_{\mathsf{foc}} n : X^{-}}$$

Focused normal forms: decisions

"focused" judgment ($\Gamma \vdash_{\mathsf{foc}} t : A$).

Either kind of neutrals:

$$\frac{\Gamma \vdash_{\mathsf{ne}} p \Uparrow P}{\Gamma \vdash_{\mathsf{foc}} p : P} \qquad \frac{\Gamma \vdash_{\mathsf{ne}} n \Downarrow X^-}{\Gamma \vdash_{\mathsf{foc}} n : X^-}$$

Problem: β-normal forms may be non-neutral.

$$\pi_i \left(\begin{array}{l} \texttt{match}\ f\ ()\ \texttt{with} \\ \mid \sigma_1\ x_1 \to x_1 \\ \mid \sigma_2\ x_2 \to x_2 \end{array} \right) \qquad \begin{array}{l} \texttt{match}\ f\ ()\ \texttt{with} \\ \mid \sigma_1\ x \to \pi_1\ x \\ \mid \sigma_2\ x \to \sigma_1\ () \end{array} \qquad g\ (\texttt{absurd}(f\ ()))$$

Solution: we can decide to extrude a neutral:

Focused normal forms: decisions

"focused" judgment ($\Gamma \vdash_{\text{foc}} t : A$).

Either kind of neutrals:

$$\frac{\Gamma \vdash_{\text{ne}} p \Uparrow P}{\Gamma \vdash_{\text{foc}} p : P} \qquad \frac{\Gamma \vdash_{\text{ne}} n \Downarrow X^-}{\Gamma \vdash_{\text{foc}} n : X^-}$$

Problem: β-normal forms may be non-neutral.

$$\pi_i \begin{pmatrix} \texttt{match}\ f\ ()\ \texttt{with} \\ \mid \sigma_1\ x_1 \to x_1 \\ \mid \sigma_2\ x_2 \to x_2 \end{pmatrix} \qquad \begin{array}{l} \texttt{match}\ f\ ()\ \texttt{with} \\ \mid \sigma_1\ x \to \pi_1\ x \\ \mid \sigma_2\ x \to \sigma_1\ () \end{array} \qquad g\ (\texttt{absurd}(f\ ()))$$

Solution: we can decide to extrude a neutral:

$$\frac{\Gamma \vdash_{\text{ne}} n \Downarrow P \qquad P \text{ positive} \qquad \Gamma, x : P \vdash_{\text{inv}} v : A}{\Gamma \vdash_{\text{foc}} \texttt{let}\ x = n\ \texttt{in}\ v : A}$$

That's it!

Focused normal forms: summary

(Grammar with type annotations)

$$
\begin{aligned}
v ::=\ & \text{values} \\
\mid\ & \lambda x.\, v \\
\mid\ & (v_1, v_2) \\
\mid\ & \texttt{match}\ x\ \texttt{with} \left|\begin{array}{l} \sigma_1\ x \to v_1 \\ \sigma_2\ x \to v_2 \end{array}\right. \\
\mid\ & \texttt{absurd}(x) \\
\mid\ & (f : P_{\text{at}})
\end{aligned}
$$

$$
\begin{aligned}
f ::=\ & \text{focused forms} \\
\mid\ & \texttt{let}\ (x : P) = n\ \texttt{in}\ v \\
\mid\ & (n : X^-) \\
\mid\ & (p : P)
\end{aligned}
$$

$$
\begin{aligned}
n ::=\ & \text{negative neutrals} \\
\mid\ & (x : N) \\
\mid\ & \pi_i\ n \\
\mid\ & n\ p
\end{aligned}
$$

$$
\begin{aligned}
p ::=\ & \text{positive neutrals} \\
\mid\ & (x : X^+) \\
\mid\ & \sigma_i\ p \\
\mid\ & (v : N)
\end{aligned}
$$

Focused normal forms: summary

(Grammar with type annotations)

$$
\begin{aligned}
v ::= & \text{ values} \\
& \mid \lambda x.\, v \\
& \mid (v_1, v_2) \\
& \mid \texttt{match } x \texttt{ with} \left| \begin{array}{l} \sigma_1\, x \to v_1 \\ \sigma_2\, x \to v_2 \end{array} \right. \\
& \mid \texttt{absurd}(x) \\
& \mid (f : P_{\text{at}})
\end{aligned}
$$

$$
\begin{aligned}
f ::= & \text{ focused forms} \\
& \mid \texttt{let } (x : P) = n \texttt{ in } v \\
& \mid (n : X^-) \\
& \mid (p : P)
\end{aligned}
$$

$$
\begin{aligned}
n ::= & \text{ negative neutrals} \\
& \mid (x : N) \\
& \mid \pi_i\, n \\
& \mid n\, p
\end{aligned}
$$

$$
\begin{aligned}
p ::= & \text{ positive neutrals} \\
& \mid (x : X^+) \\
& \mid \sigma_i\, p \\
& \mid (v : N)
\end{aligned}
$$

(Cut-free)

Focused normal forms: summary

(Grammar with type annotations)

$$
\begin{aligned}
v ::=\ & \text{values} \\
\mid\ & \lambda x.\, v \\
\mid\ & (v_1, v_2) \\
\mid\ & \texttt{match } x \texttt{ with} \left| \begin{array}{l} \sigma_1\ x \to v_1 \\ \sigma_2\ x \to v_2 \end{array} \right. \\
\mid\ & \texttt{absurd}(x) \\
\mid\ & (f : P_{\mathrm{at}})
\end{aligned}
$$

$$
\begin{aligned}
f ::=\ & \text{focused forms} \\
\mid\ & \texttt{let } (x : P) = n \texttt{ in } v \\
\mid\ & (n : X^-) \\
\mid\ & (p : P)
\end{aligned}
$$

$$
\begin{aligned}
n ::=\ & \text{negative neutrals} \\
\mid\ & (x : N) \\
\mid\ & \pi_i\ n \\
\mid\ & n\ p
\end{aligned}
$$

$$
\begin{aligned}
p ::=\ & \text{positive neutrals} \\
\mid\ & (x : X^+) \\
\mid\ & \sigma_i\ p \\
\mid\ & (v : N)
\end{aligned}
$$

(Cut-free) (System L)

