MPRI – Cours 2-12-2

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Lecture I: Primality algorithms

2009/11/23

Plan

I. Rules of the game.

II. Warming up.

III. Compositeness tests.

IV. Primality tests.

V. Conclusions.

Good reading

- G. H. Hardy and E. M. Wright. *An introduction to the theory of numbers*. Clarendon Press, 5th edition, 1985.
- D. E. Knuth. *The Art of Computer Programming: Seminumerical Algorithms*. Addison-Wesley, 2nd edition, 1981.
- H. Cohen. *A course in algorithmic algebraic number theory*, volume 138 of *Graduate Texts in Mathematics*. Springer–Verlag, 1996.
- P. Ribenboim. *The new book of prime number records*. Springer-Verlag, 1996.
- R. Crandall and C. Pomerance. *Primes – A Computational Perspective*. Springer Verlag, 2000.
- FM. La primalité en temps polynomial [d'après Adleman, Huang; Agrawal, Kayal, Saxena]. Séminaire Bourbaki, Mars 2003.

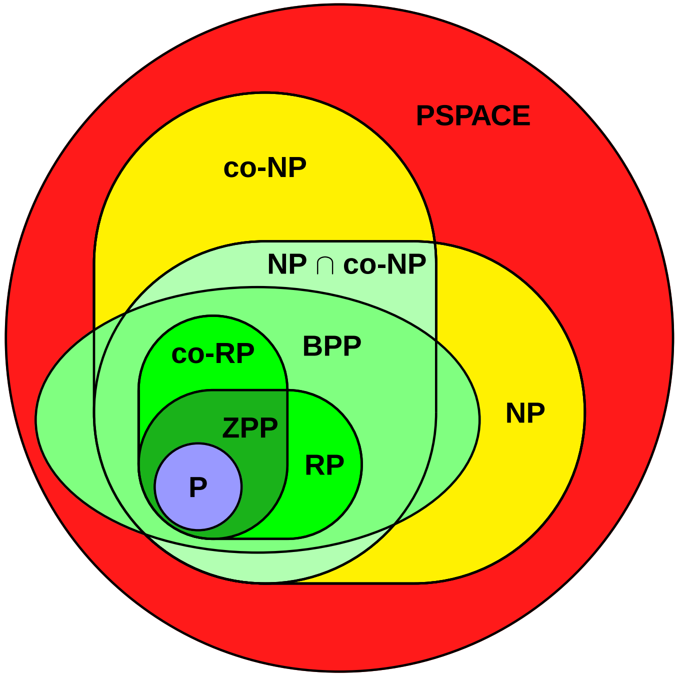
I. Rules of the game

$$N = \prod_{i=1}^{k} p_i^{\alpha_i}.$$

- What do we do in practice? Which size is doable?
  Factorization : number field sieve $O(\exp(c(\log N)^{1/3}(\log\log N)^{2/3}))$; 200 decimal digits (Bahr/Boehm/Franke/Kleinjung, 2005).
  Primality: hopefully without too much factoring, past some easy trial division; 20,000 decimal digits.
- Complexity question: to which class does **isPrime?** belong?

Best : **P** (e.g., integer multiplication).
At least : **RP**.
And: what about a proof?

Complexity classes



II. Warming up

A) $\mathbb{Z}/N\mathbb{Z}$ and $(\mathbb{Z}/N\mathbb{Z})^*$

B) Quadratic reciprocity

C) Finite fields

A) $\mathbb{Z}/N\mathbb{Z}$

$\mathbb{Z}/N\mathbb{Z}$ is a quotient ring; representatives of $\equiv$ are generally chosen as $\{0, 1, \ldots, N-1\}$

$$(\mathbb{Z}/N\mathbb{Z})^* = \{a \in \mathbb{Z}/N\mathbb{Z}, \gcd(a, N) = 1\}$$

Thm. (Chinese remaindering theorem)

$$\mathbb{Z}/N\mathbb{Z} \sim \prod_{p^e || N} \mathbb{Z}/p^e\mathbb{Z}$$

Coro.

$$(\mathbb{Z}/N\mathbb{Z})^* \sim \prod_{p^e || N} (\mathbb{Z}/p^e\mathbb{Z})^*$$

Thm. $(\mathbb{Z}/N\mathbb{Z})^*$ is cyclic iff $N = 2, 4, p^e, 2p^e$ for p odd prime.

Def. Euler totient function:

$$\varphi(N) = \#(\mathbb{Z}/N\mathbb{Z})^* = N \prod_{\substack{p|N \\ p \text{ prime}}} (1 - 1/p)$$

Thm. (Euler, Fermat) $\forall\, a \in (\mathbb{Z}/N\mathbb{Z})^*$, $a^{\varphi(N)} \equiv 1 \bmod N$.

