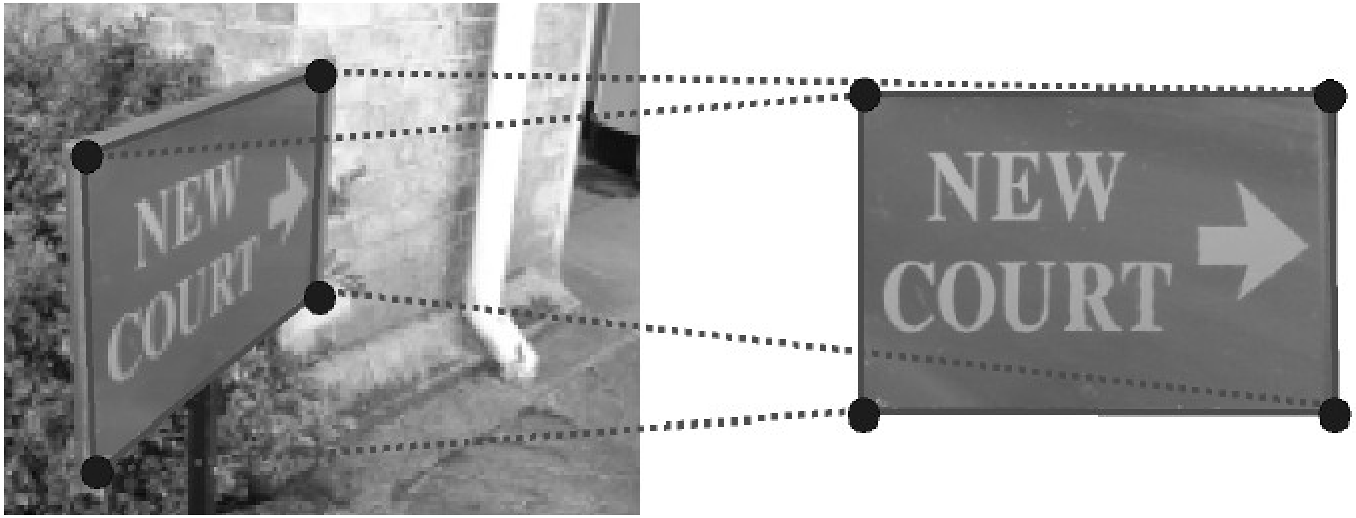
INF555



Fundamentals of 3D

Lecture 4:

Debriefing: ICP (kD-trees)
Homography
Graphics pipeline

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ICP: Algorithm at a glance

- Start from a not too far initial transformation

Do iterations until the mismatch error goes below a threshold:
- Match the point of the target to the source
- Compute the best transformation from point correspondence

In practice, this is a **very fast** registration method...

A Method for Registration of 3-D Shapes.Paul J *Besl*, Neil D Mckay.
IEEE Trans. Pattern Anal. Mach. Intell., Vol. 14, No. 2. (February 1992

ICP: Finding the best rigid transformation

Given point correspondences, find the best rigid transformation.

$$X = \{x_1, ..., x_n\}$$

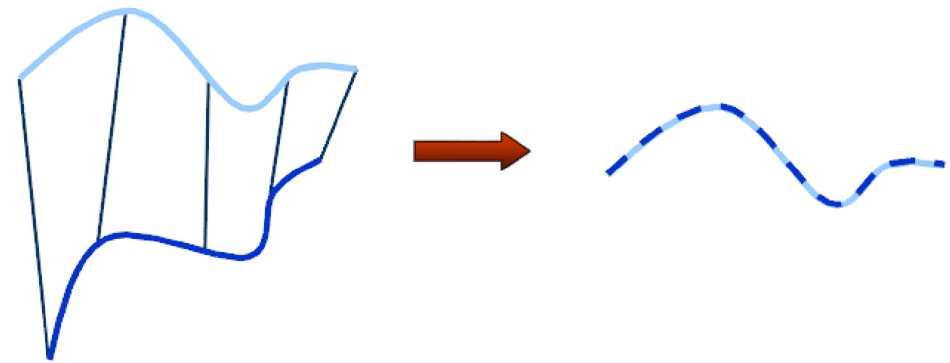
Observation/Target

$$P = \{p_1, ..., p_n\}$$

Source/Model

Find (R,t) that minimizes the squared euclidean error:

$$E(R,t) = \frac{1}{N_p} \sum_{i=1}^{N_p} ||x_i - Rp_i - t||^2$$



Align the center of mass of sets:

$$\mu_x = \frac{1}{N_x}\sum_{i=1}^{N_x} x_i \quad \text{and} \quad \mu_p = \frac{1}{N_p}\sum_{i=1}^{N_p} p_i$$

$$X = \{x_1, ..., x_n\}$$
$$P = \{p_1, ..., p_n\}$$

$$X' = \{x_i - \mu_x\} = \{x'_i\}$$
$$P' = \{p_i - \mu_p\} = \{p'_i\}$$

Finding the rotation matrix:

$$W = \sum_{i=1}^{N_p} x'_i p'^T_i$$

Compute the singular value decomposition

$$W = U \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} V^T$$

$$\sigma_1 \geq \sigma_2 \geq \sigma_3$$

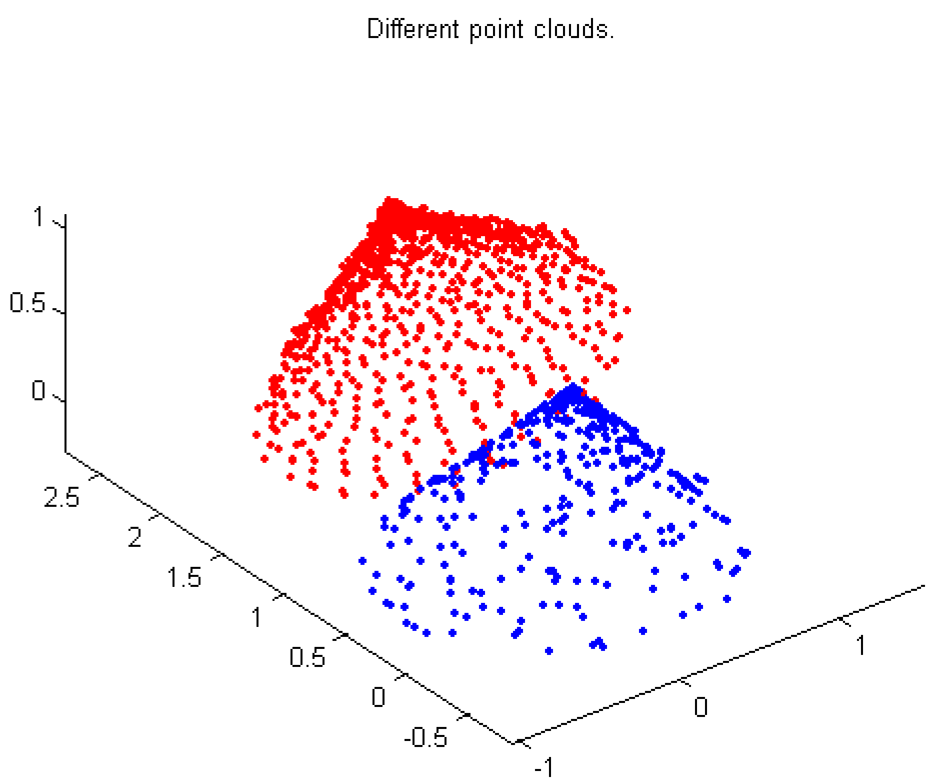
Optimal transformation:

$$R = UV^T$$
$$t = \mu_x - R\mu_p$$

ICP: Monotonicity and convergence

The average squared Euclidean distance decreases monotonously

In fact:
Each correspondence pair distance decreases



Drawback:
When does the local minimum is global?

Difficult to handle symmetry

(use texture, etc.)

Best 3D transformation (with quaternions)

With respect to least squares...

SVD take into account reflections...

$\vec{q}_R = [q_0 q_1 q_2 q_3]^t$ $\vec{q}_T = [q_4 q_5 q_6]^t$ $\vec{q} = [\vec{q}_R | \vec{q}_T]^t$

$$\boldsymbol{R} = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1 q_2 - q_0 q_3) & 2(q_1 q_3 + q_0 q_2) \\ 2(q_1 q_2 + q_0 q_3) & q_0^2 + q_2^2 - q_1^2 - q_3^2 & 2(q_2 q_3 - q_0 q_1) \\ 2(q_1 q_3 - q_0 q_2) & 2(q_2 q_3 + q_0 q_1) & q_0^2 + q_3^2 - q_1^2 - q_2^2 \end{bmatrix}$$

$$f(\vec{q}\,) = \frac{1}{N_p} \sum_{i=1}^{N_p} \| \vec{x}_i - \boldsymbol{R}(\vec{q}_R)\vec{p}_i - \vec{q}_T \|^2$$

Best 3D transformation (with quaternions)

$$\vec{\mu}_p = \frac{1}{N_p}\sum_{i=1}^{N_p}\vec{p}_i \quad \text{and} \quad \vec{\mu}_x = \frac{1}{N_x}\sum_{i=1}^{N_x}\vec{x}_i$$

Cross-covariance matrix:

$$\Sigma_{px} = \frac{1}{N_p}\sum_{i=1}^{N_p}[(\vec{p}_i - \vec{\mu}_p)(\vec{x}_i - \vec{\mu}_x)^t] = \frac{1}{N_p}\sum_{i=1}^{N_p}[\vec{p}_i\vec{x}_i^t] - \vec{\mu}_p\vec{\mu}_x^{\,t}.$$

$$A_{ij} = (\Sigma_{px} - \Sigma_{px}^T)_{ij}$$

Anti-symmetric matrix

$$\Delta = [A_{23} \quad A_{31} \quad A_{12}]^T$$

$$Q(\Sigma_{px}) = \begin{bmatrix} \mathrm{tr}(\Sigma_{px}) & \Delta^T \\ \Delta & \Sigma_{px} + \Sigma_{px}^T - \mathrm{tr}(\Sigma_{px})\boldsymbol{I}_3 \end{bmatrix}$$

Best 3D transformation (with quaternions)

$$Q(\Sigma_{px}) = \begin{bmatrix} \operatorname{tr}(\Sigma_{px}) & \Delta^T \\ \Delta & \Sigma_{px} + \Sigma_{px}^T - \operatorname{tr}(\Sigma_{px})\boldsymbol{I}_3 \end{bmatrix}$$

Take the unit eigenvector corresponding to the maximal eigenvalue:

$$\vec{q_R} = [q_0 \quad q_1 \quad q_2 \quad q_3]^t$$

Get the remaining translation as:

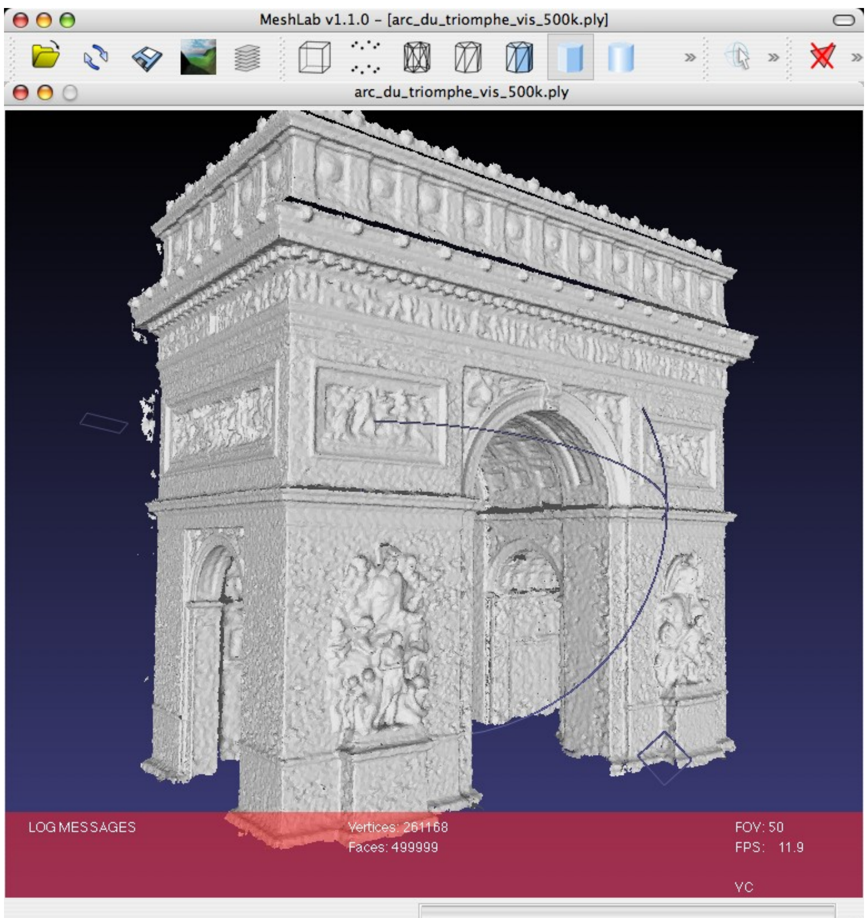
$$\vec{q}_T = \vec{\mu}_x - \boldsymbol{R}(\vec{q}_R)\vec{\mu}_p.$$

