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A brief overview of pairings attacks

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Notations

Elliptic curves

- equation (in Edwards form): $x^2 + y^2 = c^2(1 + dx^2y^2)$ where $c, d \in K$ and $cd(1 - c^4d) \neq 0$
- group law (when odd cardinality): $(x_1, y_1) + (x_2, y_2) = (\frac{x_1y_2+x_2y_1}{c(1+dx_1x_2y_1y_2)}, \frac{y_1y_2-x_1y_2}{c(1-dx_1x_2y_1y_2)})$
- cardinality (Hasse) :
$$|\#\{(x : y : z) \in \mathbb{P}^2(\mathbb{F}_q) : x^2z^2 + y^2z^2 = c^2(z^4 + dx^2y^2)\} - q - 1| \leq 2\sqrt{q}$$
- scalar product : for any r and P , $[r]P = P + \cdots + P$ (r times)

Finding elliptic curves

Use in cryptography

- Elliptic curves are used in all group-based cryptography : ElGamal, Diffie-Hellman, DSA. They are standardized since 1999.

- Curves are constructed as follows
  - select the good size
  - pick a random prime q of the good size
  - pick random parameters c and d which define a curve E
  - use the Schoof algorithm to compute the cardinality r
  - test primality of r (if desired test primality of $2(q+1)-r$)

Pairings

Definition

- E an elliptic curve over a field K
- r an integer
- P(x,y) a point on E so that $[r]P = (0, c)$ (neutral element).
- μ a unit of Φ_r in the algebraic closure of K

$$e_{E,r,P,\mu} : \begin{array}{ccc} \frac{\mathbb{Z}}{r\mathbb{Z}}P \times \frac{\mathbb{Z}}{r\mathbb{Z}}P & \rightarrow & \mu^{\mathbb{Z}/r\mathbb{Z}} \\ ([a]P, [b]P) & \mapsto & \mu^{ab}. \end{array}$$

Properties of a pairing e

Non-degenerate bilinear map.

Computations of pairings

1. Theorem of Weil (1948): pairings can be defined in terms of divisors, without computing a,b
2. Algorithm of Miller (1985): pairings evaluation is related to a "fast exponentiation" and has a polynomial complexity

Three-party Diffie-Hellman

Problem

Alice, Bob and Carol use a public elliptic curve E and a pairing e with respect to a point P. Each of the participants broadcast simultaneously an information in a public channel. How can they agree on a common key ?

Joux's protocol (2000)

1. Simultaneously, each participant generates a random integer in $[0, r-1]$ and broadcasts a multiple of P:
   - Alice generates a and computes $[a]P$;
   - Bob generates b and computes $[b]P$;
   - Carol generates c and computes $[c]P$;
2. Simultaneously, each participant computes the pairing of the received information and computes the common key:
   - Alice computes $e([b]P, [c]P)^a$;
   - Bob computes $e([c]P, [a]P)^b$;
   - Carol computes $e([a]P, [b]P)^c$;

Common secret key: μ^{abc}.

Embedding degree

Definition

Given E, K and r the embedding degree is the degree of the extension of K which contains an r-th root of unity.

Pariring friendly elliptic curves

Let q be selected so that the discrete logarithm problem is just hard enough in the elliptic curve. Then

- if k is too large, computations are slow (arithmetic in $\mathbb{F}_{q^k}$)
- if k is too small, the discrete logrithm in $\mathbb{F}_{q^k}$ is too easy and the pairing is not safe.

Key sizes

security (bits)	key size RSA $\log_2(q^k)$	key size ECDSA $\log_2 r \approx \log_2 q$	quotient
80	1024	160	6
128	3072	256	12
256	15360	512	30

We need curves such that

- cardinality $r = c \times$ prime with $c \leq 10$
- k donné

CM method

Motivation

Theorem of Köblitz and Balusubramanian : a proportion of $1 - o(1)$ of the curves defined over $\mathbb{F}_q$ have $k \approx q$.

We cannot take random curves, we must find families

Constructing pairings

Given an embedding degree k we construct a pairing-friendly curve E as follows:

1. find q, r and t subject to the CM equations in next slide; they are
   - $\mathbb{F}_q$ is the field of coefficients
   - E has $q + 1 - t$ points
   - E has a subgroup of order r.

2. apply the complex method (Morain 1990) to construct a curve E corresponding to q,r,t. The cost is $O(h_D^{2+\epsilon})$ where h_D is the class number of $\mathbb{Q}(\sqrt{D})$ (for a random D, $h_D \simeq \sqrt{D}$).