Focused forms do not solve positive η-equivalence

$$x : 1 \to (X + X) \vdash_{\mathsf{inv}} ? : 0 + (X, X)$$

Focused forms do not solve positive η-equivalence

$$x : 1 \to (X + X) \vdash_{\mathsf{inv}} ? : 0 + (X, X)$$

$$\texttt{let } y = x\,() \texttt{ in match } y \texttt{ with } \left|\begin{array}{l} \sigma_1\ y \to \sigma_2\ (y, y) \\ \sigma_2\ y \to \sigma_2\ (y, y) \end{array}\right.$$

$$\sigma_2 \left(\begin{array}{l} \texttt{let } y_1 = x\,() \texttt{ in match } y_1 \texttt{ with } \left|\begin{array}{l} \sigma_1\ y_1 \to y_1 \\ \sigma_2\ y_1 \to y_1 \end{array}\right. \\ , \\ \texttt{let } y_2 = x\,() \texttt{ in match } y_2 \texttt{ with } \left|\begin{array}{l} \sigma_1\ y_2 \to y_2 \\ \sigma_2\ y_2 \to y_2 \end{array}\right. \end{array}\right)$$

Neutrals of positive type may be eliminated at difference places, a different number of times.

Canonical neutral introduction

Idea: introduce each neutral ($n : P$) as **early** as possible.

Canonical neutral introduction

Idea: introduce each neutral $(n : P)$ as **early** as possible.

This suggests an "even more normal" form, **maximally** focused.

Canonical neutral introduction

Idea: introduce each neutral $(n : P)$ as **early** as possible.

This suggests an "even more normal" form, **maximally** focused.

To normalize a focused form f: compute **all** neutral subterms n that are typeable in the current context, and bind them now. Draft rule:

Canonical neutral introduction

Idea: introduce each neutral $(n : P)$ as **early** as possible.

This suggests an "even more normal" form, **maximally** focused.

To normalize a focused form f: compute **all** neutral subterms n that are typeable in the current context, and bind them now. Draft rule:

$$\frac{\bar{n} \stackrel{\text{def}}{=} \{n \in f \mid \Gamma \vdash_{\mathsf{ne}} n \Downarrow P\}}{\Gamma \vdash f \rhd_{max} \texttt{let } \bar{x} = \bar{n} \texttt{ in } f[\bar{x}/\bar{n}] : A}\ \text{DRAFT-DO-NOT-DISTRIBUTE}$$

Canonical neutral introduction

Idea: introduce each neutral $(n : P)$ as **early** as possible.

This suggests an "even more normal" form, **maximally** focused.

To normalize a focused form f: compute **all** neutral subterms n that are typeable in the current context, and bind them now. Draft rule:

$$\frac{\bar{n} \overset{\text{def}}{=} \{n \in f \mid \Gamma \vdash_{\text{ne}} n \Downarrow P\}}{\Gamma \vdash f \rhd_{max} \texttt{let}\ \bar{x} = \bar{n}\ \texttt{in}\ f[\bar{x}/\bar{n}] : A}\ \text{DRAFT-DO-NOT-DISTRIBUTE}$$

This relates to two different ideas, points of view:

1. a **permutation** algorithm for equivalence of $(\to, \times, +)$ (no 0)

Combining both gives equivalence with 0.

Canonical neutral introduction

Idea: introduce each neutral $(n : P)$ as **early** as possible.

This suggests an "even more normal" form, **maximally** focused.

To normalize a focused form f: compute **all** neutral subterms n that are typeable in the current context, and bind them now. Draft rule:

$$\frac{\bar{n} \stackrel{\text{def}}{=} \{n \in f \mid \Gamma \vdash_{\mathsf{ne}} n \Downarrow P\}}{\Gamma \vdash f \rhd_{max} \mathtt{let}\ \bar{x} = \bar{n}\ \mathtt{in}\ f[\bar{x}/\bar{n}] : A}\ \text{DRAFT-DO-NOT-DISTRIBUTE}$$

This relates to two different ideas, points of view:

1. a **permutation** algorithm for equivalence of $(\rightarrow, \times, +)$ (no 0)
2. a **saturation** approach for canonical goal-directed search

Combining both gives equivalence with 0.

Equivalence : intro

Algorithm for equivalence of focused values.

Mutual judgments:

- $\Gamma \vdash_{\text{inv}} v \sim_{\text{alg}} v' : A$
- $\Gamma \vdash_{\text{sat}} f \sim_{\text{alg}} f' : A$
- $\Gamma \vdash_{\text{ne}} n \sim_{\text{alg}} n' \Downarrow A$
- $\Gamma \vdash_{\text{ne}} p \sim_{\text{alg}} p' \Uparrow A$

Depends on a consistency checking judgment $\Gamma \vdash 0$
Definable as $\exists t, (\Gamma \vdash t : 0)$.
Decidable in propositional logic (key!)

Equivalence: inversion

(Terms modulo commuting conversions, or enforced ordering)

$$\frac{\Gamma, x : A \vdash_{\mathsf{inv}} v \sim_{\mathsf{alg}} v' : B}{\Gamma \vdash_{\mathsf{inv}} \lambda x.\, v \sim_{\mathsf{alg}} \lambda x.\, v' : A \to B}$$

Equivalence: inversion

(Terms modulo commuting conversions, or enforced ordering)

$$\frac{\Gamma, x : A \vdash_{\text{inv}} v \sim_{\text{alg}} v' : B}{\Gamma \vdash_{\text{inv}} \lambda x.\, v \sim_{\text{alg}} \lambda x.\, v' : A \to B}
\qquad
\frac{\Gamma \vdash_{\text{inv}} v_1 \sim_{\text{alg}} v_1' : A_1 \quad \Gamma \vdash_{\text{inv}} v_2 \sim_{\text{alg}} v_2' : A_2}{\Gamma \vdash_{\text{inv}} (v_1, v_2) \sim_{\text{alg}} (v_1', v_2') : A_1 \times A_2}$$