Def. Carmichael function:

$$\lambda(N) = \mathrm{Exp}((\mathbb{Z}/N\mathbb{Z})^*) = \mathrm{LCM}_{a \in (\mathbb{Z}/N\mathbb{Z})^*} \mathrm{ord_N(a)}$$

$$\lambda(p^e) = \begin{cases} \varphi(p^e) & \text{if } p \text{ is odd} \\ 2^{e-1} & \text{if } p = 2 \text{ and } e \in \{1, 2\} \\ 2^{e-2} & \text{if } p = 2 \text{ and } e > 2. \end{cases}$$

Prop. $\forall\, a \in (\mathbb{Z}/N\mathbb{Z})^*$, $a^{\lambda(N)} \equiv 1 \bmod N$, and $\lambda(N)$ is the smallest integer with this property.

Quadratic reciprocity

Legendre symbol: for prime odd p and $a \in \mathbb{Z}$

$$\left(\frac{a}{p}\right) = \begin{cases} 0 & \text{if } p \mid a \\ 1 & \text{if } \exists\, x \text{ s.t. } a \equiv x^2 \bmod p \\ -1 & \text{otherwise.} \end{cases}$$

Easy properties:
(i) $\left(\frac{a}{p}\right) \equiv a^{(p-1)/2} \bmod p$;
(ii) $\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2}$;
(iii) $\left(\frac{a}{p}\right) = \left(\frac{a \bmod p}{p}\right)$;
(iv) $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$;

Not so easy properties:
(v) $\left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8}$;
(vi) (Quadratic reciprocity law) p and q odd primes:

$$\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2} \times \frac{q-1}{2}} \left(\frac{p}{q}\right).$$

Jacobi symbol: $n \in \mathbb{Z}$, $m = \prod_{i=1}^{k} p_i \in \mathbb{Z}$ odd,

$$\left(\frac{n}{m}\right) = \prod_{i=1}^{k} \left(\frac{n}{p_i}\right).$$

Properties: same as for the Legendre symbol.

Ex. Show that $\left(\frac{n}{m}\right) = 0$ iff $\gcd(n, m) > 1$.

Constructing finite fields

Thm. (the canonical way) Let $f(X)$ be an irreducible polynomial of degree n over $\mathbb{F}_p$. Then $\mathbb{F}_p[X]/(f(X))$ is a finite field of degree n and cardinality p^n, noted $\mathbb{F}_{p^n}$.

Ex. Build $\mathbb{F}_{41^2}$, using a quadratic non-residue modulo 41.

$$\begin{aligned} \left(\frac{7}{41}\right) &= (-1)^{(41-1)/2 \times (7-1)/2} \left(\frac{41}{7}\right) \\ &= \left(\frac{41}{7}\right) = \left(\frac{41 \bmod 7}{7}\right) \\ &= \left(\frac{6}{7}\right) = \left(\frac{2}{7}\right)\left(\frac{3}{7}\right) = \left(\frac{3}{7}\right) = (-1)\left(\frac{7}{3}\right) \\ &= -\left(\frac{1}{3}\right) = -1 \end{aligned}$$

Hence use $\mathbb{F}_{41^2} \sim \mathbb{F}_{41}[X]/(X^2 - 7)$. Any element writes $u + vX$ with operations modulo $X^2 - 7$.

One application

Pb. Given $\left(\frac{a}{p}\right) = 1$, compute $\sqrt{a} \bmod p$.

Case $p \equiv 3 \bmod 4$**:** $r = a^{(p+1)/4} \bmod p$.

Case $p \equiv 1 \bmod 4$**:** find b s.t. $\Delta = b^2 - 4a$ is not a square.

$$\alpha = (-b + \sqrt{\Delta})/2 \quad \Rightarrow \alpha^p = (-b - \sqrt{\Delta})/2 \quad \Rightarrow \alpha\alpha^p = a$$

since $\sqrt{\Delta}^p = \left(\frac{\Delta}{p}\right)\sqrt{\Delta}$.
Let $\beta = \alpha^{(p+1)/2} \bmod (p, X^2 + bX + a)$. Then

$$\beta^2 = \alpha^{p+1} = a.$$

Moreover

$$\beta^p = \beta(\beta^2)^{(p-1)/2} = \beta a^{(p-1)/2} = \beta$$

and β is in $(\mathbb{Z}/p\mathbb{Z})^*$.

III. Compositeness tests

Z) Definition and classification.