CM equations

k given but some exceptions are allowed

Two primes q and r and a square-free integer D satisfy the CM conditions if

1. $\Phi_k(t-1) \equiv 0 \pmod{r}$
2. $q + 1 - t \equiv 0 \pmod{r}$
3. $\exists y,\ 4q = Dy^2 + t^2$

Super-singular curves

Idea

Take $t = 0$ and $k = 2$. Indeed,

1. $\Phi_k(t-1) \equiv 0 \pmod{r}$ (true for all r because $\Phi_2(-1) = 0$)
2. $q + 1 - t \equiv 0 \pmod{r}$ (true for any divisor r of $q+1$)
3. $\exists y,\ 4q = Dy^2 + t^2$ (true for any q)

Limits

- if $q = 2$ or $q = 3$ we can have $k \in \{1, 2, 3, 4, 6\}$ (but small characteristic and hence subject to the quasi-polynomial time attack)
- if $q \geq 5$ we have two possibilities
  - $k = 2$ OK
  - $k = 1$ but $q = p^{2s}$ and E or its twist are isomorphic to a pairing of embedding degree 2 defined over p^s ($\mathbb{F}_{(p^{2s})^1 = \mathbb{F}_{(p^s)^2}}$).

Cocks-Pinch

CM equations

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Method

1. replace (2) by an equivalent equation

Cocks-Pinch

CM equations

1. ~~$\Phi_k(t-1) \equiv 0 \pmod{r}$~~
2. $Dy^2 + (t-2)^2 \equiv 0 \pmod{r} \Leftrightarrow (\sqrt{-D}y + (t-2))(\sqrt{-D}y - (t-2) \equiv 0(r)$
3. $\exists y,\ 4q = Dy^2 + t^2$

Method

1. replace (2) by an equivalent equation
2. select r so that $r \equiv 1 \mod k$ and $(\frac{-D}{r}) = 1$

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3. solve (2) for y

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Method

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2. select r so that $r \equiv 1 \mod k$ and $(\frac{-D}{r}) = 1$
3. solve (2) for y
4. solve (3) for q

Dupont-Enge-Morain

CM equations

1. $\Phi_k(t-1) \equiv 0 \pmod{r}$
2. $q+1-t \equiv 0 \pmod{r}$ $a+(t-2)^2 \equiv 0 \pmod{r}$ where $a = Dy^2$
3. $\exists y,\ 4q = Dy^2 + t^2$

Method

1. replace (2) by an equivalent equation
2. compute $R(a) = \mathrm{Res}_t(\Phi_k(t-1), a+(t-2)^2)$; enumerate a's and take
   - r a prime factor of $R(a)$
   - compute $\gcd(\Phi_k(t-1) \bmod r, a+(t-2)^2 \bmod r)$ and obtain t if it is linear
3. solve (3) for q

Sparse families (e.g. MNT)

CM equations

1. $\Phi_k(t-1) \equiv 0 \pmod{r}$
2. $q + 1 - t \equiv 0 \pmod{r}$
3. $\exists y,\ 4q = Dy^2 + t^2$ generalized Pell equation (e.g. $X^2 - 3Dy^2 = 24$, where $X = 6x \pm 3$)

Method when $\varphi(k) = 2$ (example when $k = 3$)

1. put $r = \Phi_k(t-1)$, which satisfies (1)
2. put $q = r + t - 1$, which satisfies (2)
3. put $t = t(x)$, t linear, and note that this forces $q = q(x)$, quadratic polynomial q (e.g. $t(x) = -1 \pm 6x$ and $q(x) = 12x^2 - 1$). This transforms (3) into a generalized Pell equation
4. solve the generalized Pell equation to get y and x, and therefor q

Was generalized by Freeman to $k = 10$, where $\varphi(k) = 4$

Complete families (e.g. BN)

