A contractor which is minimal for narrow boxes

L. Jaulin

18-19 april 2023
Séminaire AID/CIEDS, Palaiseau

Abstract. Centered form is one of the most fundamental brick in interval analysis. It is traditionally used to enclose the range of a function over narrow intervals. The quadratic approximation property guarantees an asymptotically small overestimation for sufficiently narrow boxes. In this presentation, I will propose to use the centered form to build efficient contractors that are optimal when the intervals are narrow. The method is based on the centered form combined with a Gauss Jordan band diagonalization preconditioning.

1. Stability of a linear systems

Consider the system [1]

$$\dddot{x} + \sin(p_1 p_2) \cdot \ddot{x} + p_1^2 \cdot \dot{x} + p_1 p_2 \cdot x = 0$$

Its characteristic function is

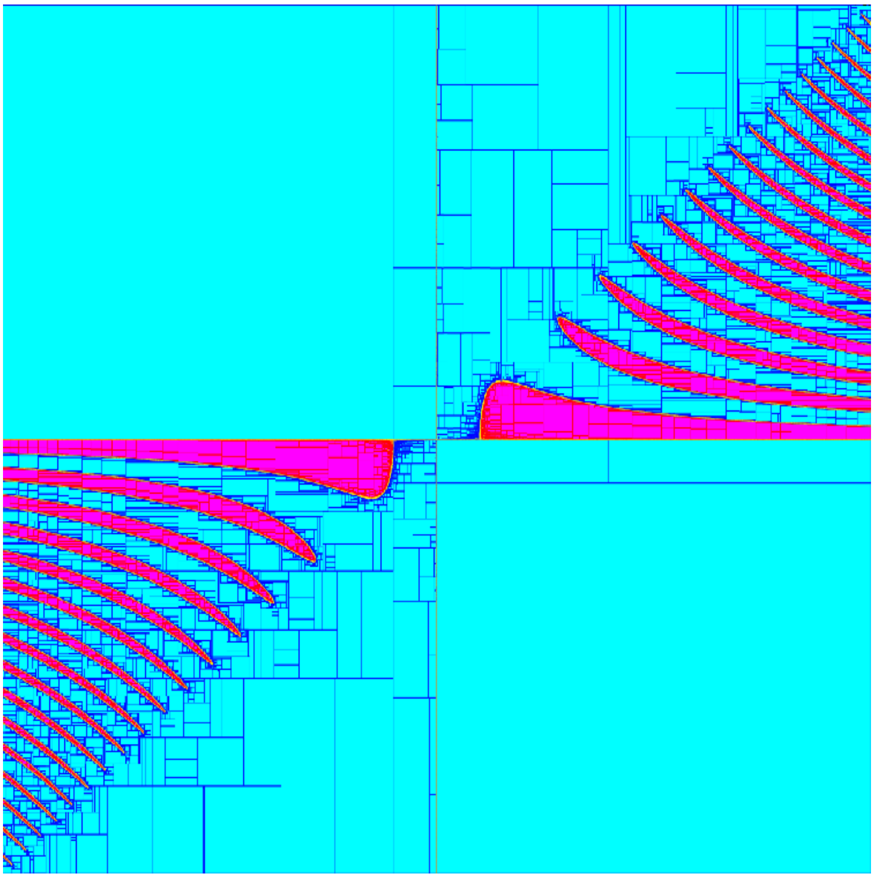
$$\theta(\mathbf{p}, s) = s^3 + \sin(p_1 p_2) \cdot s^2 + p_1^2 \cdot s + p_1 p_2$$