Example for curve registrations:

Time complexity of ICP

Linear (fixed dimension) to find least square transformation
At each iteration, perform n nearest neighbor queries

Naive implementation: O(I*n*n) => slow for large n



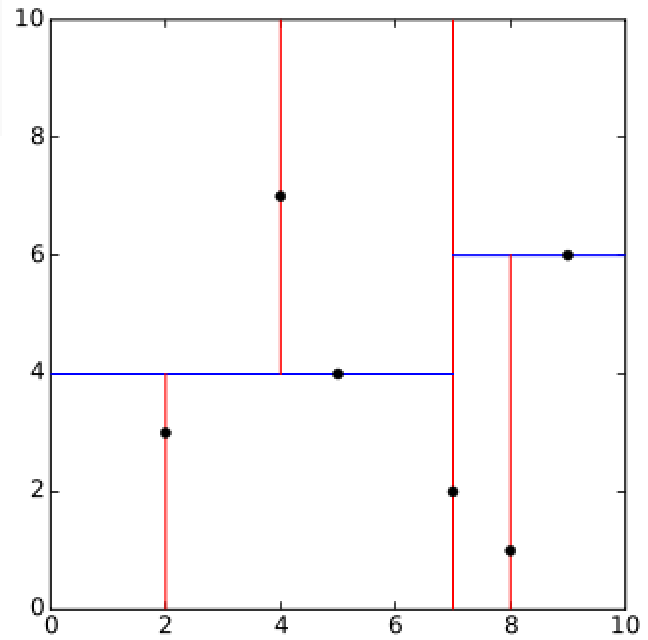
http://meshlab.sourceforge.net/

kD-trees for fast NN queries

```
function kdtree (list of points pointList, int depth)
{
    if pointList is empty
        return nil;
    else
    {
        // Select axis based on depth so that axis cycles through all valid values
        var int axis := depth mod k;

        // Sort point list and choose median as pivot element
        select median from pointList;

        // Create node and construct subtrees
        var tree_node node;
        node.location := median;
        node.leftChild := kdtree(points in pointList before median, depth+1);
        node.rightChild := kdtree(points in pointList after median, depth+1);
        return node;
    }
}
```



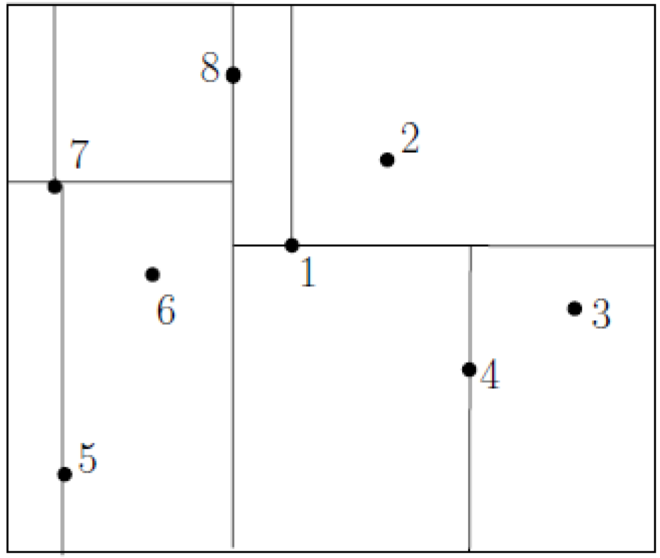
Nearest neighbor (NN) queries in small dimensions...

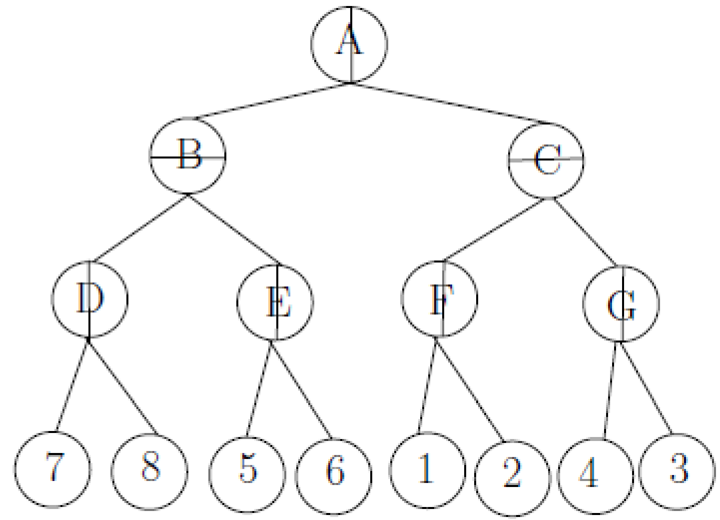
http://en.wikipedia.org/wiki/Kd-tree

KDTREE($\mathcal{P}, l$)

◁ Build a 2D kD-tree ▷
◁ l denote the level. Initially, $l = 0$ ▷
if $|\mathcal{P}| = 1$
    then return LEAF($\mathcal{P}$)
    else **if** Even(l)
        then ◁ Compute the median x-abscissa (vertical split) ▷
            $x_l = \text{MEDIANX}(\mathcal{P})$
            $\mathcal{P}_{\text{left}} = \{\mathbf{p} \in \mathcal{P} \mid x(\mathbf{p}) \leq x_l\}$
            $\mathcal{P}_{\text{right}} = \{\mathbf{p} \in \mathcal{P} \mid x(\mathbf{p}) > x_l\}$
            return TREE(x_l, KDTREE($\mathcal{P}_{\text{left}}, l+1$), KDTREE($\mathcal{P}_{\text{right}}, l+1$));
        else ◁ Compute the median y-abscissa (horizontal split) ▷
            $y_l = \text{MEDIANY}(\mathcal{P})$
            $\mathcal{P}_{\text{bottom}} = \{\mathbf{p} \in \mathcal{P} \mid y(\mathbf{p}) \leq y_l\}$
            $\mathcal{P}_{\text{top}} = \{\mathbf{p} \in \mathcal{P} \mid y(\mathbf{p}) > y_l\}$
            return TREE(y_l, KDTREE($\mathcal{P}_{\text{bottom}}, l+1$), KDTREE($\mathcal{P}_{\text{top}}, l+1$));

Build time: O(dn log n) with O(dn) memory


8
7
2
1
6
3
4
5


A
B
C
D
E
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G
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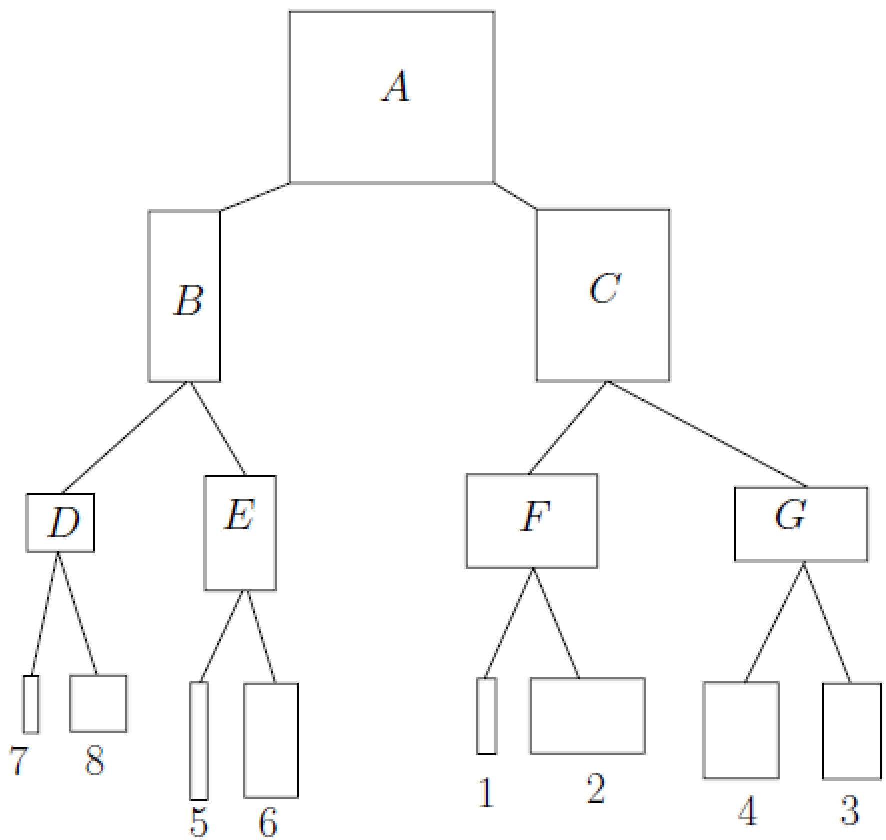
8
5
6
1
2
4
3