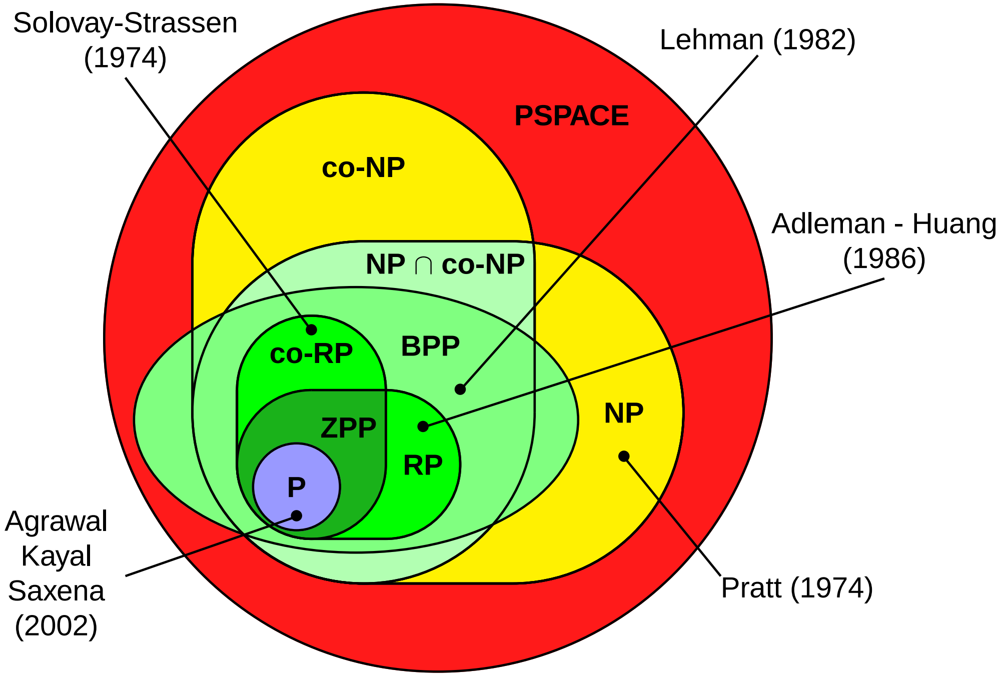
A) Fermat.

B) Euler-Solovay/Strassen.

C) Artjuhov-Miller-Rabin.

D) Other tests.

Z) Definition and classification



My view of randomized algorithms

Def. A Monte Carlo algorithm for deciding that $X \in \mathbb{A}$ returns yes or I don't know:
Proba("yes" $|X \notin \mathbb{A}) = 0$
Proba("I don't know" $|X \in \mathbb{A}) \leq 1 - \delta$, for absolute $0 < \delta < 1$.

Def. A decision problem is in **RP** if there exists a polynomial time Monte Carlo algorithm that solves it.

Rem. $\neq$ error on the answer; or a failure in the computer.

Def. A Las Vegas algorithm answers yes, no or I don't know:
Proba("yes" $|X \notin \mathbb{A}) = 0$, Proba("no" $|X \in \mathbb{A}) = 0$
Proba("I don't know") $\leq 1 - \delta$.

Def. ZPP = **RP**∩ **co-RP**.

Compositeness test: deciding that N is composite.

Primality test: deciding that N is prime.

A) Fermat

Idea: if $\gcd(a, N) = 1$, then $a^{N-1} \equiv 1 \bmod N$.

But: $2^{340} \equiv 1 \bmod 341$: pseudoprime to base 2 (psp-2).

Thm. There exists an infinite number of psp-2 numbers.

Thm. (Pomerance) For $x \geq x_0$: $x^{2/7} \ll P_2(x) \ll xL(x)^{-1/2}$ with $L(x) = \exp\{\log x \log\log\log x / \log\log x\}$.

Def. $P(N) = \#\{a \in (\mathbb{Z}/N\mathbb{Z})^*, a^{N-1} \equiv 1 \bmod N\}$.

Thm. If $N = \prod_i p_i^{\alpha_i}$, $P(N) = \prod_i \gcd(p_i - 1, N - 1)$.

Proof: ■

The test

function isComposite(N)

1. Choose a at random in $\mathbb{Z}/N\mathbb{Z} - \{0\}$.
2. Compute $g = \gcd(a, N)$; **if** $g > 1$, **then** return $(\texttt{yes}, g \mid N)$.
3. **if** $a^{N-1} \not\equiv 1 \bmod N$, **then** return $(\texttt{yes}, a)$
   otherwise return I don't know.

Cost. $O((\log N)\mathrm{M}(\log N))$; typically $O((\log N)^3)$, asymptotically $\tilde{O}((\log N)^2)$.

Prop. Proba("I don't know") $= P(N)/(N-1)$.