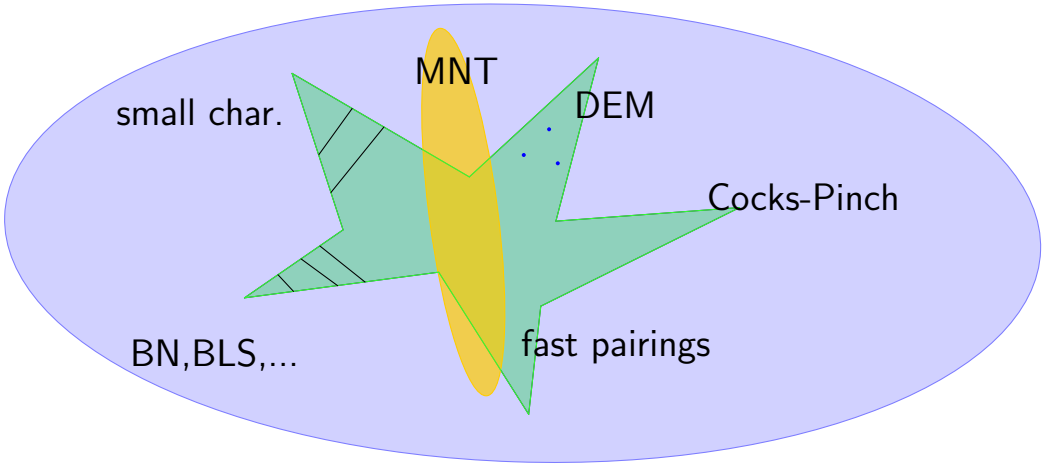
CM equations

1. $\Phi_k(t-1) \equiv 0 \pmod{r}$
2. $q+1-t \equiv 0 \pmod{r}$ $Dy^2+(t-2)^2 \equiv 0 \pmod{r}$ $Dy^2+(t-2)^2 \equiv 0 \pmod{r} \Leftrightarrow (\sqrt{-D}y+(t-2))(\sqrt{-D}y-(t-2) \equiv 0(\mod r)$
3. $\exists y,\ 4q = Dy^2+t^2$

1. replace (2) by an equivalent equation
2. - select $r(x) \in \mathbb{Q}[x]$ so that $\mathbb{Q}[x]/r(x)$ contains a root of x^2-D and a root of $\Phi_k(x)$
   - take $t = t(x)$ to be such that $t-1$ is a kth root of unity mod $r(x)$
3. put $y = t(x)/\sqrt{-D}$ which satisfies (2)
4. solve (3) for q

Note that we generate a large number of elliptic curves very quickly.

Summary



- Pinch-Cocks constructs all the fast pairings, but it is never in the fast case.
- Sparse families (e.g. MNT) construct many pairings but $k = 2$ and they are not fast for the ≥ 80 bits of security.
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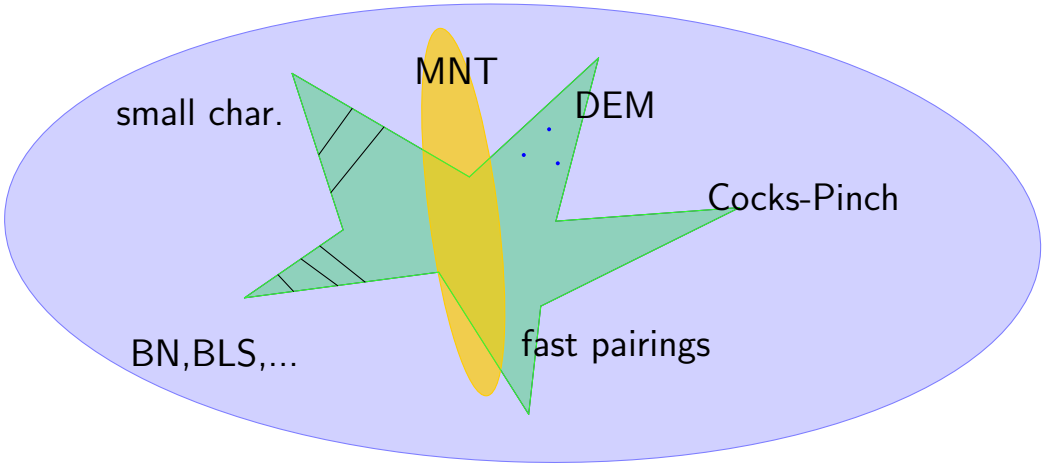

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We are left with small char and parametrized families (e.g. BN, BLS).

Discrete logarithm problem (DLP)

DLP

Given P and $[a]P$ find a.

Generic algorithm

A combination of Pohlig-Hellman reduction and Pollard's rho solves DLP in a generic group G after $O(\sqrt{r})$ operations, where r is the largest prime factor of $\#G$.

Relation to pairings

A pairing $e : \langle P \rangle \times \langle P \rangle \to K(\mu)$ is safe only if

1. DLP in $E[r]$ is hard; (DLP on elliptic curves) **if** $\log_2 \#G = n$, **cost**$=2^{\frac{n}{2}}$
2. DLP in $K(\mu)$ is hard. (DLP in finite fields) **if** $\log_2 \#K(\mu) = n$, **cost**$\approx \exp(\sqrt[3]{n})$

Small characteristic

Chronology

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- Two 128 bit pairings are broken in char 2 (Granger, Kleinjung, Zumbrägel 2014) and 3 (Canales-Martínez, Cortés, Menezes, Oliveira, Rivera-Zamarripa and Rodríguez-Henríquez 2016).