$$\frac{\Gamma, x : A_1 \vdash_{\text{inv}} v_1 \sim_{\text{alg}} v_1' : C \quad \Gamma, x : A_2 \vdash_{\text{inv}} v_2 \sim_{\text{alg}} v_2' : C}{\Gamma, x : A_1 + A_2 \vdash_{\text{inv}} \begin{array}{l}\texttt{match}\ x\ \texttt{with} \\ \mid \sigma_1\ x \to v_1 \\ \mid \sigma_2\ x \to v_2\end{array} \sim_{\text{alg}} \begin{array}{l}\texttt{match}\ x\ \texttt{with} \\ \mid \sigma_1\ x \to v_1' \\ \mid \sigma_2\ x \to v_2'\end{array} : C}$$

$$\frac{}{\Gamma \vdash_{\text{inv}} () \sim_{\text{alg}} () : 1}
\qquad
\frac{}{\Gamma, x : 0 \vdash_{\text{inv}} \texttt{absurd}(x) \sim_{\text{alg}} \texttt{absurd}(x) : A}$$

$$\frac{\Gamma \text{ negative or atomic} \quad \Gamma \vdash_{\text{sat}} v \sim_{\text{alg}} v' : A \quad A \text{ positive or atomic}}{\Gamma \vdash_{\text{inv}} v \sim_{\text{alg}} v' : A}$$

Equivalence: neutrals

Just follow the constructors. (Failure cases)

$$\frac{(x : N) \in \Gamma}{\Gamma \vdash_{\mathsf{ne}} x \sim_{\mathsf{alg}} x \Downarrow N} \qquad \frac{\Gamma \vdash_{\mathsf{ne}} n \sim_{\mathsf{alg}} n' \Downarrow A_1 \times A_2}{\Gamma \vdash_{\mathsf{ne}} \pi_i\ n \sim_{\mathsf{alg}} \pi_i\ n' \Downarrow A_i}$$

Equivalence: neutrals

Just follow the constructors. (Failure cases)

$$\frac{(x : N) \in \Gamma}{\Gamma \vdash_{\text{ne}} x \sim_{\text{alg}} x \Downarrow N} \qquad \frac{\Gamma \vdash_{\text{ne}} n \sim_{\text{alg}} n' \Downarrow A_1 \times A_2}{\Gamma \vdash_{\text{ne}} \pi_i\ n \sim_{\text{alg}} \pi_i\ n' \Downarrow A_i}$$

$$\frac{\Gamma \vdash_{\text{ne}} n \sim_{\text{alg}} n' \Downarrow A \to B \qquad \Gamma \vdash_{\text{ne}} p \sim_{\text{alg}} p' \Uparrow A}{\Gamma \vdash_{\text{ne}} n\ p \sim_{\text{alg}} n'\ p' \Downarrow B}$$

$$\frac{\Gamma \vdash_{\text{ne}} p \sim_{\text{alg}} p' \Uparrow A}{\Gamma \vdash_{\text{ne}} \sigma_i\ p \sim_{\text{alg}} \sigma_i\ p' \Uparrow A_1 + A_2} \qquad \frac{(x : X^+) \in \Gamma}{\Gamma \vdash_{\text{ne}} x \sim_{\text{alg}} x \Uparrow X^+}$$

$$\frac{\Gamma \vdash_{\text{inv}} v \sim_{\text{alg}} v' : N \qquad N \text{ negative}}{\Gamma \vdash_{\text{ne}} v \sim_{\text{alg}} v' \Uparrow N}$$

Equivalence: saturation

The hard stuff.

$$\frac{n \in f_i \qquad \Gamma \vdash_{\mathsf{ne}} n \Downarrow P \qquad \bar{n'} \stackrel{\mathrm{def}}{=} \{n' \in f_j \mid \Gamma \vdash_{\mathsf{ne}} n \sim_{\mathsf{alg}} n' \Downarrow P\} \qquad \Gamma, x : P \vdash_{\mathsf{inv}} f_1[x/\bar{n'}] \sim_{\mathsf{alg}} f_2[x/\bar{n'}] : A}{\Gamma \vdash_{\mathsf{sat}} f_1 \sim_{\mathsf{alg}} f_2 : A}$$

Equivalence: saturation

The hard stuff.

$$\frac{n \in f_i \qquad \Gamma \vdash_{\text{ne}} n \Downarrow P \qquad \bar{n'} \stackrel{\text{def}}{=} \{n' \in f_j \mid \Gamma \vdash_{\text{ne}} n \sim_{\text{alg}} n' \Downarrow P\} \qquad \Gamma, x : P \vdash_{\text{inv}} f_1[x/\bar{n'}] \sim_{\text{alg}} f_2[x/\bar{n'}] : A}{\Gamma \vdash_{\text{sat}} f_1 \sim_{\text{alg}} f_2 : A}$$

$$\frac{\Gamma \vdash 0}{\Gamma \vdash_{\text{sat}} f_1 \sim_{\text{alg}} f_2 : A}$$

Equivalence: saturation

The hard stuff.

$$\frac{n \in f_i \qquad \Gamma \vdash_{\mathrm{ne}} n \Downarrow P \qquad \bar{n'} \stackrel{\mathrm{def}}{=} \{n' \in f_j \mid \Gamma \vdash_{\mathrm{ne}} n \sim_{\mathrm{alg}} n' \Downarrow P\} \qquad \Gamma, x : P \vdash_{\mathrm{inv}} f_1[x/\bar{n'}] \sim_{\mathrm{alg}} f_2[x/\bar{n'}] : A}{\Gamma \vdash_{\mathrm{sat}} f_1 \sim_{\mathrm{alg}} f_2 : A}$$

$$\frac{\Gamma \vdash 0}{\Gamma \vdash_{\mathrm{sat}} f_1 \sim_{\mathrm{alg}} f_2 : A}$$

$$\frac{\neg(\exists n \in f_i,\ \Gamma \vdash_{\mathrm{ne}} n \Downarrow P) \qquad \neg(\Gamma \vdash 0) \qquad \Gamma \vdash_{\mathrm{ne}} f_1 \sim_{\mathrm{alg}} f_2 \Downarrow X^-}{\Gamma \vdash_{\mathrm{sat}} f_1 \sim_{\mathrm{alg}} f_2 : X^-}$$

$$\frac{\neg(\exists n \in f_i,\ \Gamma \vdash_{\mathrm{ne}} n \Downarrow P) \qquad \neg(\Gamma \vdash 0) \qquad \Gamma \vdash_{\mathrm{ne}} f_1 \sim_{\mathrm{alg}} f_2 \Uparrow P}{\Gamma \vdash_{\mathrm{sat}} f_1 \sim_{\mathrm{alg}} f_2 : P}$$