Stability domain

$$\mathbb{S} = \{\mathbf{p} \,|\, \theta(\mathbf{p}, s)\, \text{Hurwitz}\}\,.$$

We have

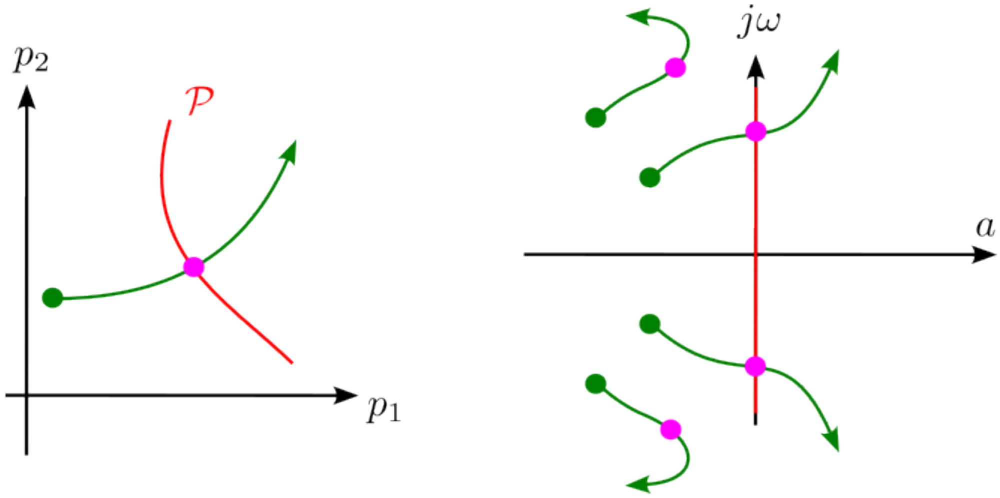
$$\mathbb{S} : \begin{cases} p_1 p_2 & \geq 0 \\ \sin(p_1 p_2) & \geq 0 \\ p_1^2 \sin(p_1 p_2) - p_1 p_2 & \geq 0 \end{cases}$$



Value set approach

The roots of $\theta(\mathbf{p}, s) = 0$ change continuously with $\mathbf{p}$.
We define the *value set*

$$\mathscr{P} = \{\mathbf{p} \,|\, \exists \omega > 0,\, \theta(\mathbf{p}, j\omega) = 0\}.$$



Zero exclusion theorem

Cut off frequency. The roots of

$$P(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

are in the disk with center 0 and radius

$$\omega_c = 1 + \max\{\|a_0\|, \|a_1\|, \ldots, \|a_{n-1}\|\}$$

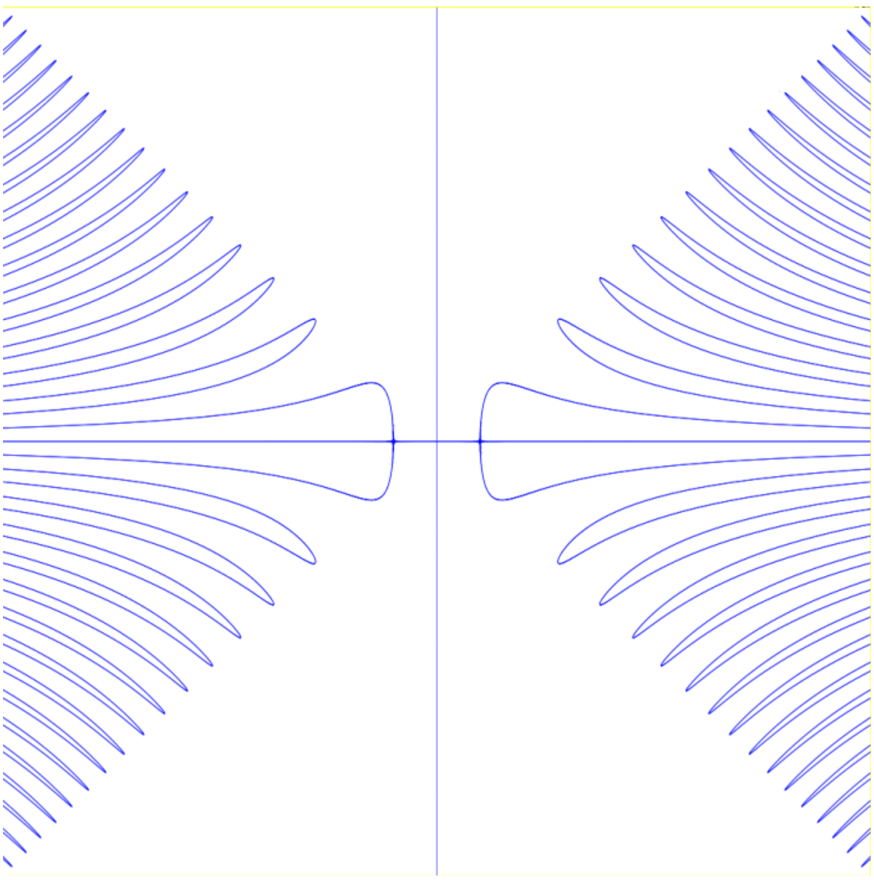
which is the Cauchy bound.