The case of $\mathbb{F}_p^k$ with non-small characteristic p

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- 2000: Schirokauer creates a variant of NFS where integers are replaced by complex numbers of the form $a + ib$ where a and b are integers. He asks if this applies to parametrized primes (SNFS).

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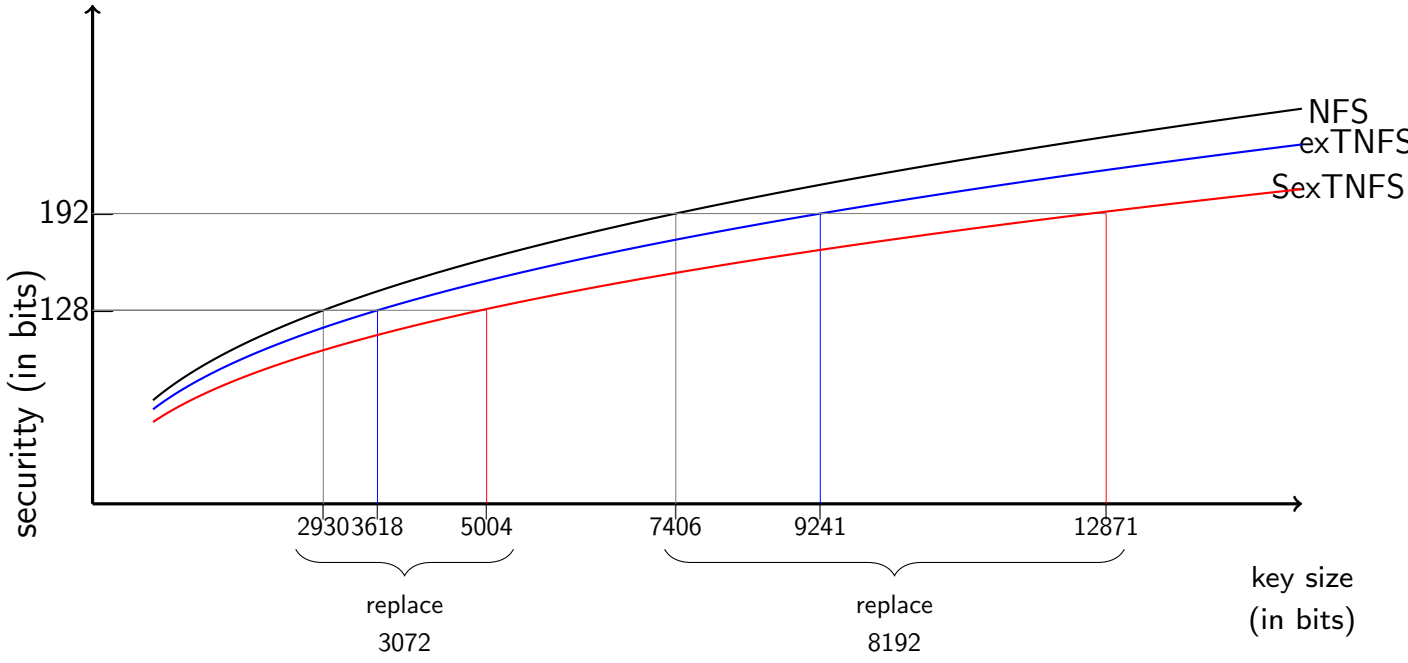
- 2015: Barbulescu, Gaudry, Kleinjung rehabilitate Schirokauer's TNFS.
- 2015-2016: Kim and later Barbulescu combine TNFS and the recent methods of polynomial selection to obtain very good asymptotic complexities when k has a factor 2 or 3 and, particularly good when additionally p has parametrized form.

Change of keysizes by ignoring o(1)