(negative side-conditions: no backtracking)

Equivalence: demo

$$\emptyset \vdash \lambda f.\, \lambda x.\, f\ x \approx_{\beta\eta} \lambda f.\, \lambda x.\, f\ (f\ (f\ x)) : (\mathbb{B} \to \mathbb{B}) \to (\mathbb{B} \to \mathbb{B})$$

$f : \mathbb{B} \to \mathbb{B}, x : \mathbb{B} \vdash_{\mathsf{inv}}$

Equivalence: demo

$$\emptyset \vdash \lambda f.\, \lambda x.\, f\ x \approx_{\beta\eta} \lambda f.\, \lambda x.\, f\ (f\ (f\ x)) : (\mathbb{B} \to \mathbb{B}) \to (\mathbb{B} \to \mathbb{B})$$

$f : \mathbb{B} \to \mathbb{B}, x : \mathbb{B} \vdash_{\mathsf{inv}}$

$\texttt{match}\ x\ \texttt{with}\ \sigma_i\ x \to^{i}$

Equivalence: demo

$$\emptyset \vdash \lambda f.\, \lambda x.\, f\ x \approx_{\beta\eta} \lambda f.\, \lambda x.\, f\ (f\ (f\ x)) : (\mathbb{B} \to \mathbb{B}) \to (\mathbb{B} \to \mathbb{B})$$

$f : \mathbb{B} \to \mathbb{B}, x : \mathbb{B} \vdash_{\mathsf{inv}}$

$\mathtt{match}\ x\ \mathtt{with}\ \sigma_i\ x \to^{i}$

$\quad \mathtt{let}\ x_1 = f\ (\sigma_i\ x)\ \mathtt{in}$

Equivalence: demo

$$\emptyset \vdash \lambda f.\, \lambda x.\, f\ x \approx_{\beta\eta} \lambda f.\, \lambda x.\, f\ (f\ (f\ x)) : (\mathbb{B} \to \mathbb{B}) \to (\mathbb{B} \to \mathbb{B})$$

$f : \mathbb{B} \to \mathbb{B}, x : \mathbb{B} \vdash_{\mathsf{inv}}$

$$\begin{array}{l} \texttt{match}\ x\ \texttt{with}\ \sigma_i\ x \to^i \\ \quad \texttt{let}\ x_1 = f\ (\sigma_i\ x)\ \texttt{in} \\ \quad\quad \texttt{match}\ x_1\ \texttt{with} \\ \quad\quad \left|\begin{array}{l} \sigma_1\ x_1 \to \sigma_1\ x_1 \\ \sigma_2\ x_1 \to \sigma_2\ x_1 \end{array}\right. \end{array} \qquad \sim_{\mathsf{alg}}$$

Equivalence: demo

$$\emptyset \vdash \lambda f.\, \lambda x.\, f\ x \approx_{\beta\eta} \lambda f.\, \lambda x.\, f\ (f\ (f\ x)) : (\mathbb{B} \to \mathbb{B}) \to (\mathbb{B} \to \mathbb{B})$$

$f : \mathbb{B} \to \mathbb{B}, x : \mathbb{B} \vdash_{\mathsf{inv}}$

$$\begin{array}{l} \texttt{match}\ x\ \texttt{with}\ \sigma_i\ x \to^i \\ \quad \texttt{let}\ x_1 = f\ (\sigma_i\ x)\ \texttt{in} \\ \quad\quad \texttt{match}\ x_1\ \texttt{with} \\ \quad\quad \mid \sigma_1\ x_1 \to \sigma_1\ x_1 \\ \quad\quad \mid \sigma_2\ x_1 \to \sigma_2\ x_1 \end{array} \quad \sim_{\mathsf{alg}} \quad \begin{array}{l} \texttt{match}\ x\ \texttt{with}\ \sigma_i\ x \to^i \\ \quad \texttt{let}\ x_1 = f\ (\sigma_i\ x)\ \texttt{in} \\ \quad\quad \texttt{match}\ x_1\ \texttt{with}\ \sigma_j\ x_1 \to^j \end{array}$$

Equivalence: demo

$$\emptyset \vdash \lambda f.\, \lambda x.\, f\ x \approx_{\beta\eta} \lambda f.\, \lambda x.\, f\ (f\ (f\ x)) : (\mathbb{B} \to \mathbb{B}) \to (\mathbb{B} \to \mathbb{B})$$

$f : \mathbb{B} \to \mathbb{B}, x : \mathbb{B} \vdash_{\mathsf{inv}}$

$$\begin{array}{l} \texttt{match}\ x\ \texttt{with}\ \sigma_i\ x \to^i \\ \quad \texttt{let}\ x_1 = f\ (\sigma_i\ x)\ \texttt{in} \\ \quad\quad \texttt{match}\ x_1\ \texttt{with} \\ \quad\quad \mid \sigma_1\ x_1 \to \sigma_1\ x_1 \\ \quad\quad \mid \sigma_2\ x_1 \to \sigma_2\ x_1 \end{array} \sim_{\mathsf{alg}} \begin{array}{l} \texttt{match}\ x\ \texttt{with}\ \sigma_i\ x \to^i \\ \quad \texttt{let}\ x_1 = f\ (\sigma_i\ x)\ \texttt{in} \\ \quad\quad \texttt{match}\ x_1\ \texttt{with}\ \sigma_j\ x_1 \to^j \\ \quad\quad\quad \texttt{let}\ x_2 = f\ (\sigma_j\ x_1)\ \texttt{in} \end{array}$$

Equivalence: demo

$$\emptyset \vdash \lambda f.\, \lambda x.\, f\ x \approx_{\beta\eta} \lambda f.\, \lambda x.\, f\ (f\ (f\ x)) : (\mathbb{B} \to \mathbb{B}) \to (\mathbb{B} \to \mathbb{B})$$