Proof. Probability of yes is:

$$\left(1 - \frac{\varphi(N)}{N-1}\right) + \frac{\varphi(N)}{N-1}\left(1 - \frac{P(N)}{\varphi(N)}\right). \square$$

Rem. if N is prime, proba is 1...!

Carmichael numbers

Def. composite N s.t. $P(N) = \varphi(N)$.

Ex. 541.

Rem. $P(N)/(N-1) = \varphi(N)/(N-1)$ close to 1.

Thm.(Alford, Granville, Pomerance, 1992) There are infinitely many Carmichael numbers.

More properties of Carmichael numbers:
1. N is squarefree.
2. For all $p \mid N$, $p - 1 \mid N - 1$ (equivalently $\lambda(N) \mid N - 1$).
3. N has at least three prime factors.

B) Euler and Solovay-Strassen

Idea: (Euler) if N is prime and $\gcd(a, N) = 1$, then $a^{(N-1)/2} \equiv \left(\frac{a}{N}\right) \bmod N$.

Pb: $2^{(1105-1)/2} \equiv \left(\frac{2}{1105}\right) \bmod 1105$; this is an Euler pseudoprime to base 2 (epsp-2). There are an infinite number of them.

Prop. $E_2(x) \le P_2(x)$.

Def. $\mathcal{E}(N) = \{a \in (\mathbb{Z}/N\mathbb{Z})^*, a^{(N-1)/2} \equiv \left(\frac{a}{N}\right) \bmod N\}$; $E(N) = \#\mathcal{E}(N)$.

Prop. $\mathcal{E}(N)$ is proper subgroup of $(\mathbb{Z}/N\mathbb{Z})^*$.

Coro. $E(N)/\varphi(N) \le 1/2$.

The exact value of $E(N)$

Thm. (Monier) Write $N = \prod_{i=1}^{k} p_i^{\alpha_i}$ where p_i are distinct odd primes, $\alpha_i \ge 1$. Write $N = 1 + 2^s t$ with t odd and $p_i = 1 + 2^{s_i} t_i$ with t_i odd. Assume $s_1 \le s_2 \le \cdots \le s_k$ and put $T_i = \gcd(t, t_i)$, $n_i = \gcd((N-1)/2, p_i - 1)$ and $\mathcal{N} = \prod_i n_i$. Then

$$E(N) = \delta(N)\mathcal{N}$$

where

$$\delta(N) = \begin{cases} 2 & \text{if } s = s_1 \\ 1/2 & \text{if } \exists i, \alpha_i \text{ odd and } s_i < s \\ 1 & \text{otherwise.} \end{cases}$$

Proof. exercise.

The test

function isComposite2(N)

1. Choose a at random in $\mathbb{Z}/N\mathbb{Z} - \{0\}$.

2. Compute $g = \gcd(a, N)$; **if** $g > 1$, **then** return $(\texttt{yes}, g \mid N)$.

3. **If** $a^{(N-1)/2} \not\equiv \left(\frac{a}{N}\right) \bmod N$ **then** return $(\texttt{yes}, a)$
   else return I don't know.

Prop. Proba("I don't know")$= E(N)/(N-1) \leq 1/2$.

Coro. isComposite? $\in$ **RP** (hence **isPrime?**$\in$ **co** $-$ **RP**).

Miller (1975): $a = 2, 3, \ldots$; Ankeny–Montgomery–Lenstra–Bach: if an adequate Riemann hypothesis is true, then the smallest witness is $< 2(\log N)^2$, yielding a deterministic $O((\log N)^3 \mathsf{M}(\log N))$ algorithm.

C) Artjuhov-Miller-Rabin

N being odd, write $N - 1 = 2^s t$ with $s \geq 1$ and odd t.

$$a^{N-1} - 1 = (a^t - 1)(a^t + 1)(a^{2t} + 1) \cdots (a^{2^{s-1}t} + 1)$$

$$(AMR_a) : a^t \equiv 1 \bmod N \text{ or } \exists j, 0 \leq j < s, a^{2^j t} \equiv -1 \bmod N.$$

Pb: $N = 2047 = 23 \times 89$ is s.t. $N - 1 = 2 \times 1023$ and $2^{(N-1)/2} \equiv 1 \bmod N$: strong-pseudoprime to base 2 (spsp-2).

Thm. spsp-$a \Rightarrow$ epsp-a.

Def. $F(N) = \#\{a \in (\mathbb{Z}/N\mathbb{Z})^*, (AMR_a) \text{ is satisfied}\}$.

Thm. (Monier)

$$F(N) = \left[1 + \frac{2^{ks_1} - 1}{2^k - 1}\right] \prod_{i=1}^{k} T_i.$$

Coro. $F(N)/(N-1) \leq 1/4$.