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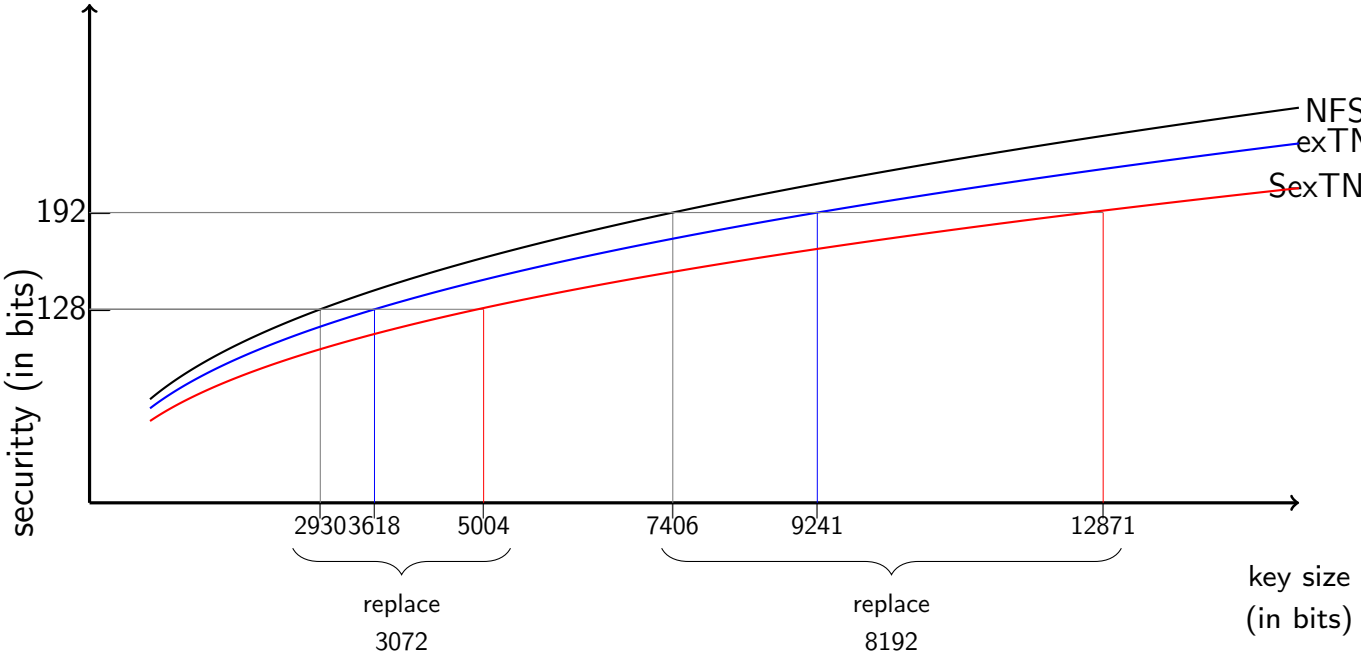
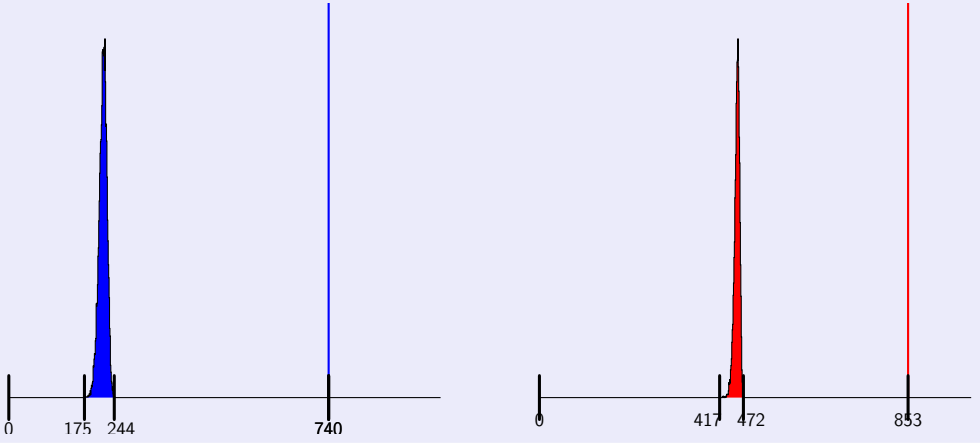


This suggests that for BN one has to replace 3072 by 5004.

Updating keysizes (joint work with S. Duquesne)

Unpreciseness of a previous work: Menezes, Sarkar, Singh 2016



New key sizes

Family	$\log_2(p^k)$	κ	A	$\log_2 B$
obsolète sizes	3072			
BN	5534	2	1145	74.00
BLS12	5530	2	1098	73.65
KSS16	$\approx 4400^*$	1	9	76.5
KSS18	$\approx 4300^*$	1	9	76

*: curve side is weaker, we need 5410 and resp. 6257 bits.

Conclusion

- Only small char and parametrized pairings have been considered for industrial application.
- New attacks in small char. lead ECRYPT to forbid small characteristic.
- New attacks in non-small char demand to update the key sizes.
- The practical improvements of NFS which have asted over 30 years transformed $o(1)$ from positive to negative and made the new attacks practical.
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Post quantun alternatives might be introduced.