$f : \mathbb{B} \to \mathbb{B}, x : \mathbb{B} \vdash_{\mathsf{inv}}$

$$\begin{array}{l} \texttt{match}\ x\ \texttt{with}\ \sigma_i\ x \to^i \\ \quad \texttt{let}\ x_1 = f\ (\sigma_i\ x)\ \texttt{in} \\ \quad\quad \texttt{match}\ x_1\ \texttt{with} \\ \quad\quad \mid \sigma_1\ x_1 \to \sigma_1\ x_1 \\ \quad\quad \mid \sigma_2\ x_1 \to \sigma_2\ x_1 \end{array} \quad \sim_{\mathsf{alg}} \quad \begin{array}{l} \texttt{match}\ x\ \texttt{with}\ \sigma_i\ x \to^i \\ \quad \texttt{let}\ x_1 = f\ (\sigma_i\ x)\ \texttt{in} \\ \quad\quad \texttt{match}\ x_1\ \texttt{with}\ \sigma_j\ x_1 \to^j \\ \quad\quad\quad \texttt{let}\ x_2 = f\ (\sigma_j\ x_1)\ \texttt{in} \\ \quad\quad\quad\quad \texttt{match}\ x_2\ \texttt{with}\ \sigma_k\ x_2 \to^k \end{array}$$

Equivalence: demo

$$\emptyset \vdash \lambda f.\, \lambda x.\, f\ x \approx_{\beta\eta} \lambda f.\, \lambda x.\, f\ (f\ (f\ x)) : (\mathbb{B} \to \mathbb{B}) \to (\mathbb{B} \to \mathbb{B})$$

$f : \mathbb{B} \to \mathbb{B}, x : \mathbb{B} \vdash_{\mathsf{inv}}$

$$\begin{array}{l} \texttt{match}\ x\ \texttt{with}\ \sigma_i\ x \to^{i} \\ \quad \texttt{let}\ x_1 = f\ (\sigma_i\ x)\ \texttt{in} \\ \quad\quad \texttt{match}\ x_1\ \texttt{with} \\ \quad\quad \left|\begin{array}{l} \sigma_1\ x_1 \to \sigma_1\ x_1 \\ \sigma_2\ x_1 \to \sigma_2\ x_1 \end{array}\right. \end{array} \sim_{\mathsf{alg}} \begin{array}{l} \texttt{match}\ x\ \texttt{with}\ \sigma_i\ x \to^{i} \\ \quad \texttt{let}\ x_1 = f\ (\sigma_i\ x)\ \texttt{in} \\ \quad\quad \texttt{match}\ x_1\ \texttt{with}\ \sigma_j\ x_1 \to^{j} \\ \quad\quad\quad \texttt{let}\ x_2 = f\ (\sigma_j\ x_1)\ \texttt{in} \\ \quad\quad\quad\quad \texttt{match}\ x_2\ \texttt{with}\ \sigma_k\ x_2 \to^{k} \\ \quad\quad\quad\quad\quad \texttt{let}\ x_3 = f\ (\sigma_k\ x_2)\ \texttt{in} \end{array}$$

Equivalence: demo

$$\emptyset \vdash \lambda f.\, \lambda x.\, f\ x \approx_{\beta\eta} \lambda f.\, \lambda x.\, f\ (f\ (f\ x)) : (\mathbb{B} \to \mathbb{B}) \to (\mathbb{B} \to \mathbb{B})$$

$f : \mathbb{B} \to \mathbb{B}, x : \mathbb{B} \vdash_{\text{inv}}$

$$\begin{array}{l}
\texttt{match}\ x\ \texttt{with}\ \sigma_i\ x \to^i \\
\quad \texttt{let}\ x_1 = f\ (\sigma_i\ x)\ \texttt{in} \\
\quad\quad \texttt{match}\ x_1\ \texttt{with} \\
\quad\quad \mid \sigma_1\ x_1 \to \sigma_1\ x_1 \\
\quad\quad \mid \sigma_2\ x_1 \to \sigma_2\ x_1
\end{array}
\sim_{\text{alg}}
\begin{array}{l}
\texttt{match}\ x\ \texttt{with}\ \sigma_i\ x \to^i \\
\quad \texttt{let}\ x_1 = f\ (\sigma_i\ x)\ \texttt{in} \\
\quad \texttt{match}\ x_1\ \texttt{with}\ \sigma_j\ x_1 \to^j \\
\quad\quad \texttt{let}\ x_2 = f\ (\sigma_j\ x_1)\ \texttt{in} \\
\quad\quad \texttt{match}\ x_2\ \texttt{with}\ \sigma_k\ x_2 \to^k \\
\quad\quad\quad \texttt{let}\ x_3 = f\ (\sigma_k\ x_2)\ \texttt{in} \\
\quad\quad\quad \texttt{match}\ x_3\ \texttt{with} \\
\quad\quad\quad \mid \sigma_1\ x_3 \to \sigma_1\ x_3 \\
\quad\quad\quad \mid \sigma_2\ x_3 \to \sigma_2\ x_3
\end{array}$$

$\mathbb{B}$

Equivalence: demo

$$\emptyset \vdash \lambda f.\, \lambda x.\, f\ x \approx_{\beta\eta} \lambda f.\, \lambda x.\, f\ (f\ (f\ x)) : (\mathbb{B} \to \mathbb{B}) \to (\mathbb{B} \to \mathbb{B})$$

$f : \mathbb{B} \to \mathbb{B}, x : \mathbb{B} \vdash_{\text{inv}}$

$$\begin{array}{l} \texttt{match}\ x\ \texttt{with}\ \sigma_i\ x \to^i \\ \quad \texttt{let}\ x_1 = f\ (\sigma_i\ x)\ \texttt{in} \\ \quad\quad \texttt{match}\ x_1\ \texttt{with} \\ \quad\quad \mid \sigma_1\ x_1 \to \sigma_1\ x_1 \\ \quad\quad \mid \sigma_2\ x_1 \to \sigma_2\ x_1 \end{array} \sim_{\text{alg}} \begin{array}{l} \texttt{match}\ x\ \texttt{with}\ \sigma_i\ x \to^i \\ \quad \texttt{let}\ x_1 = f\ (\sigma_i\ x)\ \texttt{in} \\ \quad\quad \texttt{match}\ x_1\ \texttt{with}\ \sigma_j\ x_1 \to^j \\ \quad\quad\quad \texttt{let}\ x_2 = f\ (\sigma_j\ x_1)\ \texttt{in} \\ \quad\quad\quad\quad \texttt{match}\ x_2\ \texttt{with}\ \sigma_k\ x_2 \to^k \\ \quad\quad\quad\quad\quad \texttt{let}\ x_3 = f\ (\sigma_k\ x_2)\ \texttt{in} \\ \quad\quad\quad\quad\quad\quad \texttt{match}\ x_3\ \texttt{with} \\ \quad\quad\quad\quad\quad\quad \mid \sigma_1\ x_3 \to \sigma_1\ x_3 \\ \quad\quad\quad\quad\quad\quad \mid \sigma_2\ x_3 \to \sigma_2\ x_3 \end{array}$$