A
B
C
D
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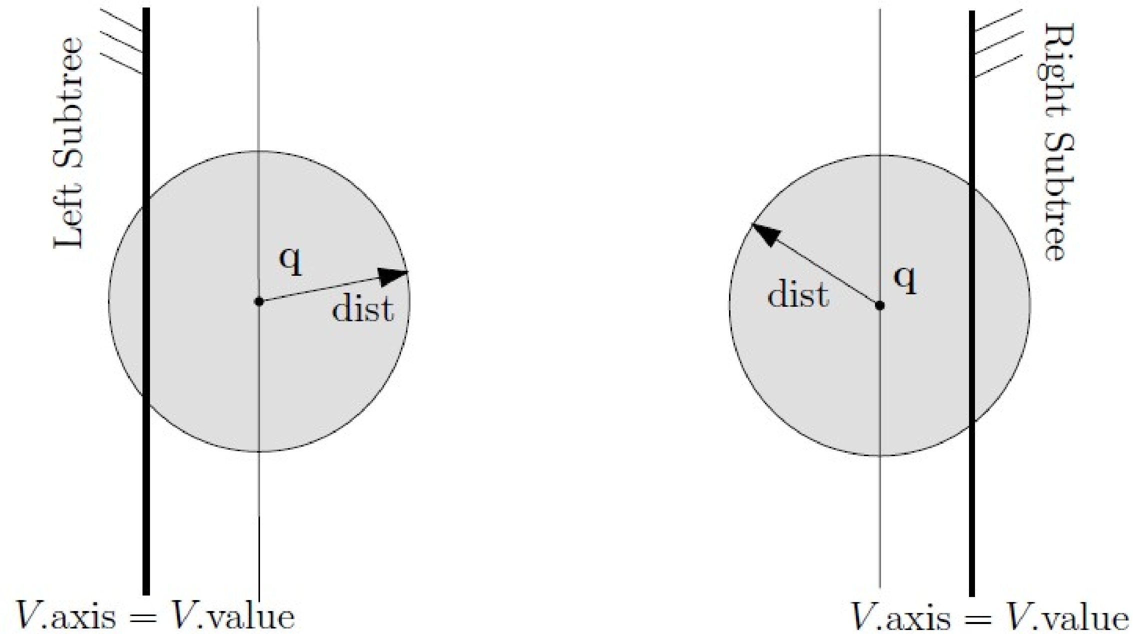
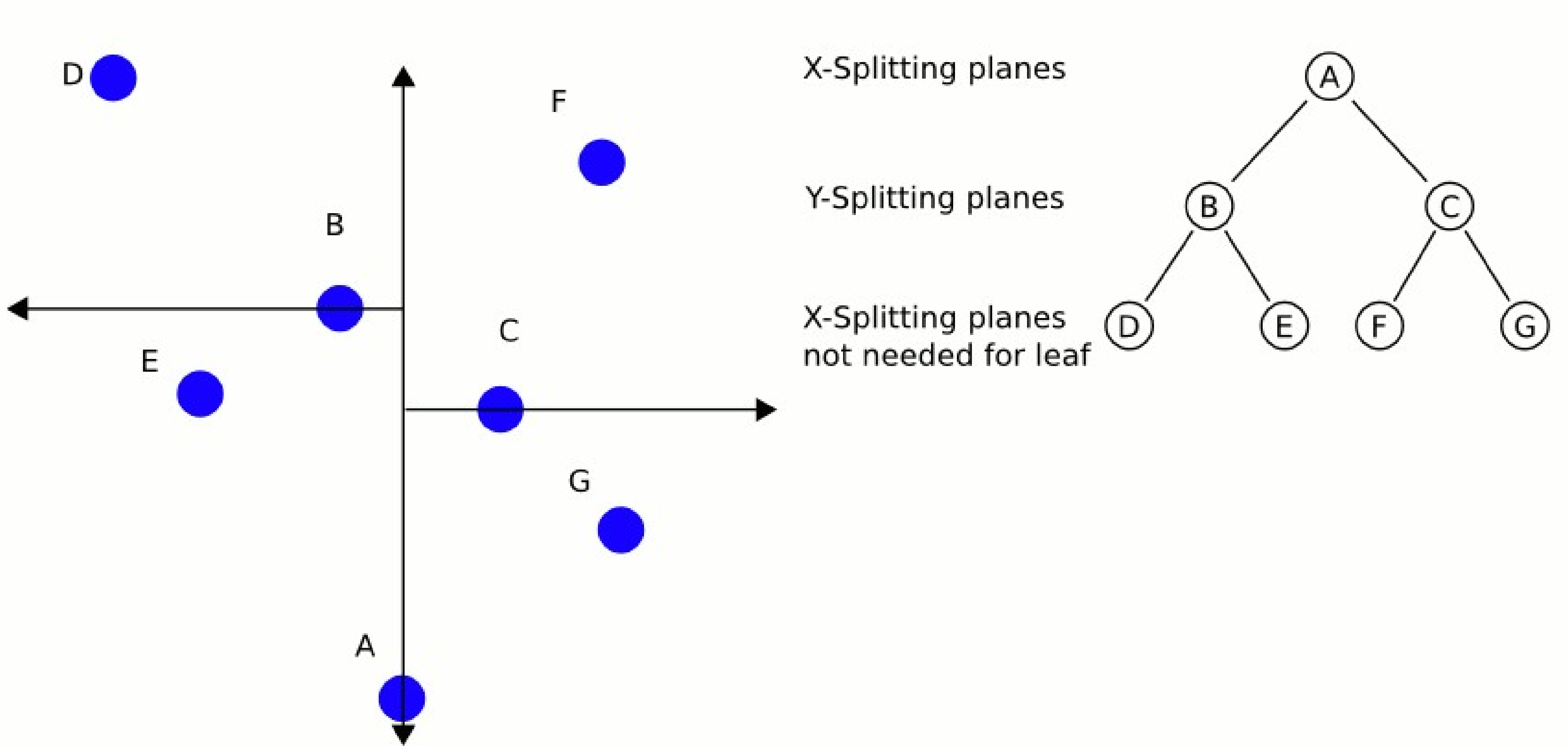
7
8
5
6
1
2
4
3

```
SearchNNinKDTree(q, V; p, dist)
◁ Input: ▷
◁ V: a kD-Tree node ▷
◁ q: a query point ▷
◁ Output: ▷
◁ p: nearest neighbor point ▷
◁ dist: distance to the nearest neighbor ▷
if V.left = V.right = NULL
   then ◁ Leaf of a kD-Tree ▷
         dist' = ||q − V.point||
         if dist' < dist
            then dist = dist'
                 p = V.point
   else  if q_{V.axis} ≤ V.value
            then ◁ Search on the left subtree first ▷
                 SearchNNinKDTree(q, V.left; p, dist)
                 if q_{V.axis} + dist > V.value
                    then SearchNNinKDTree(q, V.right; p, dist)
            else ◁ Search on the right subtree first ▷
                 SearchNNinKDTree(q, V.right; p, dist)
                 if q_{V.axis} − dist ≤ V.value
                    then SearchNNinKDTree(q, V.left; p, dist)
```



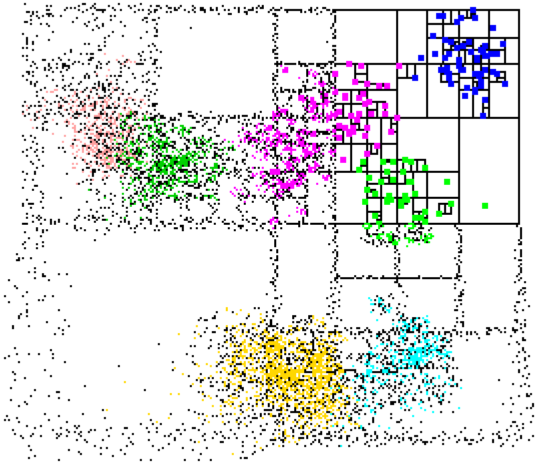
Query complexity: From O(dlog n) to O(dn)

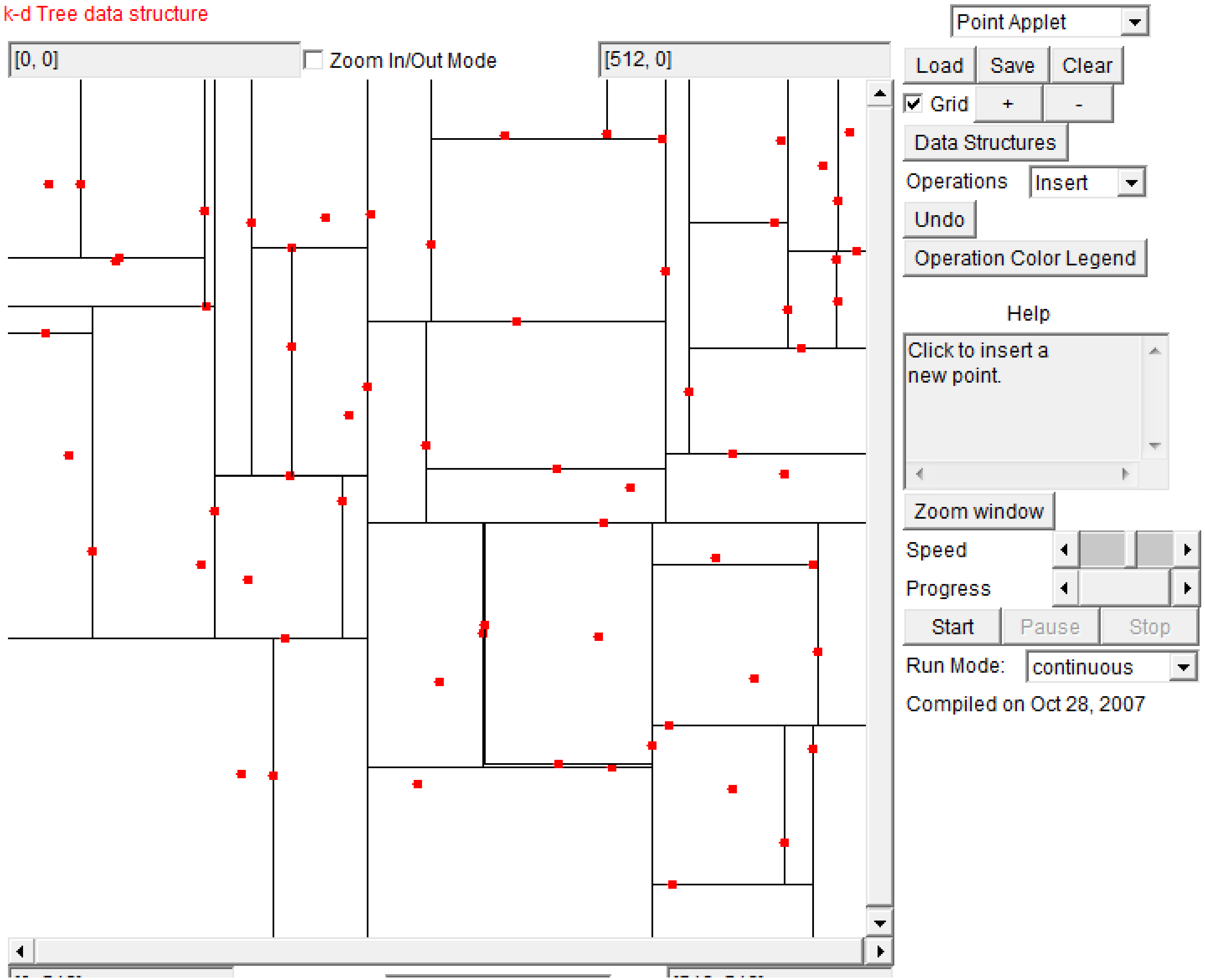
kD-trees for fast NN queries



Kd-Trees are extremely useful data-structures (many applications)

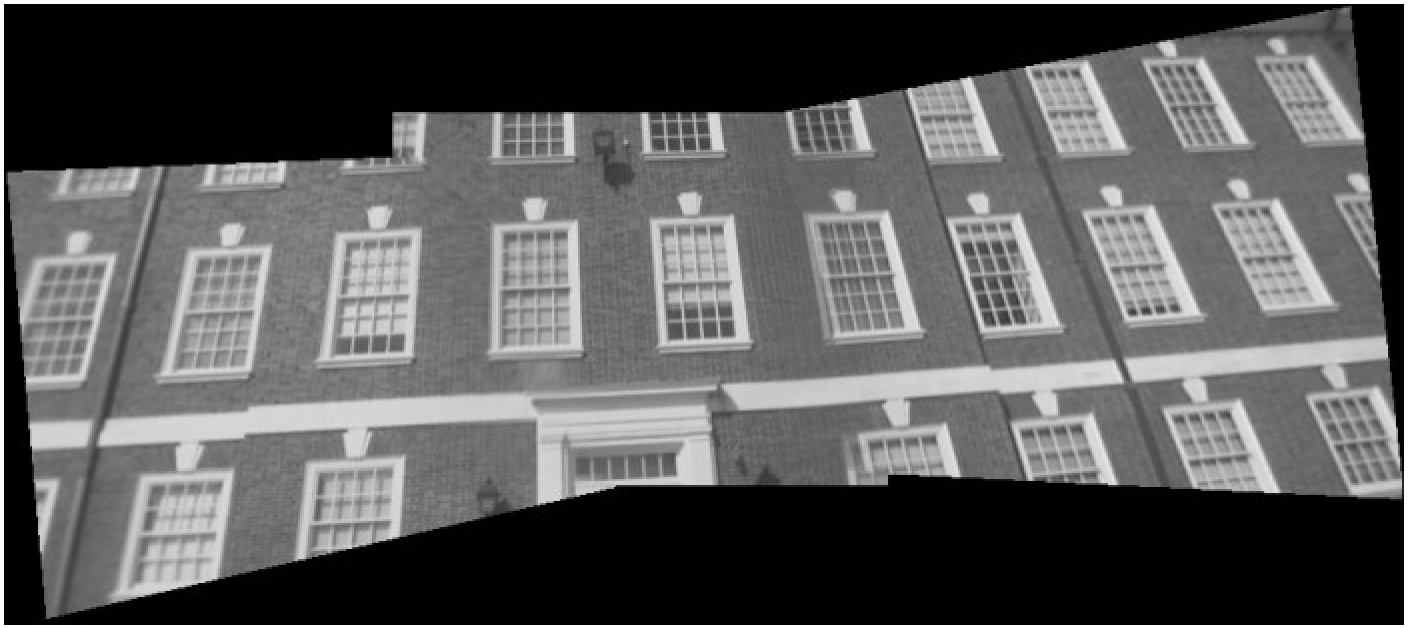
But also: Approximate nearest neighbors
http://www.cs.umd.edu/~mount/ANN/