The test

function isComposite3(N)

1. Choose a at random in $\mathbb{Z}/N\mathbb{Z} - \{0\}$.

2. Compute $g = \gcd(a, N)$; **if** $g > 1$, **then** return $(\texttt{yes}, g \mid N)$.

3. **If** (AMR_a) **then** return $(\texttt{yes}, a)$
   else return I don't know.

Prop. Proba("I don't know")$= F(N)/(N-1)$.

Numerical tables

x	$P_2(x)$	$E_2(x)$	$F_2(x)$	$C(x)$	$\pi(x)$
10^4	22	12	5	7	1229
10^5	78	36	16	16	9592
10^6	245	114	46	43	78498
10^7	750	375	162	105	664579
10^8	2057	1071	488	255	5761455
10^9	5597	2939	1282	646	50847534
10^{10}	14884	7706	3291	1547	455052511
25×10^9	21853	11347	4842	2163	1091987405
10^{11}	38975	20417	8607	3605	4118054813
10^{12}	101629	53332	22407	8241	37607912018
10^{13}	264239	139597	58897	19279	346065536839
10^{14}				44706	3204941750802
10^{15}				105212	29844570422669
10^{16}				246683	279238341033925

Building primes?

function randomProbablePrime(b)

repeat
  choose odd N at random in $[2^{b-1}, 2^b[$
until N passes k tests.

$p_{b,k} =$ Proba($X = N$ is composite$|Y_k = N$ passes k tests) $=$?

Rem. What we know is
Proba($Y_k = N$ passes k tests$|X = N$ is composite) $\leq (1/4)^k$.

Thm. (Burthe, 1996) $\forall b \geq 2, \forall k \geq 1, p_{b,k} \leq 4^{-k}$.

Other tests

Goal: reduce the non-answer probability while keeping the computations fast.

- Algebraic extensions: Lucas (degree 2), Adams & Shanks, Gurak.
- Elliptic curves: Gordon.
- Combinations of the preceding: no examples known of spsp-a and Lucas pseudoprime, for instance.
- Frobenius pseudoprimes à la Grantham: $\leq 1/7710$. Cf. also Zhang.

IV. Primality tests

A) Fermat

B) En route for **P**.

C) Agrawal, Kayal, Saxena.

A) Fermat

Thm. N is prime if and only if $(\mathbb{Z}/N\mathbb{Z})^*$ is cyclic of ordre $N-1$:

$$\left.\begin{array}{l} a^{N-1} \equiv 1 \bmod N \\ \forall\, p \mid N-1,\ a^{\frac{N-1}{p}} \not\equiv 1 \bmod N \end{array}\right\} \Rightarrow N \text{ is prime}$$

Certificate: $(N, \{p \mid N-1\}, a)$ $\Rightarrow$**isPrime?**$\in$**NP**.

Thm. (**Pocklington**, 1914) Let s s.t. $s \mid N-1$

$$\left.\begin{array}{l} a^{N-1} \equiv 1 \bmod N \\ \forall\, q \text{ prime } \mid s, \gcd(a^{\frac{N-1}{q}} - 1, N) = 1 \end{array}\right\} \Rightarrow \forall\, p \mid N,\ p \equiv 1 \bmod s$$

Coro. $s > \sqrt{N} \Rightarrow N$ is prime.

Rem. factorisation is not polynomial time in the classical world (see later), but polynomial quantic; search for a is not either (except if Riemann is true or randomized approach).

Example of use

Hyp. We know how to find all prime factors < 20.

$$\begin{array}{rcl} N_0 & = & 100003, \quad N_0 - 1 = 2 \times 3 \times 7 \times N_1, \\ N_1 & = & 2381, \quad N_1 - 1 = 2^2 \times 5 \times 7 \times 17 \end{array}$$

p	2	5	7	17
$3^{(N_1-1)/p} \bmod N_1$	2380	1347	1944	949

$\Rightarrow N_1$ is prime

$$s = N_1 > \sqrt{N_0}$$

$$2^{N_0-1} \equiv 1 \bmod N_0, \gcd(2^{(N_0-1)/N_1} - 1, N_0) = 1$$

$\Rightarrow N_0$ is prime

Rem. We have got a (recursive) primality proof of depth $O(\log N)$.

The $N+1$ test

For a_0 and a_1 integers, let:

$$A_N = A_N(a_0, a_1) = \mathbb{Z}/N\mathbb{Z}[T]/(T^2 + a_1 T + a_0)$$

and $\Delta = a_1^2 - 4a_0$.
Elements of A_N are $u + v\alpha$ with u, v dans $\mathbb{Z}/N\mathbb{Z}$, computations made using $\alpha^2 = -a_1\alpha - a_0$.
Thm. Let p be a prime $\nmid \Delta$.