For

$$\theta(\mathbf{p},s) = s^3 + \sin(p_1p_2)\cdot s^2 + p_1^2\cdot s + p_1p_2$$

and $s = j\omega$, we get

$$\begin{array}{cc} & (j\omega)^3 + \sin(p_1p_2)\cdot(j\omega)^2 + p_1^2\cdot(j\omega) + p_1p_2 = 0 \\ \Leftrightarrow & -j\omega^3 - \sin(p_1p_2)\cdot\omega^2 + jp_1^2\cdot\omega + p_1p_2 = 0 \\ \Leftrightarrow & \left\{ \begin{array}{rcl} -\sin(p_1p_2)\cdot\omega^2 + p_1p_2 & = & 0 \\ -\omega^2 + p_1^2 & = & 0 \end{array} \right. \end{array}$$



Linear systems with delays

Periodic system

$$x(t+1) - x(t) = 0$$

The characteristic function is

$$\theta(s) = e^s - 1$$

The roots are

$$s = 2\pi kj, k \in \mathbb{N}$$

Turkulov system. Consider the system [7]:

$$\ddot{x}(t) + 2\dot{x}(t - p_1) + x(t - p_2) = 0$$

Its characteristic function is

$$\theta(\mathbf{p}, s) = s^2 + 2se^{-sp_1} + e^{-sp_2}.$$

We define

$$\mathscr{P} = \{\mathbf{p} \mid \exists \omega > 0,\, \theta(\mathbf{p}, j\omega) = 0\}.$$