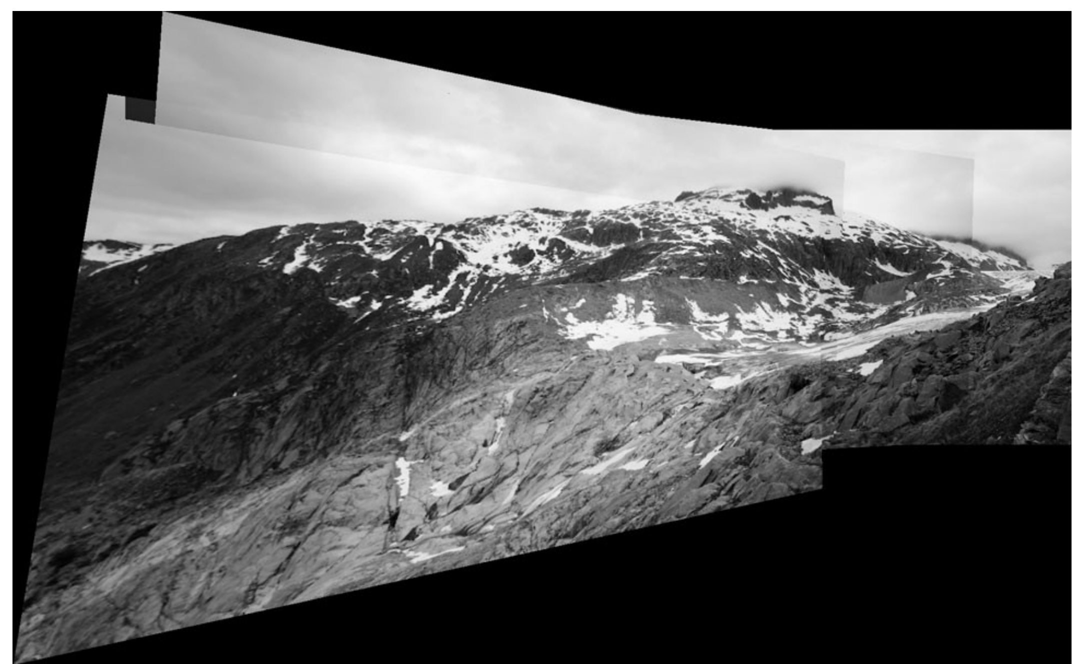




http://donar.umiacs.umd.edu/quadtree/points/kdtree.html

Homography (Collineation)





Homography (Collineation)

$$\mathbf{r_i} = \begin{bmatrix} x'_i \\ y'_i \\ w'_i \end{bmatrix} = \mathbf{Hl_i} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ w_i \end{bmatrix}$$

$$\mathbf{l_i} = \begin{bmatrix} \frac{x_i}{w_i} & \frac{y_i}{w_i} \end{bmatrix}^T \qquad \mathbf{r}_i = \begin{bmatrix} \frac{x'_i}{w'_i} & \frac{y'_i}{w'_i} \end{bmatrix}^T$$

Assuming h33 is not zero, set it to 1 and get:

$$x'_i = \frac{h_{11}x_i + h_{12}y_i + h_{13}}{h_{31}x_i + h_{32}y_i + 1},$$

$$y'_i = \frac{h_{21}x_i + h_{22}y_i + h_{23}}{h_{31}x_i + h_{32}y_i + 1}.$$

Homography (Collineation)

$$x_i' = \frac{h_{11}x_i + h_{12}y_i + h_{13}}{h_{31}x_i + h_{32}y_i + 1},$$

$$y_i' = \frac{h_{21}x_i + h_{22}y_i + h_{23}}{h_{31}x_i + h_{32}y_i + 1}.$$

$$x_i' = h_{11}x_i + h_{12}y_i + h_{13} - x_i'(h_{31}x_i + h_{32}y_i),$$

$$y_i' = h_{21}x_i + h_{22}y_i + h_{23} - y_i'(h_{31}x_i + h_{32}y_i).$$

Homography (Collineation)

From 4 pairs of point correspondences:

$$\begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 & -x_1x'_1 & -y_1x'_1 \\ 0 & 0 & 0 & x_1 & y_1 & 1 & -x_1y'_1 & -y_1y'_1 \\ x_2 & y_2 & 1 & 0 & 0 & 0 & -x_2x'_2 & -y_2x'_2 \\ 0 & 0 & 0 & x_2 & y_2 & 1 & -x_2y'_2 & -y_2y'_2 \\ x_3 & y_3 & 1 & 0 & 0 & 0 & -x_3x'_3 & -y_3x'_3 \\ 0 & 0 & 0 & x_3 & y_3 & 1 & -x_3y'_3 & -y_3y'_3 \\ x_4 & y_4 & 1 & 0 & 0 & 0 & -x_4x'_4 & -y_4x'_4 \\ 0 & 0 & 0 & x_4 & y_4 & 1 & -x_4y'_4 & -y_4y'_4 \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \end{bmatrix} = \begin{bmatrix} x'_1 \\ y'_1 \\ x'_2 \\ y'_2 \\ x'_3 \\ y'_3 \\ x'_4 \\ y'_4 \end{bmatrix}$$

$$\underbrace{\mathbf{A}}_{8\times 8} \times \underbrace{\mathbf{h}}_{8\times 1} = \underbrace{\mathbf{b}}_{8\times 1}.$$

$$\mathbf{Ah} = \mathbf{b} \Longrightarrow \qquad \mathbf{h} = \mathbf{A}^{-1}\mathbf{b}.$$

Homography (Collineation)

From n pairs of point correspondences:

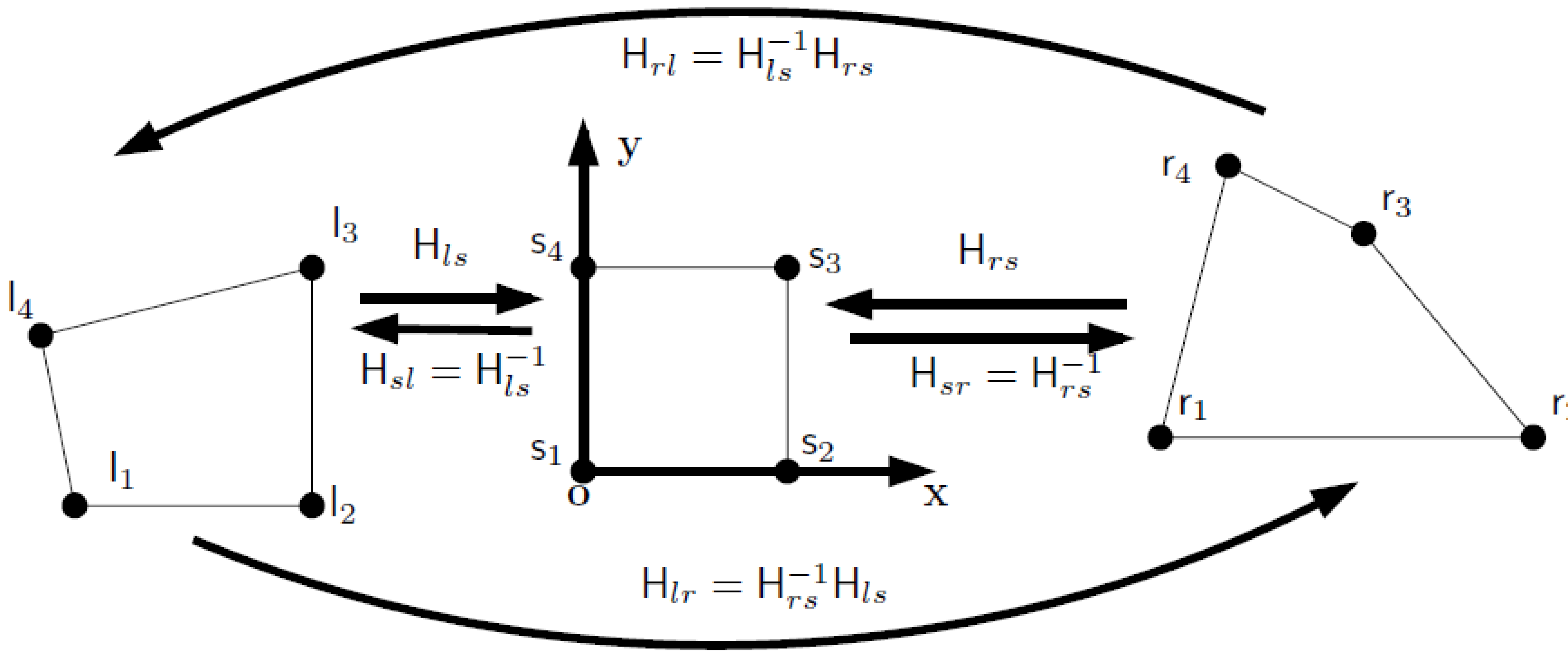
$$\begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 & -x_1x'_1 & -y_1x'_1 \\ 0 & 0 & 0 & x_1 & y_1 & 1 & -x_1y'_1 & -y_1y'_1 \\ x_2 & y_2 & 1 & 0 & 0 & 0 & -x_2x'_2 & -y_2x'_2 \\ 0 & 0 & 0 & x_2 & y_2 & 1 & -x_2y'_2 & -y_2y'_2 \\ x_3 & y_3 & 1 & 0 & 0 & 0 & -x_3x'_3 & -y_3x'_3 \\ 0 & 0 & 0 & x_3 & y_3 & 1 & -x_3y'_3 & -y_3y'_3 \\ x_4 & y_4 & 1 & 0 & 0 & 0 & -x_4x'_4 & -y_4x'_4 \\ 0 & 0 & 0 & x_4 & y_4 & 1 & -x_4y'_4 & -y_4y'_4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_n & y_n & 1 & 0 & 0 & 0 & -x_nx'_n & -y_nx'_n \\ 0 & 0 & 0 & x_n & y_n & 1 & -x_ny'_n & -y_ny'_n \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \end{bmatrix} = \begin{bmatrix} x'_1 \\ y'_1 \\ x'_2 \\ y'_2 \\ x'_3 \\ y'_3 \\ x'_4 \\ y'_4 \\ \vdots \\ x'_n \\ y'_n \end{bmatrix},$$