$\mathbb{B}$

Equivalence: demo

$$\emptyset \vdash \lambda f.\, \lambda x.\, f\ x \approx_{\beta\eta} \lambda f.\, \lambda x.\, f\ (f\ (f\ x)) : (\mathbb{B} \to \mathbb{B}) \to (\mathbb{B} \to \mathbb{B})$$

$f : \mathbb{B} \to \mathbb{B}, x : \mathbb{B} \vdash_{\text{inv}}$

$$\begin{array}{l} \texttt{match}\ x\ \texttt{with}\ \sigma_i\ x \to^i \\ \quad \texttt{let}\ x_1 = f\ (\sigma_i\ x)\ \texttt{in} \\ \quad\quad \texttt{match}\ x_1\ \texttt{with} \\ \quad\quad \mid \sigma_1\ x_1 \to \sigma_1\ x_1 \\ \quad\quad \mid \sigma_2\ x_1 \to \sigma_2\ x_1 \end{array} \sim_{\text{alg}} \begin{array}{l} \texttt{match}\ x\ \texttt{with}\ \sigma_i\ x \to^i \\ \quad \texttt{let}\ x_1 = f\ (\sigma_i\ x)\ \texttt{in} \\ \quad\quad \texttt{match}\ x_1\ \texttt{with}\ \sigma_j\ x_1 \to^j \\ \quad\quad\quad \texttt{let}\ x_2 = f\ (\sigma_j\ x_1)\ \texttt{in} \\ \quad\quad\quad\quad \texttt{match}\ x_2\ \texttt{with}\ \sigma_k\ x_2 \to^k \\ \quad\quad\quad\quad\quad \texttt{let}\ x_3 = f\ (\sigma_k\ x_2)\ \texttt{in} \\ \quad\quad\quad\quad\quad\quad \texttt{match}\ x_3\ \texttt{with} \\ \quad\quad\quad\quad\quad\quad \mid \sigma_1\ x_3 \to \sigma_1\ x_3 \\ \quad\quad\quad\quad\quad\quad \mid \sigma_2\ x_3 \to \sigma_2\ x_3 \end{array}$$

$\mathbb{B}$

Equivalence: demo

$$\emptyset \vdash \lambda f.\, \lambda x.\, f\ x \approx_{\beta\eta} \lambda f.\, \lambda x.\, f\ (f\ (f\ x)) : (\mathbb{B} \to \mathbb{B}) \to (\mathbb{B} \to \mathbb{B})$$

$f : \mathbb{B} \to \mathbb{B}, x : \mathbb{B} \vdash_{\mathsf{inv}}$

$$\begin{array}{l} \texttt{match}\ x\ \texttt{with}\ \sigma_i\ x \to^i \\ \quad \texttt{let}\ x_1 = f\ (\sigma_i\ x)\ \texttt{in} \\ \quad\quad \texttt{match}\ x_1\ \texttt{with} \\ \quad\quad \mid \sigma_1\ x_1 \to \sigma_1\ x_1 \\ \quad\quad \mid \sigma_2\ x_1 \to \sigma_2\ x_1 \end{array} \quad \sim_{\mathsf{alg}} \quad \begin{array}{l} \texttt{match}\ x\ \texttt{with}\ \sigma_i\ x \to^i \\ \quad \texttt{let}\ x_1 = f\ (\sigma_i\ x)\ \texttt{in} \\ \quad\quad \texttt{match}\ x_1\ \texttt{with}\ \sigma_j\ x_1 \to^j \\ \quad\quad\quad \texttt{let}\ x_2 = f\ (\sigma_j\ x_1)\ \texttt{in} \\ \quad\quad\quad\quad \texttt{match}\ x_2\ \texttt{with}\ \sigma_k\ x_2 \to^k \\ \quad\quad\quad\quad\quad \texttt{let}\ x_3 = f\ (\sigma_k\ x_2)\ \texttt{in} \\ \quad\quad\quad\quad\quad\quad \texttt{match}\ x_3\ \texttt{with} \\ \quad\quad\quad\quad\quad\quad \mid \sigma_1\ x_3 \to \sigma_1\ x_3 \\ \quad\quad\quad\quad\quad\quad \mid \sigma_2\ x_3 \to \sigma_2\ x_3 \end{array}$$

$\mathbb{B}$

Equivalence: demo

$$\emptyset \vdash \lambda f.\, \lambda x.\, f\ x \approx_{\beta\eta} \lambda f.\, \lambda x.\, f\ (f\ (f\ x)) : (\mathbb{B} \to \mathbb{B}) \to (\mathbb{B} \to \mathbb{B})$$

$f : \mathbb{B} \to \mathbb{B}, x : \mathbb{B} \vdash_{\mathsf{inv}}$

$$\begin{array}{l} \texttt{match}\ x\ \texttt{with}\ \sigma_i\ x \to^i \\ \quad \texttt{let}\ x_1 = f\ (\sigma_i\ x)\ \texttt{in} \\ \quad \texttt{match}\ x_1\ \texttt{with} \\ \quad \mid \sigma_1\ x_1 \to \sigma_1\ x_1 \\ \quad \mid \sigma_2\ x_1 \to \sigma_2\ x_1 \end{array} \quad \sim_{\mathsf{alg}} \quad \begin{array}{l} \texttt{match}\ x\ \texttt{with}\ \sigma_i\ x \to^i \\ \quad \texttt{let}\ x_1 = f\ (\sigma_i\ x)\ \texttt{in} \\ \quad \texttt{match}\ x_1\ \texttt{with}\ \sigma_j\ x_1 \to^j \\ \quad \texttt{let}\ x_2 = f\ (\sigma_j\ x_1)\ \texttt{in} \\ \quad \texttt{match}\ x_2\ \texttt{with}\ \sigma_k\ x_2 \to^k \\ \quad \texttt{let}\ x_3 = f\ (\sigma_k\ x_2)\ \texttt{in} \\ \quad \texttt{match}\ x_3\ \texttt{with} \\ \quad \mid \sigma_1\ x_3 \to \sigma_1\ x_3 \\ \quad \mid \sigma_2\ x_3 \to \sigma_2\ x_3 \end{array}$$