- if $(\Delta/p) = -1$, then $A_p \sim \mathbb{F}_{p^2}$;
- if $(\Delta/p) = +1$, then $A_p \sim \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

Proof : If $(\Delta/p) = -1$, $T^2 + a_1 T + a_0$ is irreducible, hence we recover the classical construction of $\mathbb{F}_{p^2}$.
If $(\Delta/p) = +1$, $T^2 + a_1 T + a_0 = (T - u)(T - v)$ with $u \not\equiv v \bmod p$.
Therefore

$$A_p \sim (\mathbb{Z}/p\mathbb{Z})[T]/(T-u)) \times (\mathbb{Z}/p\mathbb{Z})[T]/(T-v)) \sim \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}. \Box$$

Thm. Let N be an odd integer. Assume that we found a_0, a_1 s.t. $\Delta = a_1^2 - 4a_0$ satisfies $(\Delta/N) = -1$. Write $N + 1 = \prod_i q_i^{\beta_i}$. Suppose we have found $\theta \in A_N = A_N(a_0, a_1)$ s.t.

$$\theta^{N+1} = 1 \text{ in } A_N,$$

and for all i :

$$\theta^{(N+1)/q_i} = u_i + v_i\alpha \text{ with } (u_i - 1, v_i, N) = 1.$$

Then N is prime.
Proof : assume N is composite and let $p \mid N$ with $p \leq \sqrt{N}$.
Reduce A_N mod p towards A_p :

$$\tau = \theta \bmod p = (u \bmod p) + (v \bmod p)\alpha.$$

We get

$$\tau^{N+1} = 1 \text{ in } A_p,$$

and

$$\tau^{(N+1)/q_i} \neq 1 \text{ in } A_p$$

which proves τ has ordre $N + 1$ in $(A_p)^*$.
Hence $N + 1 \leq \# A_p = p^2$, contradiction. $\Box$

Choosing θ**:** using $\overline{\alpha} = -a_1 - \alpha$ (conjugate), enough to choose

$$\theta = \frac{\alpha + m}{\overline{\alpha} + m} = \frac{(m^2 - a_0) + (2m - a_1)\alpha}{m(m - a_1) + a_0}$$

for varying m.

Ex. Consider $N = 101$; $N + 1 = 2 \times 3 \times 17$. Take $a_1 = -2$, $a_0 = -1$, $\Delta = 8$ and $\left(\frac{8}{101}\right) = \left(\frac{2}{101}\right) = -1$. Take $\theta = 1 + 2\alpha$ (using $m = 1$)

$$\theta^{102} = 1, \theta^{102/2} = 100, \gcd(100 - 1, N) = 1,$$

$$\theta^{102/3} = 47T + 3, \gcd(3 - 1, 47, N) = 1,$$

$$\theta^{102/17} = 23T + 85, \gcd(85 - 1, 23, N) = 1.$$

Remarks

- Pocklington-like theorems exist.
- Deduce from this the degree 2 pseudoprimes.
- All this can be reformulated in terms of Lucas sequences (bouhhhh!).
- **Lucas-Lehmer:** $M_m = 2^m - 1$ is prime iff for $L_0 = 4$, $L_{n+1} = L_n^2 - 2 \bmod M_m$, one has $L_{m-2} = 0$ [using $\sqrt{3}$].
  $\Rightarrow$ largest known primes, e.g., $M_{43112609}$ with $12,978,189$ decimal digits. Lower bound (?) for primality proving algorithms: $O((\log N)\mathrm{M}(M_p))$ (super fast arithmetic!).

B) En route for **P**

- Gauss and Jacobi sums: L. Adleman, C. Pomerance, S. Rumely (1980, 1983); H. Cohen, H. W. Lenstra, Jr (1981 – 1984) ; H. Cohen, A. K. Lenstra (1982, 1987). W. Bosma & M.-P. van der Hulst (1990) ; P. Mihăilescu (1998). deterministic $O((\log N)^{c_1 \log\log\log N})$.
- almost **RP**: Goldwasser and Kilian using elliptic curves (1986); practical algorithm by Atkin (1986; later FM). See Smith's part.
- **RP**: Adleman and Huang using hyperelliptic curves (1986ff). See Smith's part.

C) Agrawal, Kayal, Saxena (AKS)

First idea: (Agrawal, Biswas – 1999)

Prop. N is prime iff $P(X) = (X+1)^N - X^N - 1 \equiv 0 \bmod N$.

In pratice: choose $Q(X) \in \mathbb{Z}/N\mathbb{Z}[X]$ at random of degree $O(\log N)$. If

$$(X+1)^N \not\equiv X^N + 1 \bmod (Q(X), N)$$

then N is composite.