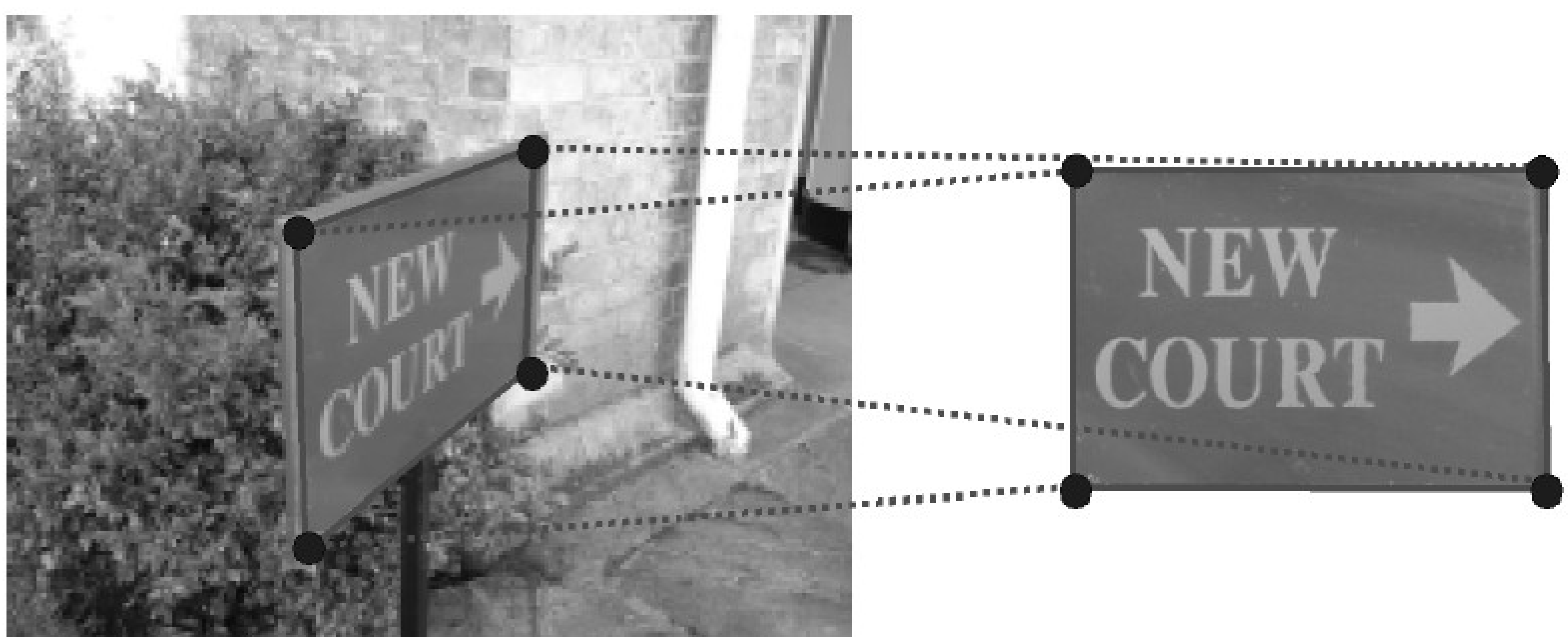
$$\underbrace{\mathbf{A}}_{2n\times 8} \times \underbrace{\mathbf{h}}_{8\times 1} = \underbrace{\mathbf{b}}_{2n\times 1}$$

$$\mathbf{A}^+ = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$$

$$\mathbf{h} = \mathbf{A}^+\mathbf{b}$$

Matrix pseudo-inverse


$H_{rl} = H_{ls}^{-1} H_{rs}$
y
l_3
l_4
l_1
l_2
H_{ls}
$H_{sl} = H_{ls}^{-1}$
s_4
s_3
s_1
s_2
o
x
H_{rs}
$H_{sr} = H_{rs}^{-1}$
r_4
r_3
r_1
r_2
$H_{lr} = H_{rs}^{-1} H_{ls}$


NEW COURT
NEW COURT

Homography (Collineation)

Matching planar surfaces...

(a) (b)

Homography (Collineation)

Matching perspective pictures acquired from the same nodal point

(a)

(b)

(c)

(d)

(e)

Homography (Collineation): Projective geometry

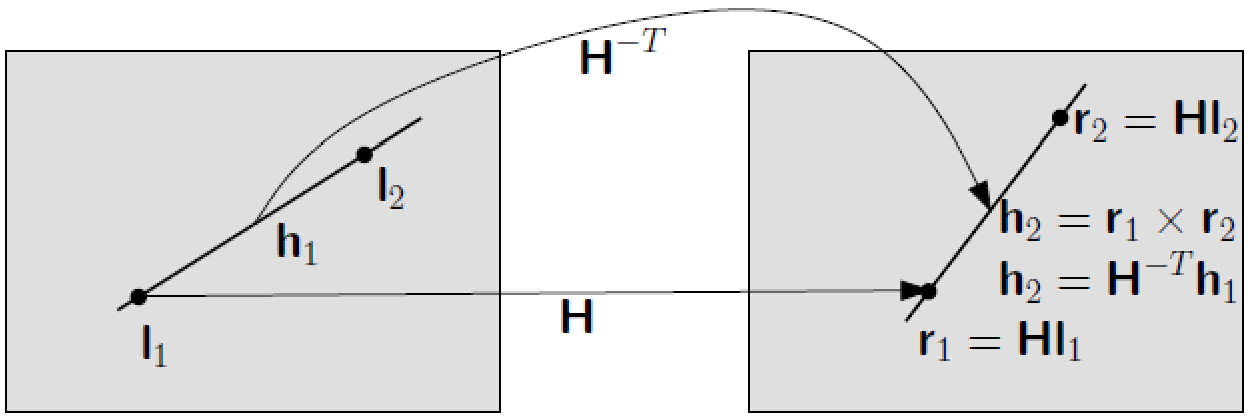
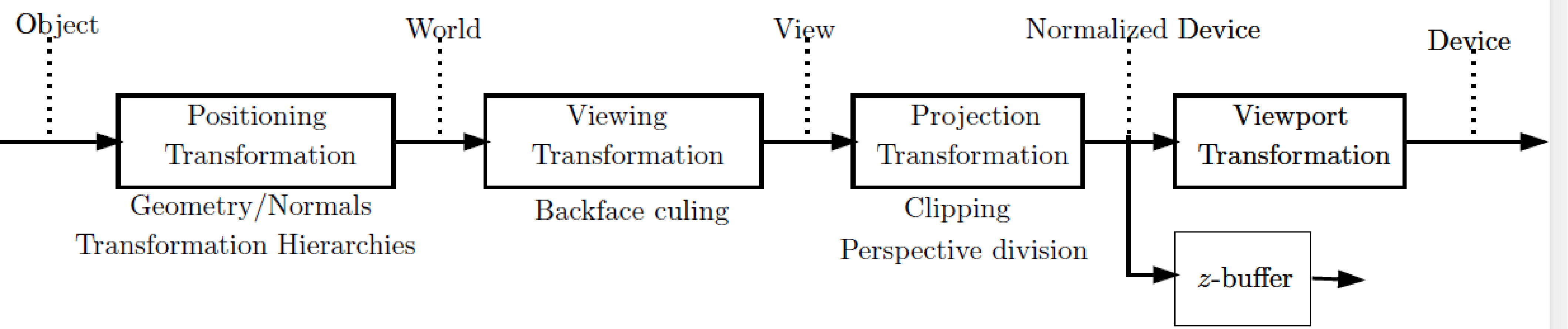
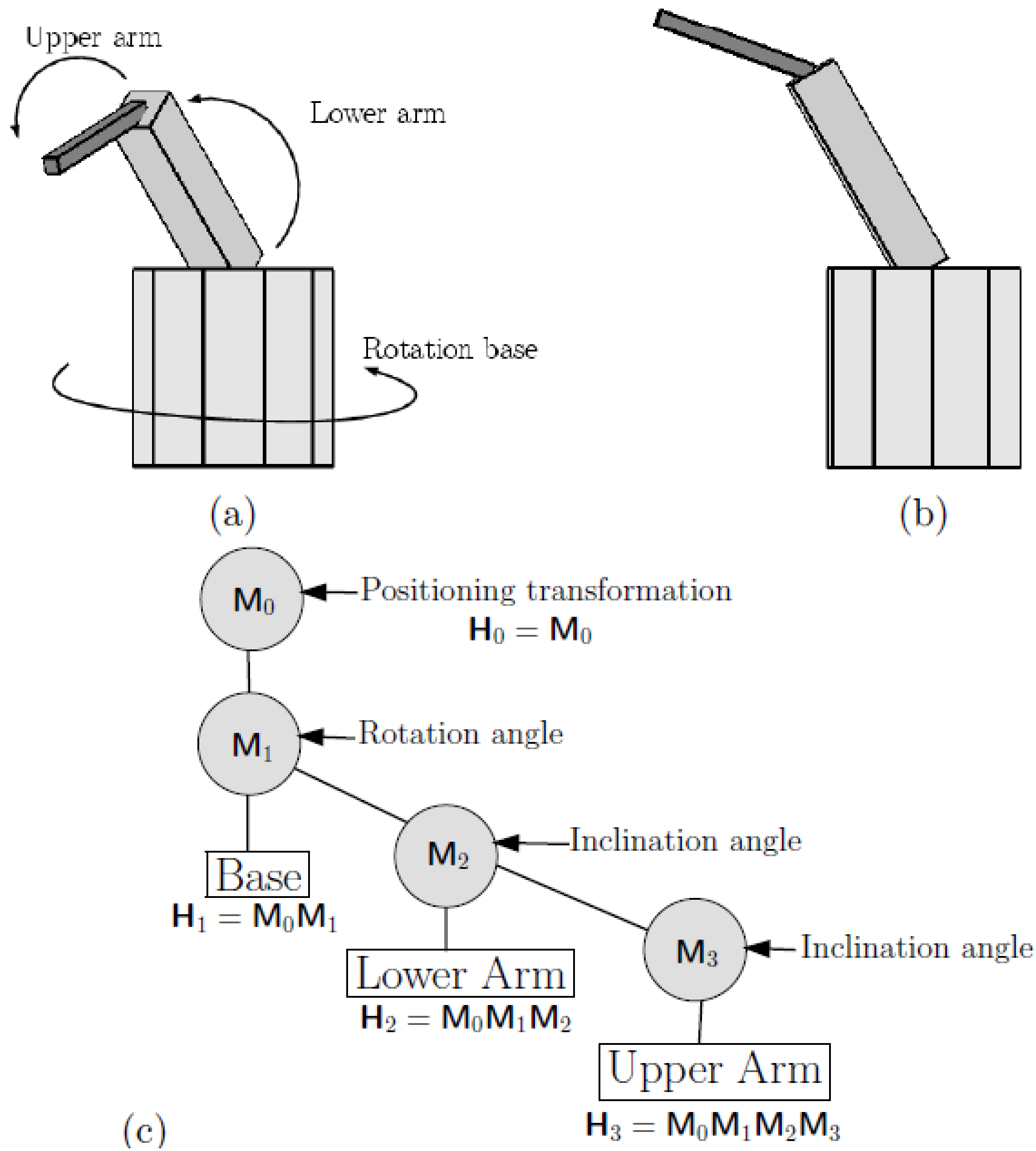


FIGURE 3.33 *Point/line mappings under a homography* $\mathbf{H}$. *Lines map by the transpose of the inverse of the homography mapping points:* $\mathbf{H}^{-T}$.