$\mathbb{B}$

$f\ (f\ (f\ x)) = f\ x \quad \Longrightarrow \quad \mathsf{ok}$

Equivalence: demo

$$\emptyset \vdash \lambda f.\, \lambda x.\, f\ x \approx_{\beta\eta} \lambda f.\, \lambda x.\, f\ (f\ (f\ x)) : (\mathbb{B} \to \mathbb{B}) \to (\mathbb{B} \to \mathbb{B})$$

$f : \mathbb{B} \to \mathbb{B}, x : \mathbb{B} \vdash_{\mathsf{inv}}$

$$\begin{array}{l}\texttt{match}\ x\ \texttt{with}\ \sigma_i\ x \to^i \\ \quad \texttt{let}\ x_1 = f\ (\sigma_i\ x)\ \texttt{in} \\ \quad\quad \texttt{match}\ x_1\ \texttt{with} \\ \quad\quad \mid \sigma_1\ x_1 \to \sigma_1\ x_1 \\ \quad\quad \mid \sigma_2\ x_1 \to \sigma_2\ x_1\end{array} \quad \sim_{\mathsf{alg}} \quad \begin{array}{l}\texttt{match}\ x\ \texttt{with}\ \sigma_i\ x \to^i \\ \quad \texttt{let}\ x_1 = f\ (\sigma_i\ x)\ \texttt{in} \\ \quad\quad \texttt{match}\ x_1\ \texttt{with}\ \sigma_j\ x_1 \to^j \\ \quad\quad\quad \texttt{let}\ x_2 = f\ (\sigma_j\ x_1)\ \texttt{in} \\ \quad\quad\quad\quad \texttt{match}\ x_2\ \texttt{with}\ \sigma_k\ x_2 \to^k \\ \quad\quad\quad\quad\quad \texttt{let}\ x_3 = f\ (\sigma_k\ x_2)\ \texttt{in} \\ \quad\quad\quad\quad\quad\quad \texttt{match}\ x_1\ \texttt{with} \\ \quad\quad\quad\quad\quad\quad \mid \sigma_1\ x_1 \to \sigma_1\ x_1 \\ \quad\quad\quad\quad\quad\quad \mid \sigma_2\ x_1 \to \sigma_2\ x_1\end{array}$$

$\mathbb{B}$

$f\ (f\ (f\ x)) = f\ x \quad \Longrightarrow \quad$ ok

Equivalence: demo

$$\emptyset \vdash \lambda f.\, \lambda x.\, f\ x \approx_{\beta\eta} \lambda f.\, \lambda x.\, f\ (f\ (f\ x)) : (\mathbb{B} \to \mathbb{B}) \to (\mathbb{B} \to \mathbb{B})$$

$f : \mathbb{B} \to \mathbb{B}, x : \mathbb{B} \vdash_{\mathsf{inv}}$

$$
\begin{array}{l}
\texttt{match}\ x\ \texttt{with}\ \sigma_i\ x \to^i \\
\quad \texttt{let}\ x_1 = f\ (\sigma_i\ x)\ \texttt{in} \\
\quad\quad \texttt{match}\ x_1\ \texttt{with} \\
\quad\quad \mid \sigma_1\ x_1 \to \sigma_1\ x_1 \\
\quad\quad \mid \sigma_2\ x_1 \to \sigma_2\ x_1
\end{array}
\quad \sim_{\mathsf{alg}} \quad
\begin{array}{l}
\texttt{match}\ x\ \texttt{with}\ \sigma_i\ x \to^i \\
\quad \texttt{let}\ x_1 = f\ (\sigma_i\ x)\ \texttt{in} \\
\quad\quad \texttt{match}\ x_1\ \texttt{with}\ \sigma_j\ x_1 \to^j \\
\quad\quad\quad \texttt{let}\ x_2 = f\ (\sigma_j\ x_1)\ \texttt{in} \\
\quad\quad\quad\quad \texttt{match}\ x_2\ \texttt{with}\ \sigma_k\ x_2 \to^k \\
\quad\quad\quad\quad\quad \texttt{let}\ x_3 = f\ (\sigma_k\ x_2)\ \texttt{in} \\
\quad\quad\quad\quad\quad\quad \texttt{match}\ x_3\ \texttt{with} \\
\quad\quad\quad\quad\quad\quad \mid \sigma_1\ x_3 \to \sigma_1\ x_3 \\
\quad\quad\quad\quad\quad\quad \mid \sigma_2\ x_3 \to \sigma_2\ x_3
\end{array}
$$

$\mathbb{B}$

$f\ (f\ x) = f\ x \quad \Longrightarrow \quad f\ (f\ (f\ x)) = f\ (f\ x) = f\ x \quad \Longrightarrow \quad$ ok

Equivalence: demo

$$\emptyset \vdash \lambda f.\, \lambda x.\, f\ x \approx_{\beta\eta} \lambda f.\, \lambda x.\, f\ (f\ (f\ x)) : (\mathbb{B} \to \mathbb{B}) \to (\mathbb{B} \to \mathbb{B})$$

$f : \mathbb{B} \to \mathbb{B}, x : \mathbb{B} \vdash_{\mathsf{inv}}$

$$\begin{array}{l}
\texttt{match}\ x\ \texttt{with}\ \sigma_i\ x \to^i \\
\quad \texttt{let}\ x_1 = f\ (\sigma_i\ x)\ \texttt{in} \\
\quad\quad \texttt{match}\ x_1\ \texttt{with} \\
\quad\quad \mid \sigma_1\ x_1 \to \sigma_1\ x_1 \\
\quad\quad \mid \sigma_2\ x_1 \to \sigma_2\ x_1
\end{array}
\quad \sim_{\mathsf{alg}} \quad
\begin{array}{l}
\texttt{match}\ x\ \texttt{with}\ \sigma_i\ x \to^i \\
\quad \texttt{let}\ x_1 = f\ (\sigma_i\ x)\ \texttt{in} \\
\quad\quad \texttt{match}\ x_1\ \texttt{with}\ \sigma_j\ x_1 \to^j \\
\quad\quad\quad \texttt{let}\ x_2 = f\ (\sigma_j\ x_1)\ \texttt{in} \\
\quad\quad\quad\quad \texttt{match}\ x_1\ \texttt{with}\ \sigma_k\ x_1 \to^k \\
\quad\quad\quad\quad\quad \texttt{let}\ x_3 = f\ (\sigma_k\ x_1)\ \texttt{in} \\
\quad\quad\quad\quad\quad\quad \texttt{match}\ x_3\ \texttt{with} \\
\quad\quad\quad\quad\quad\quad \mid \sigma_1\ x_3 \to \sigma_1\ x_3 \\
\quad\quad\quad\quad\quad\quad \mid \sigma_2\ x_3 \to \sigma_2\ x_3
\end{array}$$