The probability of failure is bounded by $1 - 1/(4 \log N)$.

Conjecture: If N is composite, there exists $1 \leq r \leq \log N$ s.t. $P(X)$ is not divisible by $X^r - 1$ modulo N.

Agrawal, Kayal, Saxena

Thm. Let N, s be integers, r a prime number and $q = P(r-1)$. If:

(0)

$$\binom{q-1+s}{s} > N^{2\lfloor\sqrt{r}\rfloor};$$

(i) $N \neq M^k$, $k > 1$;

(ii) N has no prime factor $\leq s$;

(iii) $N^{(r-1)/q} \bmod r \notin \{0, 1\}$;

(iv) $\forall a, 1 \leq a \leq s, (X-a)^N \equiv X^N - a \bmod (X^r - 1, N)$;

then N is prime.

Proof

Assume N composite. Let p prime dividing N ($p > s$ from (ii)), s.t. $p^{(r-1)/q} \not\equiv 1 \bmod r$, i.e., $q \mid d := \mathrm{ord}_r(p)$.

Prop. $\forall i, j,\ \forall a \in 1..s$: $(X-a)^{p^i N^j} = X^{p^i N^j} - a \bmod (X^r - 1, p)$.

Combinatorial argument: $L = \{p^i N^j, 0 \leq i, j \leq \lfloor \sqrt{r} \rfloor\}$; all elements of L are distinct, hence $\#L = (\lfloor \sqrt{r} \rfloor + 1)^2 > r$.

$\Rightarrow$ two elements $u_2 > u_1$ are equal modulo r:

$$u_1 = p^{i_1} N^{j_1}, u_2 = p^{i_2} N^{j_2} = u_1 + kr, (i_1, j_1) \neq (i_2, j_2).$$

$$(X-a)^{u_2} = X^{u_1 + kr} - a = X^{u_1} - a = (X-a)^{u_1} \bmod (X^r - 1, p).$$

It suffices to prove $u_1 = u_2$, hence a contradiction.

$\forall a = 1..s, (X-a)^N = X^N - a \bmod (X^r - 1, p)$.

But $(X-a)^p = X^p - a \bmod (X^r - 1, p)$.

Lemma. If $(X-a)^{m_1} = X^{m_1} - a \bmod (X^r - 1, p)$ and $(X-a)^{m_2} = X^{m_2} - a \bmod (X^r - 1, p)$, then $(X-a)^{m_1 m_2} = X^{m_1 m_2} - a \bmod (X^r - 1, p)$.

Proof. There exists $g(X) \in \mathbb{F}_p[X]$ s.t.:

$$(X-a)^{m_2} - (X^{m_2} - a) = (X^r - 1) g(X)$$

$$\begin{aligned}(X^{m_1} - a)^{m_2} - (X^{m_1 m_2} - a) &= (X^{m_1 r} - 1) g(X^{m_1}) \\ &= (X^r - 1) f(X) g(X^{m_1})\end{aligned}$$

$$(X-a)^{m_1 m_2} \equiv (X^{m_1} - a)^{m_2} \equiv X^{m_1 m_2} - a \bmod (X^r - 1, p). \square$$

Let $h(X)$ be an irreducible factor of $\Phi_r(X)$ (r-th cyclotomic polynomial) in $\mathbb{F}_p[X]$.
Classical result: $F = \mathbb{F}_p[X]/(h(X))$ is a finite field of degree $d = \mathrm{ord}_r(p) \geq q > s$.

Put $\theta = X \bmod (h(X), p)$ and $S = \{\prod_{a=1}^{s} (\theta - a)^{\alpha_a}, \alpha_a \in \mathbb{N}\}$.

Lemma. $\#S \geq \binom{q-1+s}{s}$.

Proof. all $X - a$ are irreducible and distinct in $\mathbb{F}_p[X]$, since $p > s$. All $\prod (X-a)^{\alpha_a}$ with $\sum \alpha_a < q \leq \deg(h)$, are all distinct, therefore this is true for all $\prod (\theta - a)^{\alpha_a}$. $\square$

End of proof: by construction, if $\beta \in S$: $\beta^{u_1} = \beta^{u_2}$, i.e., β is a root of $Y^{u_2} - Y^{u_1} = Y^{u_1} Q(Y)$.

$$u_2 - u_1 \leq N^{2\lfloor \sqrt{r} \rfloor} < \binom{q-1+s}{s} \leq \#S,$$

hence $Q = 0$ and $u_2 = u_1$. $\square$

A combinatorial proof

Lemma. $\#\{(\alpha_i), \sum \alpha_i < q\} = \binom{q-1+s}{s}$.