Graphics pipeline



Graphics pipeline: Scene graph



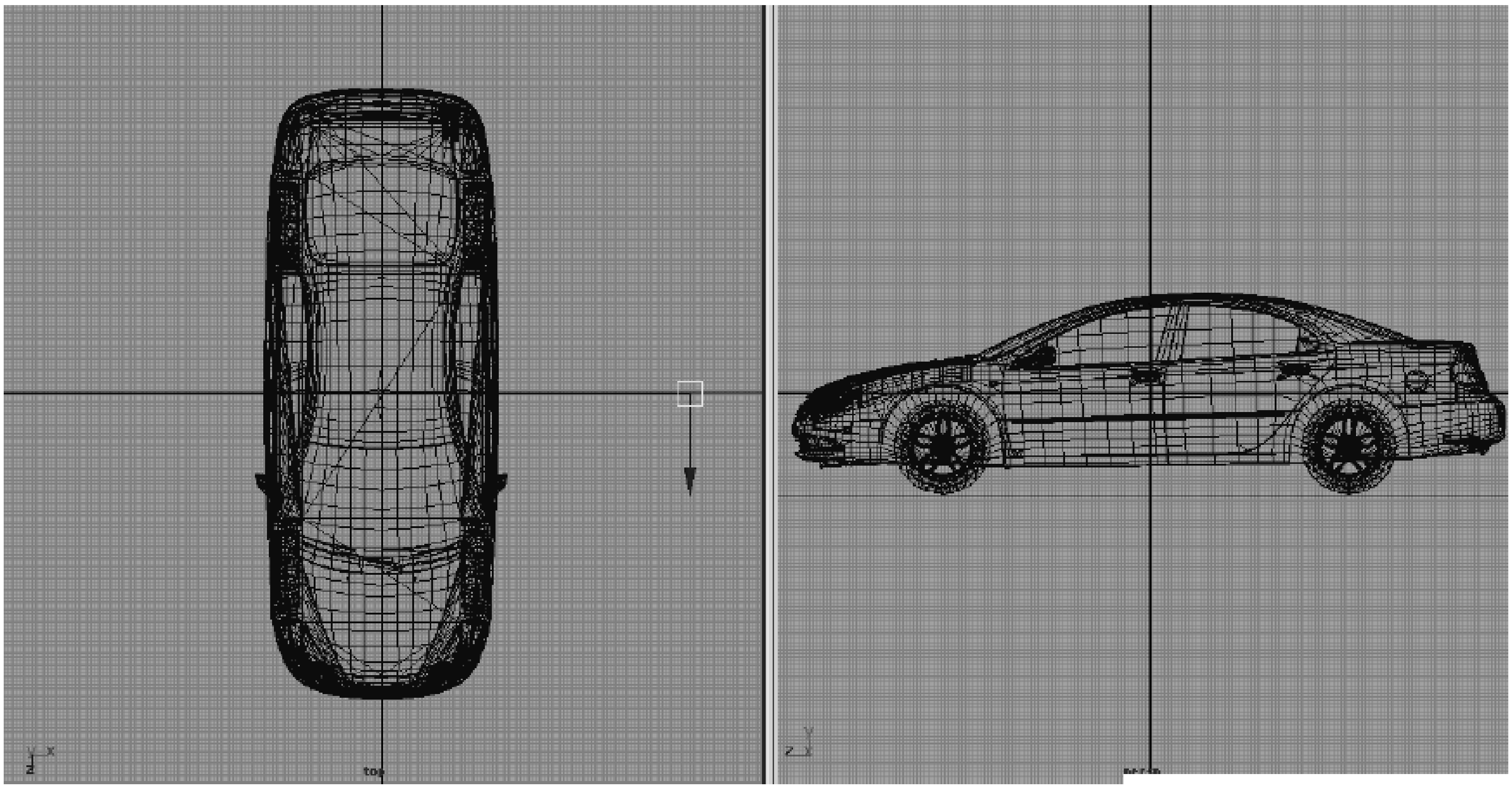
Scaled rigid transformation:

$$\mathbf{M}_k = \underbrace{\mathbf{T}_k}_{\text{Translation}} \underbrace{\mathbf{R}_k}_{\text{Rotation}} \underbrace{\mathbf{S}_k}_{\textit{Scale}}$$

Graphics pipeline: Projection

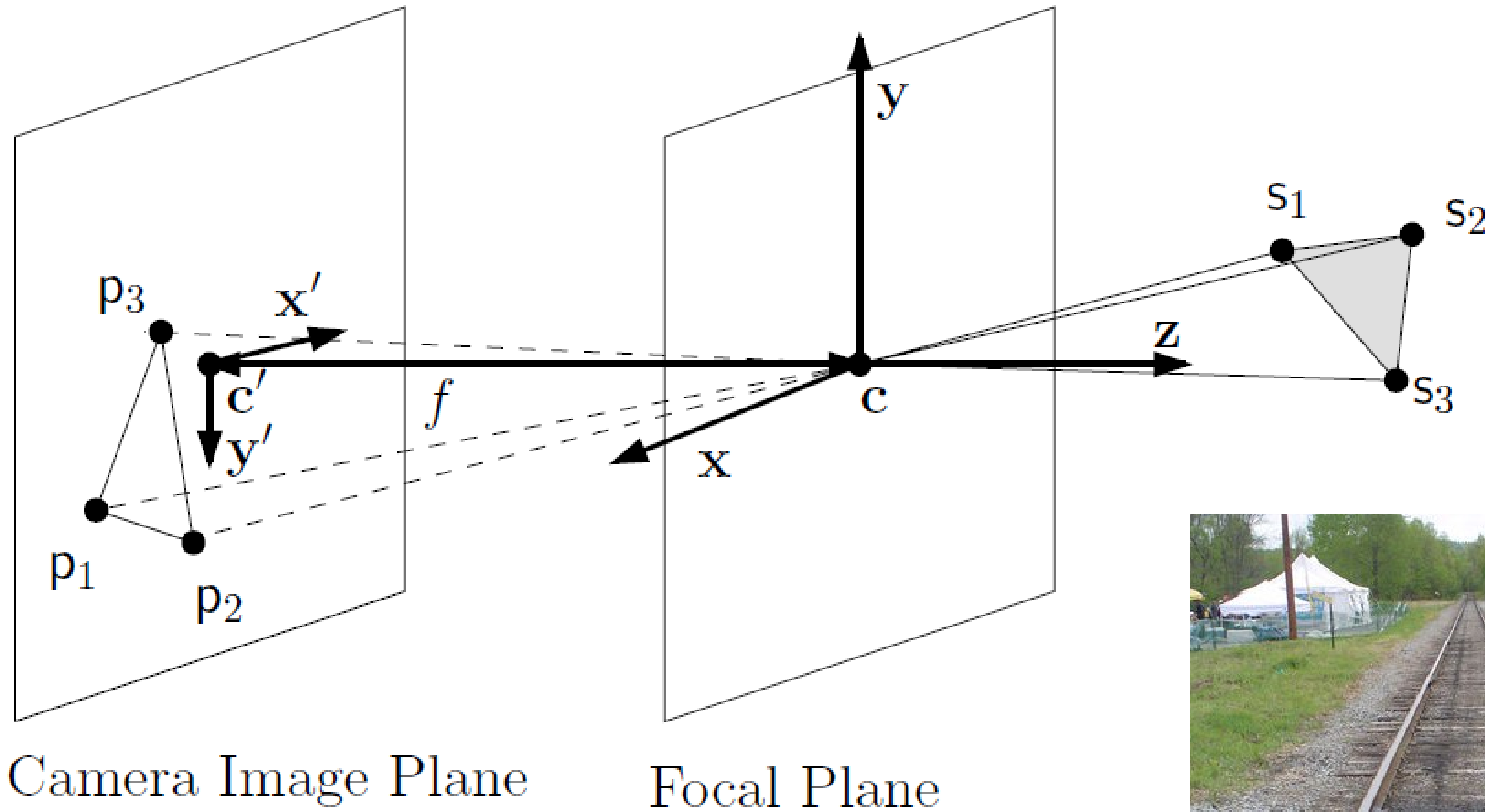
Projections are **irreversible** transformations

Orthographic projection



$$\mathbf{P}_O = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{e}_1{}^T \\ \mathbf{e}_2{}^T \\ \mathbf{0}^T \\ \mathbf{e}_4{}^T \end{bmatrix}$$

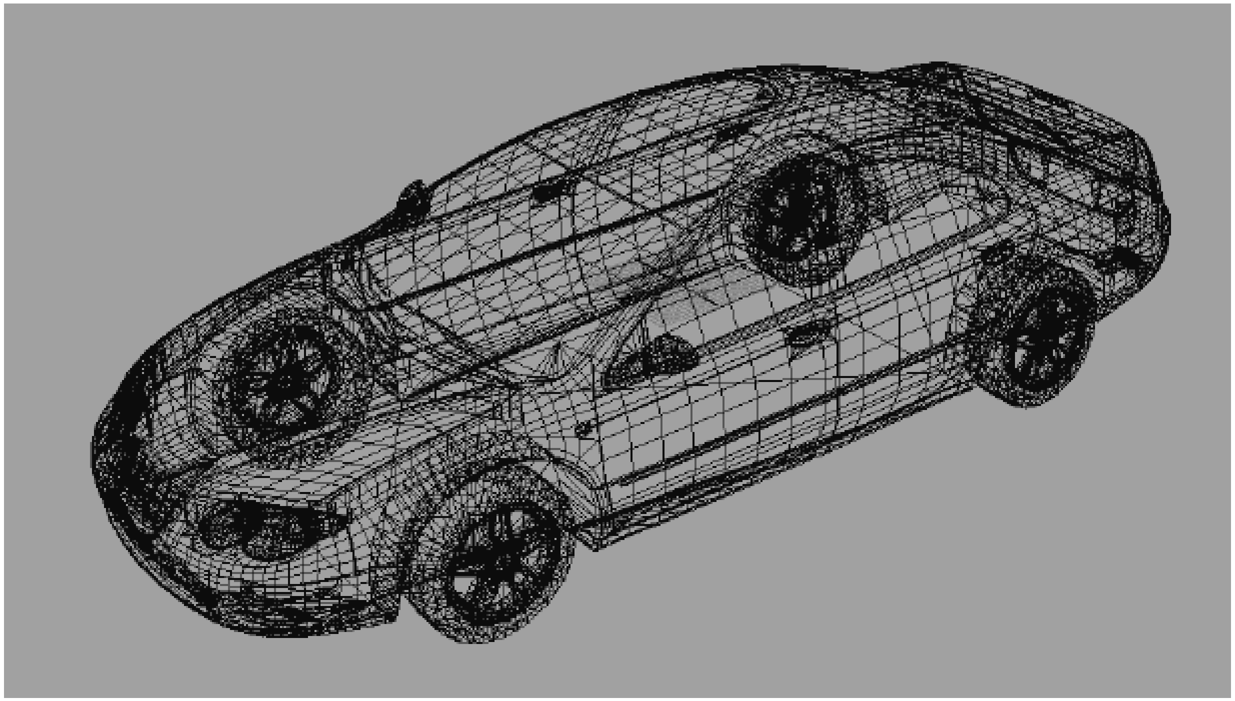
Perspective projection: Pinhole camera



$$\frac{x_p}{x_s} = \frac{y_p}{y_s} = -\frac{f}{z_s}$$

$$\frac{x_s}{x_p} = \frac{y_s}{y_p} = -\frac{z_s}{f}$$

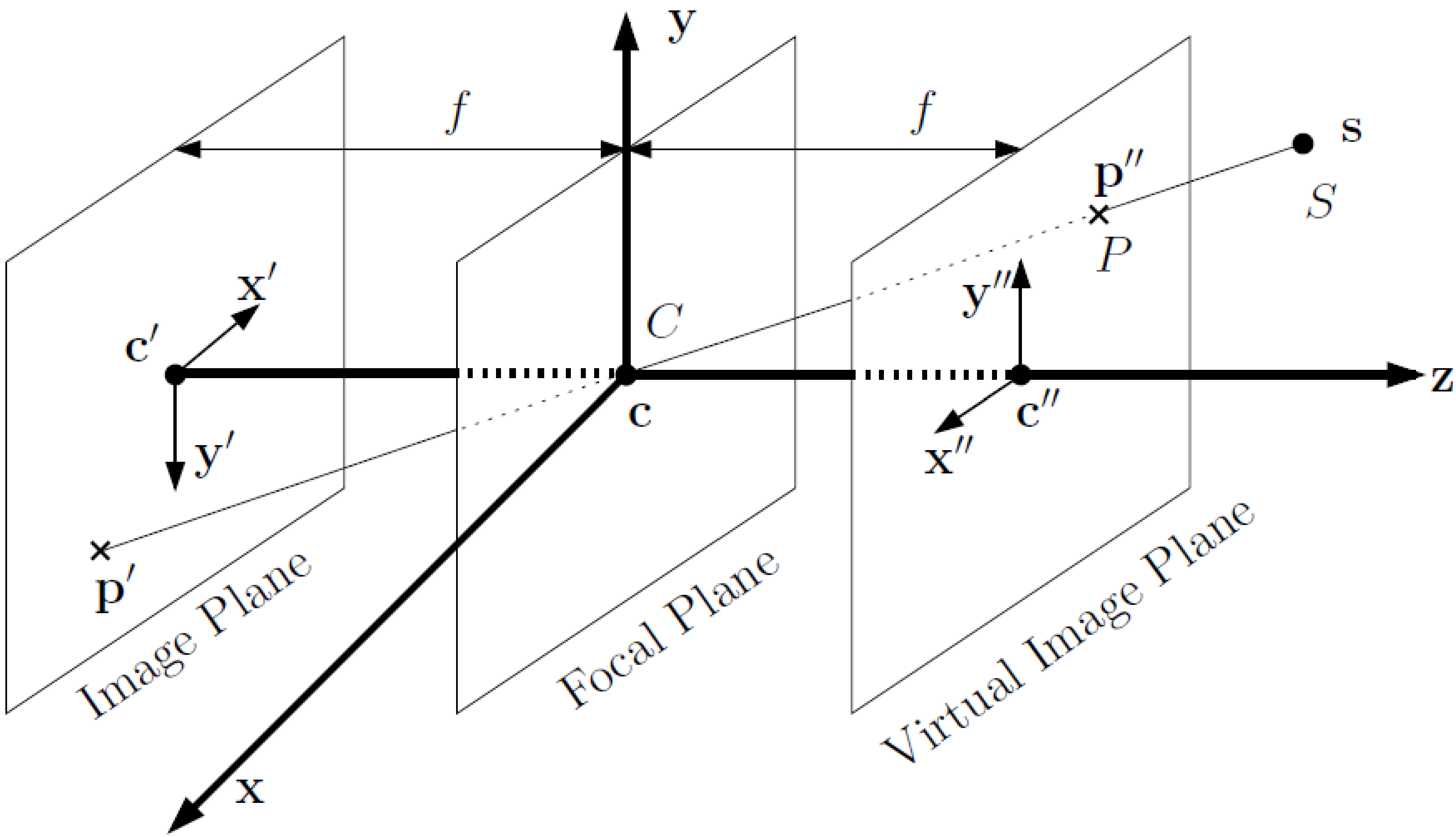
Graphics pipeline: Perspective projection