$\mathbb{B}$

$$f\ (f\ x) = f\ x \quad \Longrightarrow \quad f\ (f\ (f\ x)) = f\ (f\ x) = f\ x \quad \Longrightarrow \quad \text{ok}$$

Equivalence: demo

$$\emptyset \vdash \lambda f.\, \lambda x.\, f\ x \approx_{\beta\eta} \lambda f.\, \lambda x.\, f\ (f\ (f\ x)) : (\mathbb{B} \to \mathbb{B}) \to (\mathbb{B} \to \mathbb{B})$$

$f : \mathbb{B} \to \mathbb{B}, x : \mathbb{B} \vdash_{\mathsf{inv}}$

$$\begin{array}{l} \texttt{match}\ x\ \texttt{with}\ \sigma_i\ x \to^i \\ \quad \texttt{let}\ x_1 = f\ (\sigma_i\ x)\ \texttt{in} \\ \quad\quad \texttt{match}\ x_1\ \texttt{with} \\ \quad\quad \mid \sigma_1\ x_1 \to \sigma_1\ x_1 \\ \quad\quad \mid \sigma_2\ x_1 \to \sigma_2\ x_1 \end{array} \quad \sim_{\mathsf{alg}} \quad \begin{array}{l} \texttt{match}\ x\ \texttt{with}\ \sigma_i\ x \to^i \\ \quad \texttt{let}\ x_1 = f\ (\sigma_i\ x)\ \texttt{in} \\ \quad\quad \texttt{match}\ x_1\ \texttt{with}\ \sigma_j\ x_1 \to^j \\ \quad\quad\quad \texttt{let}\ x_2 = f\ (\sigma_j\ x_1)\ \texttt{in} \\ \quad\quad\quad\quad \texttt{match}\ x_1\ \texttt{with}\ \sigma_k\ x_1 \to^k \\ \quad\quad\quad\quad\quad \texttt{let}\ x_3 = f\ (\sigma_k\ x_1)\ \texttt{in} \\ \quad\quad\quad\quad\quad\quad \texttt{match}\ x_3\ \texttt{with} \\ \quad\quad\quad\quad\quad\quad \mid \sigma_1\ x_3 \to \sigma_1\ x_3 \\ \quad\quad\quad\quad\quad\quad \mid \sigma_2\ x_3 \to \sigma_2\ x_3 \end{array}$$

$\mathbb{B}$

$$f\ (f\ x) = f\ x \implies f\ (f\ (f\ x)) = f\ (f\ x) = f\ x \implies \text{ok}$$

Equivalence: summary

Key insight: looking for all neutrals, *intra et extra muros*.

Addition of 0 is conceptually very simple once the right point of view is in place. It does not complicate the proofs.

Correctness: immediate.

Completeness: maximal multi-focusing (WIP, will be in my thesis).

Termination: easy on the focused structure.

Obstacles to older approaches

Trying to decide equivalence by "small-step" rewriting of pieces of programs (or in fact any method relying only on comparing the two terms) does not scale to 0.
Neil Ghani, who first solved the sum case (without 0) in his 1995 PhD thesis, had the right intuition:

> *The η_0-reducts of a term are determined more by the consistency of the context in which the term was typed, rather than the term itself. [...] Context inconsistency, term typability and other important issues in the study of the η_0-rewrite rule are decidable [in our setting].*

What was lacking was the framework to make this **easy**.

Obstacle to direct approaches

(A realist pessimist way to say: "Future work")

Can we do without the intermediate focusing step? "Head focusing" on the fly?

Obstacle to direct approaches

(A realist pessimist way to say: "Future work")

Can we do without the intermediate focusing step? "Head focusing" on the fly?

Obstacle 1: current-context neutrals may be "blocked" by future-context neutrals: $\pi_1\ (\texttt{match}\ n\ \texttt{with}\ \mid \sigma_1\, x \to y \mid \sigma_2\, x \to y)$
Need: rewriting/extrusion choices in context, à la "constrained environments" [Marcelo Fiore, Alex Simpson].

Obstacle to direct approaches

(A realist pessimist way to say: "Future work")

Can we do without the intermediate focusing step? "Head focusing" on the fly?

Obstacle 1: current-context neutrals may be "blocked" by future-context neutrals: $\pi_1\ (\texttt{match}\ n\ \texttt{with}\ \mid \sigma_1\, x \to y \mid \sigma_2\, x \to y)$
Need: rewriting/extrusion choices in context, à la "constrained environments" [Marcelo Fiore, Alex Simpson].

Obstacle 2: decisions may need to be synchronized between branches.
Need: "it depends" algorithm with choice propagation, [Vincent Balat, Roberto Di Cosmo] use delimited control for this.

Obstacle to direct approaches

(A realist pessimist way to say: "Future work")

Can we do without the intermediate focusing step? "Head focusing" on the fly?

Obstacle 1: current-context neutrals may be "blocked" by future-context neutrals: $\pi_1\ (\texttt{match}\ n\ \texttt{with}\ \mid \sigma_1\, x \to y \mid \sigma_2\, x \to y)$
Need: rewriting/extrusion choices in context, à la "constrained environments" [Marcelo Fiore, Alex Simpson].

Obstacle 2: decisions may need to be synchronized between branches.
Need: "it depends" algorithm with choice propagation, [Vincent Balat, Roberto Di Cosmo] use delimited control for this.

Not so simple anymore... **Questions?**