Proof: bijection with subsets of s elements of $[1, q-1+s]$.
If (α_a) is a solution:

$$\begin{aligned}\beta_1 &= \alpha_1 + 1, \\ \beta_2 &= \alpha_1 + \alpha_2 + 2, \\ &\cdots \\ \beta_s &= \alpha_1 + \alpha_2 + \cdots + \alpha_s + s.\end{aligned}$$

$\Rightarrow 1 \leq \beta_1 < \beta_2 < \cdots < \beta_s \leq q - 1 + s.$ $\square$

Analysis

Cost: s computations of X^N modulo (X^r-1,N); one computation costs $O(\log N)$ products of degree r polynomials, hence:

$$O(s(\log N)\mathsf{M}_P(r)\mathsf{M}(\log N)).$$

Prop. If $s=\lfloor 2\lfloor\sqrt{r}\rfloor\log N/\log 2\rfloor+1$ and $q\geq 2s$, then

$$\binom{q-1+s}{s}>N^{2\lfloor\sqrt{r}\rfloor}.$$

Proof:

$$\binom{q-1+s}{s}>(q/s)^s\geq 2^s>N^{2\lfloor\sqrt{r}\rfloor}.$$

Coro. $O((\log N)^2 r^{1/2}\mathsf{M}_P(r)\mathsf{M}(\log N))$.

Analytical number theory: we can find $r=(\log N)^{2/(2\delta-1)}$ for $\delta\in]0.5,0.676]$.

At last...

Using $r=(\log N)^{2/(2\delta-1)}$.
Coro. There exists a deterministic primality proving algorithm whose running time is

$$O\left((\log N)^{(8\delta+1)/(2\delta-1)}\right)$$

using $\mathsf{M}_P(r)=r^2$, $\mathsf{M}(\log N)=(\log N)^2$; and

$$\tilde{O}\left((\log N)^{6\delta/(2\delta-1)}\right)$$

with $\mathsf{M}_P(r)=\tilde{O}(r)$, $\mathsf{M}(\log N)=\tilde{O}(\log N)$.
Proof:

$$L^2 r^{1/2}\mathsf{M}_P(r)\mathsf{M}(\log N)=L^{2+2/(2\delta-1)+4/(2\delta-1)+2}.\square$$

Ex. (AKS original) $\delta=2/3$: 19, 12; $\delta=1$ (Sophie Germain): 9, 6.

Rem. Jacobi $O((\log N)^{c\log\log\log N})$, ECPP $\tilde{O}((\log N)^4)$.

Rem. Non effective.

What next?

- cf. D. Bernstein homepage for more on the history of improvements to the basic test.
- Including: H. W. Lenstra, Jr. ($\tilde{O}_{\mathit{eff}}((\log N)^{12})$ or $\tilde{O}((\log N)^8)$), S. David.
- Cleaner version of AKS: $\tilde{O}_{\mathit{eff}}((\log N)^{10.5})$ or $\tilde{O}((\log N)^{7.5})$.
- H. W. Lenstra, C. Pomerance : $\tilde{O}_{\mathit{eff}}((\log N)^6)$.
- P. Berrizbeitia / Q. Cheng :
  Let r prime s.t. $r^\alpha \mid\mid N-1$, $r\geq\log^2 N$; $1<a<N$ s.t.
  $a^{r^\alpha}\equiv 1 \bmod N$, $\gcd(a^{r^{\alpha-1}}-1,N)=1$,
  $(X+1)^N=X^N+1 \bmod (X^r-a,N)$, then N is prime. Heuristic complexity would be $\tilde{O}((\log N)^4)$ for these numbers.
- D. Bernstein, P. Mihăilescu: use $e\mid N^d-1$; inject cyclotomic ideas, $\tilde{O}((\log N)^4)$.

V. Conclusions for primality

Which algorithm?

- easy to understand / implement, fast: compositeness tests;
- fast, proven: Jacobi;
- fast, heuristic: ECPP;
- certificate: ECPP;
- deterministic polynomial: AKS.

D. Bernstein has an AKS example for $2^{1024}+643$ (13 hours on 800 MHz PC, 200 Mb memory).

To be compared to FASTECPP:
14/07/03: FM, 7000dd with mpifastECPP.
19/08/03: J. Franke, T. Kleinjung, T. Wirth, 10000dd.
06/06: FM, 20,562 dd avec mpifastECPP.

What's left to be done?

Open questions:

- Is $\tilde{O}((\log N)^4)$ the best running time for all numbers?
  Compare: $\tilde{O}((\log N)^2)$ for Fermat or Mersenne numbers.

  Claim (Lukes, Patterson, Williams): $\tilde{O}((\log N)^3)$ under GRH? (pseudosquares or pseudocubes).

- Combination of tests?