$$\mathbf{P}_P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{f} & 0 \end{bmatrix}$$

$$\mathbf{P}_P\mathbf{p} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{f} & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ \frac{z}{f} \end{bmatrix} \xrightarrow{\text{perspective division}} \begin{bmatrix} \frac{xf}{z} \\ \frac{yf}{z} \\ f \end{bmatrix}$$

Graphics pipeline: Perspective projection

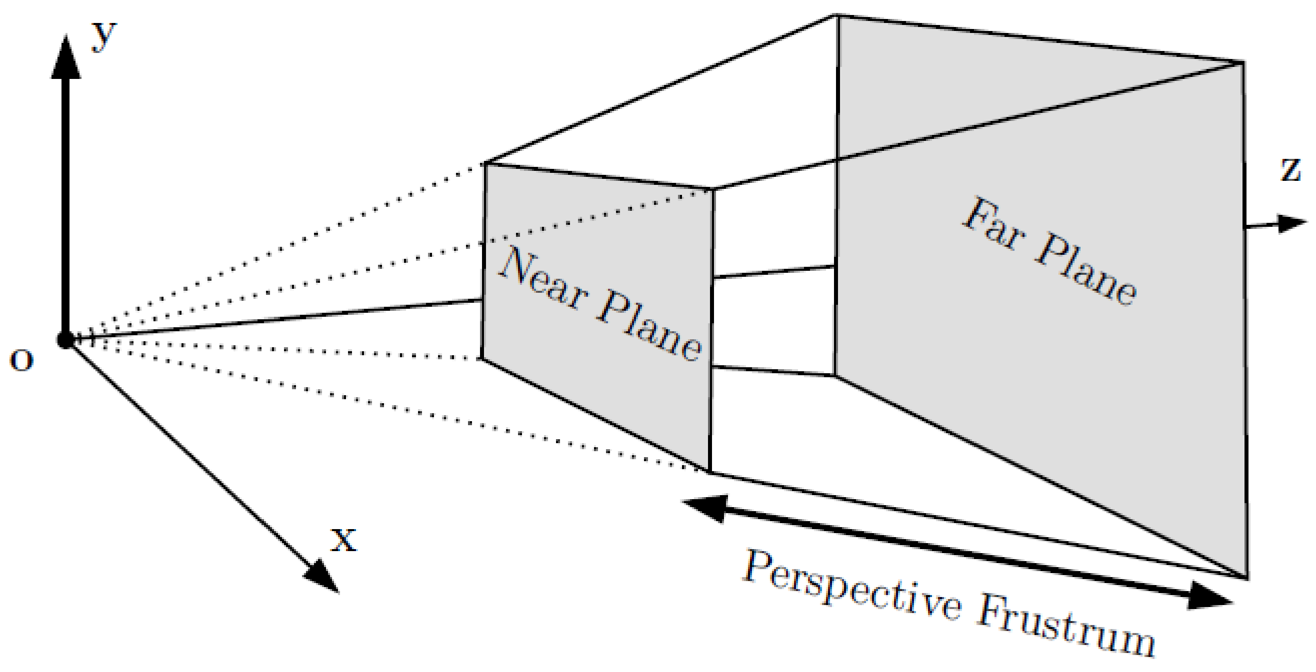


$$x_p = f\frac{x_s}{z_s} \quad \text{and} \quad y_p = f\frac{y_s}{z_s}.$$

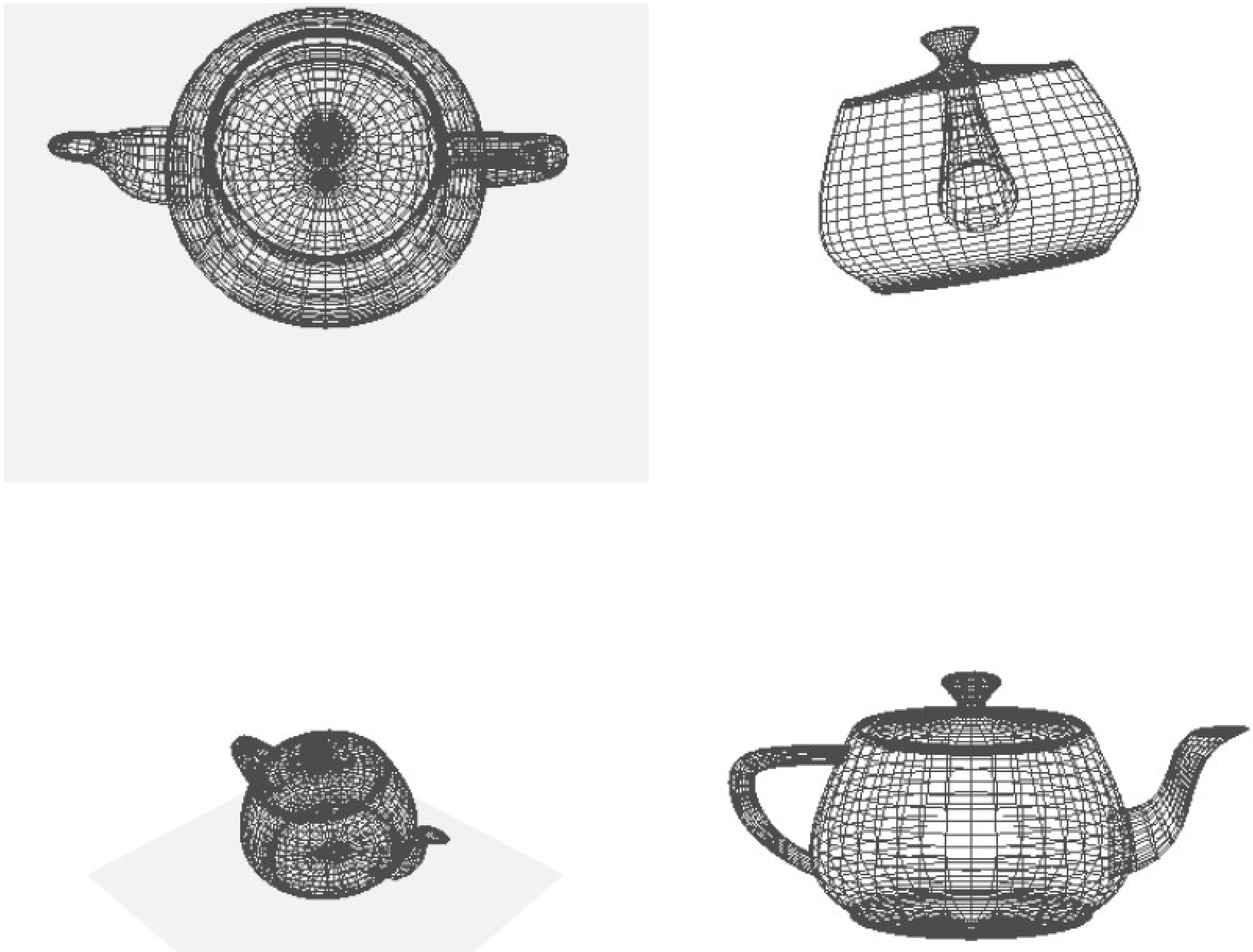
$$\begin{bmatrix} x_s f \\ y_s f \\ z_s f \\ z_s \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & f & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_s \\ y_s \\ z_s \\ 1 \end{bmatrix}$$

Graphics pipeline:
Perspective truncated pyramid
Perspective frustrum

$$\mathbf{C}_P = \begin{bmatrix} \frac{2\text{near}}{\text{right}-\text{left}} & 0 & \frac{\text{right}+\text{left}}{\text{right}-\text{left}} & 0 \\ 0 & \frac{2\text{near}}{\text{top}-\text{bottom}} & \frac{\text{top}+\text{bottom}}{\text{top}-\text{bottom}} & 0 \\ 0 & 0 & -\frac{\text{far}+\text{near}}{\text{far}-\text{near}} & -2\frac{\text{far}\times\text{near}}{\text{far}-\text{near}} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$



Graphics pipeline: Several viewports



Graphics pipeline:

Orthographic projection.

$$\mathrm{p} \longleftarrow \mathsf{M}_V \mathsf{C}_O \mathsf{V} \mathsf{M}_W \mathsf{s}.$$

Perspective projection.

$$\mathrm{p} \longleftarrow \mathsf{M}_V \mathsf{C}_P \mathsf{V} \mathsf{M}_W \mathsf{s}.$$

Mv: positionning transformation
V: viewing transformation
C: clipping transformation
Mv: viewing transformation

OpenGL

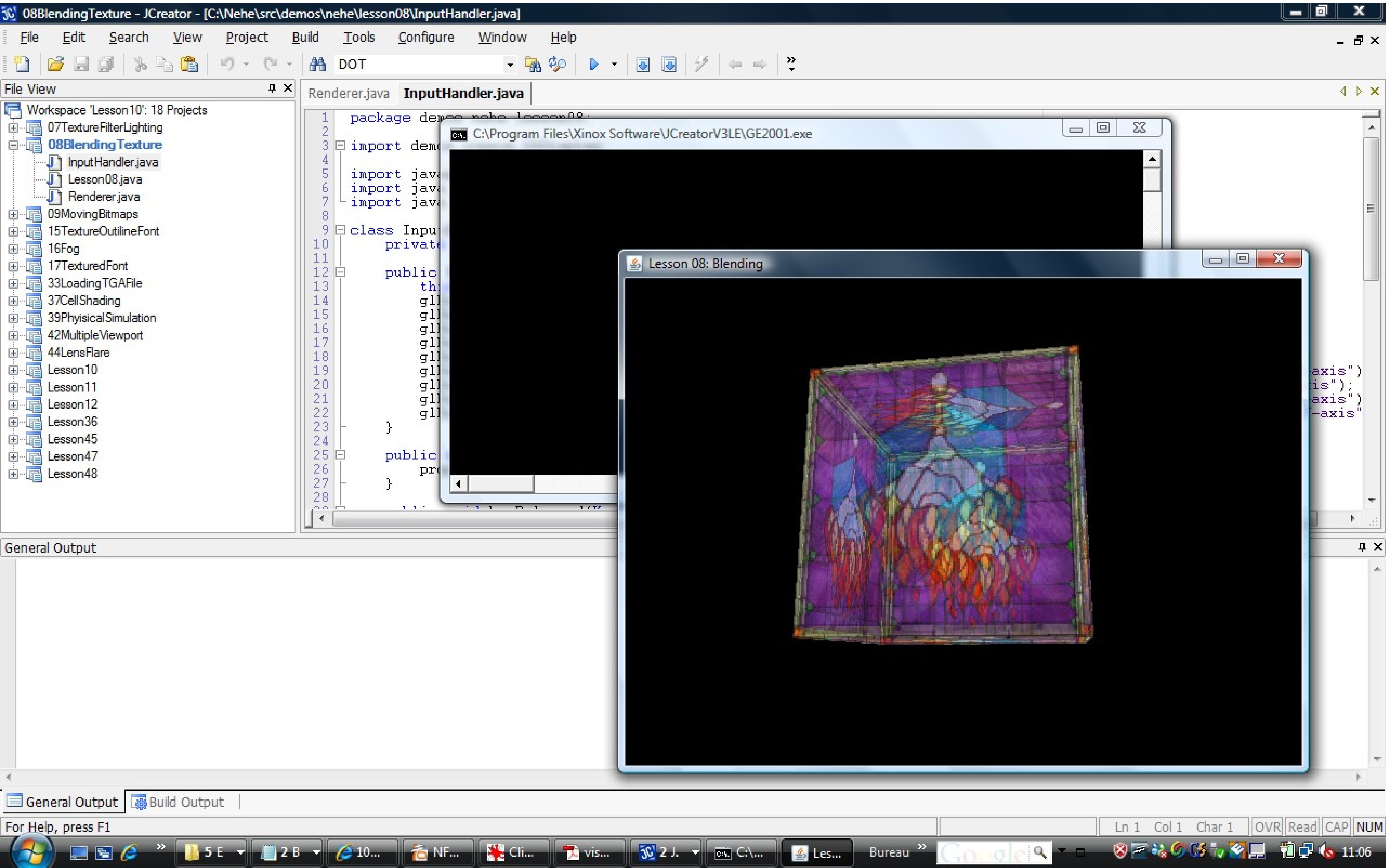
- Industry standard
- OpenGL ES
- Digital assets: Collada

Polish stack calculations

Stacks for view/geometry and textures

Shading languages

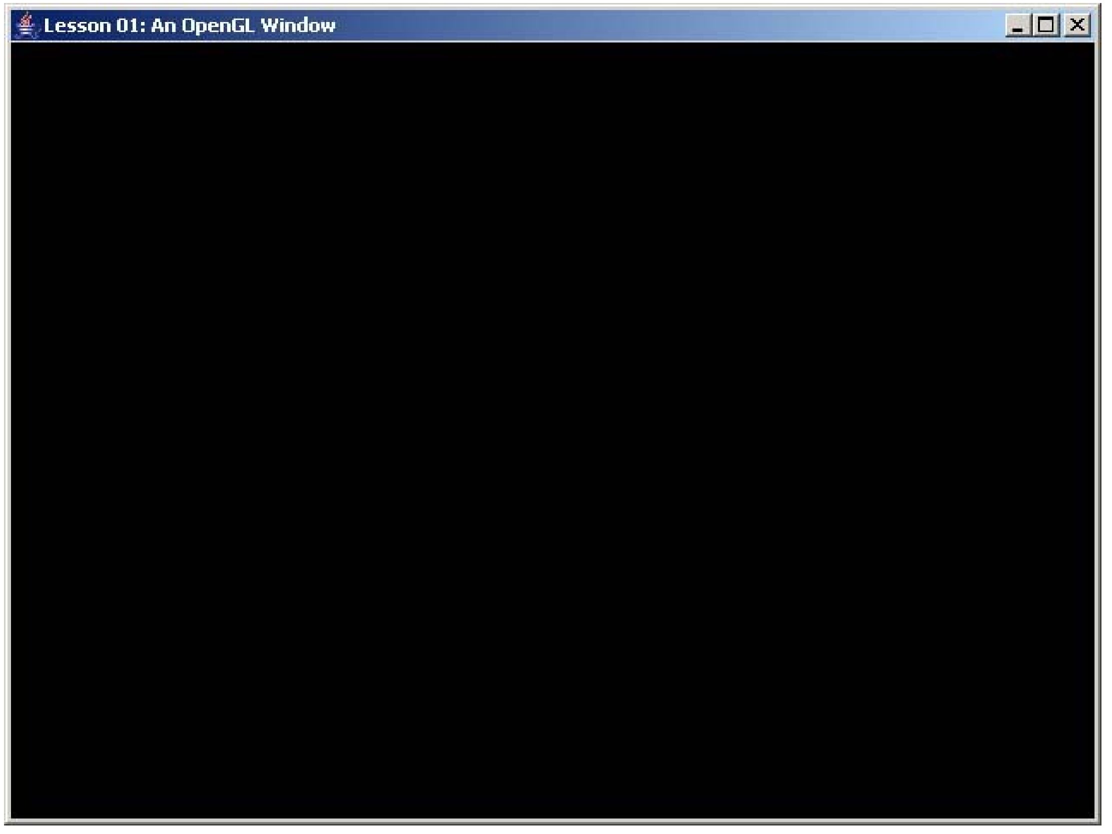
OpenGL in Java: JOGL



OpenGL in Java: JOGL

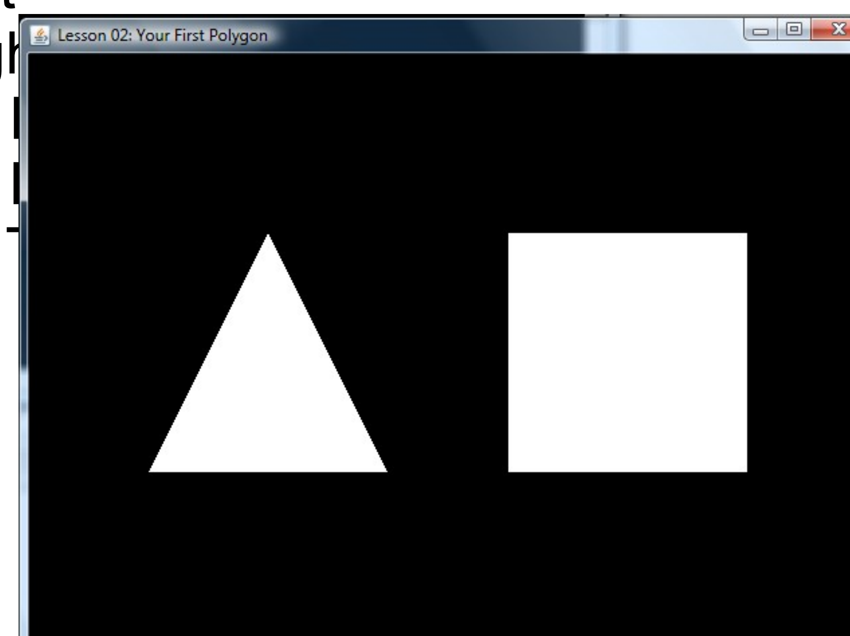
Java Web start JNLP(Java Network Launch Protocol)
Java Applets

www.java-tips.org/other-api-tips/jogl/setting-up-an-opengl-window-nehe-tutorial-jogl-port-2.html

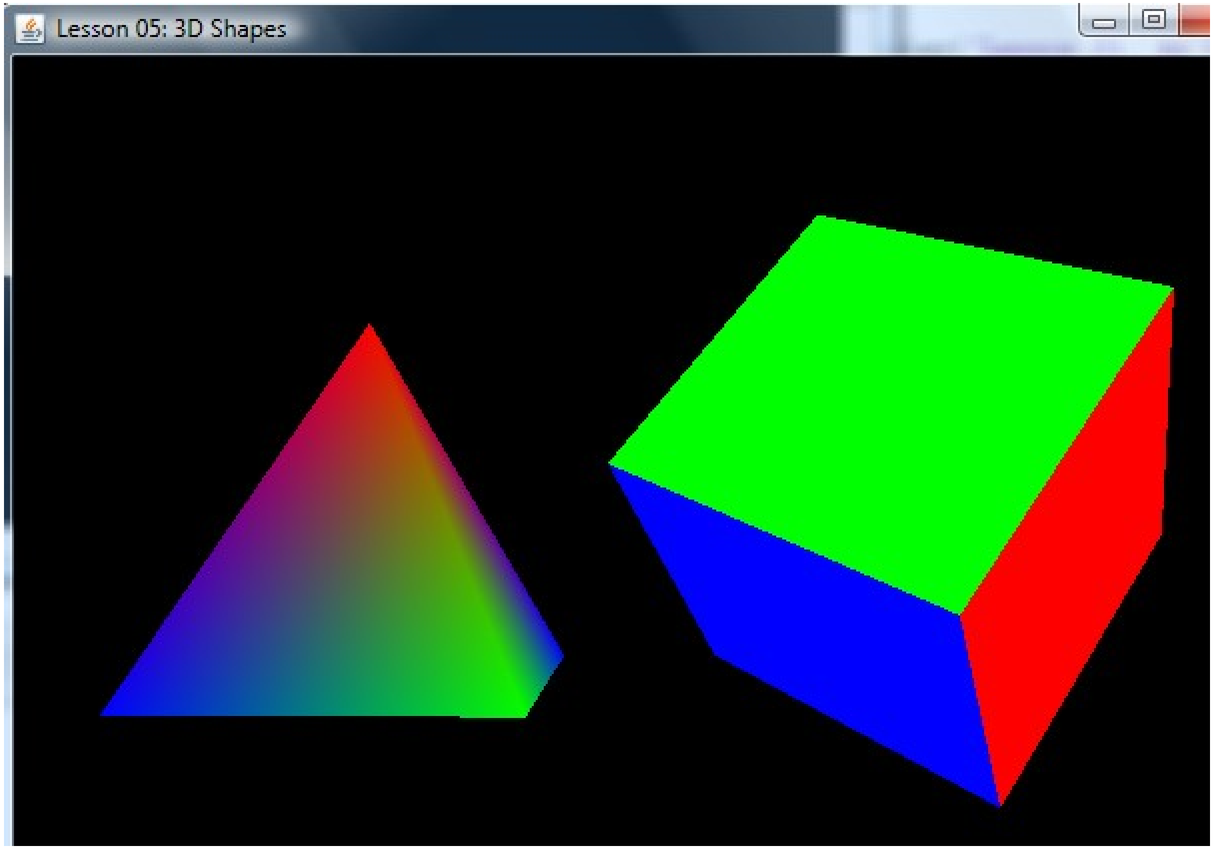


http://100town.com/web/public/products/colladaonjogl

```
public void display(GLAutoDrawable gLDrawable) {
    final GL gl = gLDrawable.getGL();
    gl.glClear(GL.GL_COLOR_BUFFER_BIT | GL.GL_DEPTH_BUFFER_BIT);
    gl.glLoadIdentity();
    gl.glTranslatef(-1.5f, 0.0f, -6.0f);
    gl.glBegin(GL.GL_TRIANGLES);        // Drawing Using Triangles
    gl.glVertex3f(0.0f, 1.0f, 0.0f);    // Top
    gl.glVertex3f(-1.0f, -1.0f, 0.0f);  // Bottom Left
    gl.glVertex3f(1.0f, -1.0f, 0.0f);   // Bottom Right
    gl.glEnd();                         // Finished Drawing The Triangle
    gl.glTranslatef(3.0f, 0.0f, 0.0f);
    gl.glBegin(GL.GL_QUADS);            // Draw A Quad
    gl.glVertex3f(-1.0f, 1.0f, 0.0f);   // Top Left
    gl.glVertex3f(1.0f, 1.0f, 0.0f);    // Top Righ
    gl.glVertex3f(1.0f, -1.0f, 0.0f);   // Bottom
    gl.glVertex3f(-1.0f, -1.0f, 0.0f);  // Bottom
    gl.glEnd();                         // Done Drawing
    gl.glFlush();
}
```



JOGL: 3D color shapes



GLU (Utility)
GLUT (Utility Toolkit, including user interfaces.)

http://www.cs.umd.edu/~meesh/kmconroy/JOGLTutorial/