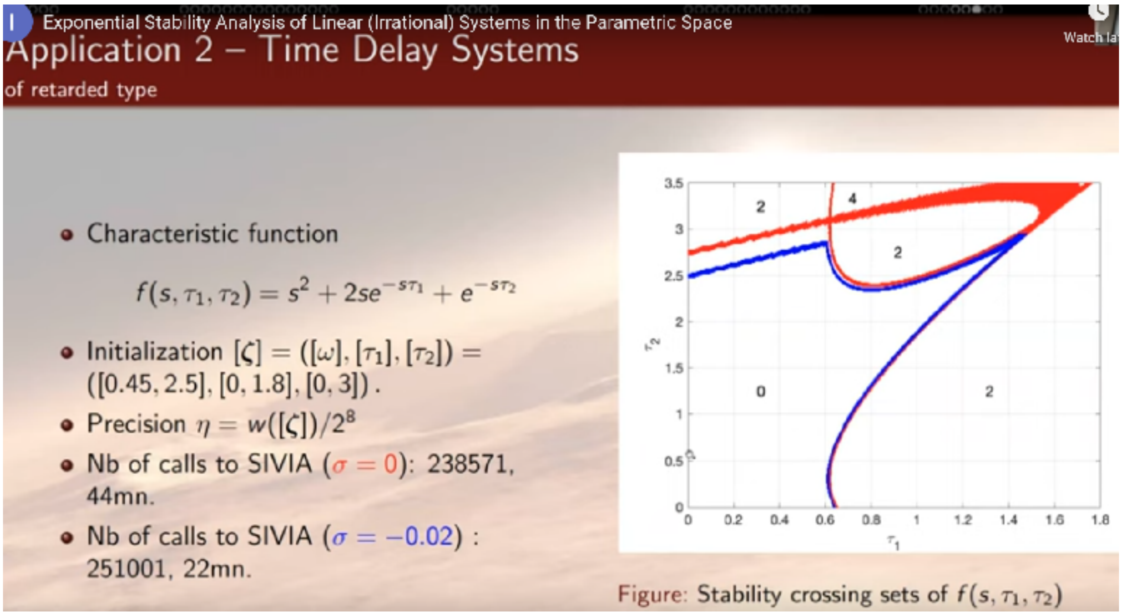
Now

$$\begin{array}{rl} & \theta(p_1, p_2, j\omega) \\ = & -\omega^2 + 2j\omega e^{-j\omega p_1} + e^{-j\omega p_2} \\ = & -\omega^2 + 2j\omega(\cos(\omega p_1) - j\sin(\omega p_1)) \\ & +\cos(\omega p_2) - j\sin(-\omega p_2) \\ = & -\omega^2 + 2\omega\sin(\omega p_1) + \cos(\omega p_2) \\ & +j\cdot(2\omega\cos(\omega p_1) - \sin(\omega p_2)) \end{array}$$

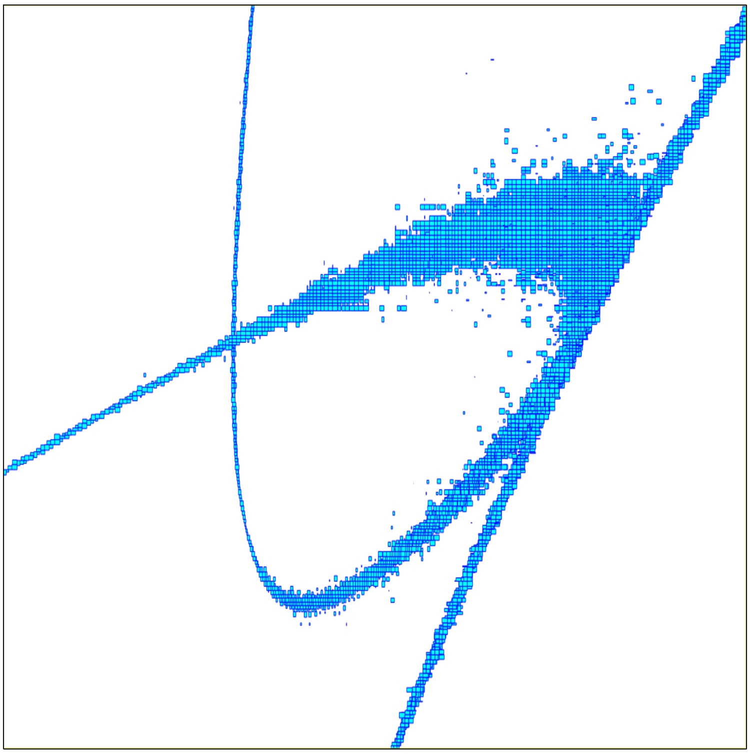
We have

$$\theta(p_1, p_2, j\omega) = 0$$
$$\Leftrightarrow \underbrace{\begin{pmatrix} -\omega^2 + 2\omega\sin(\omega p_1) + \cos(\omega p_2) \\ 2\omega\cos(\omega p_1) - \sin(\omega p_2) \end{pmatrix}}_{\mathbf{f}(p_1, p_2, \omega)} = \mathbf{0}$$

With $[p_1] = [0, 2.5]$, $[p_2] = [1, 4]$, $[\omega] = [0, 10]$, with a Matlab implementation, with a forward-backward contractor, and $\varepsilon = 2^{-8}$, [2] got:



https://youtu.be/DaR2NZZIV10?t=2453



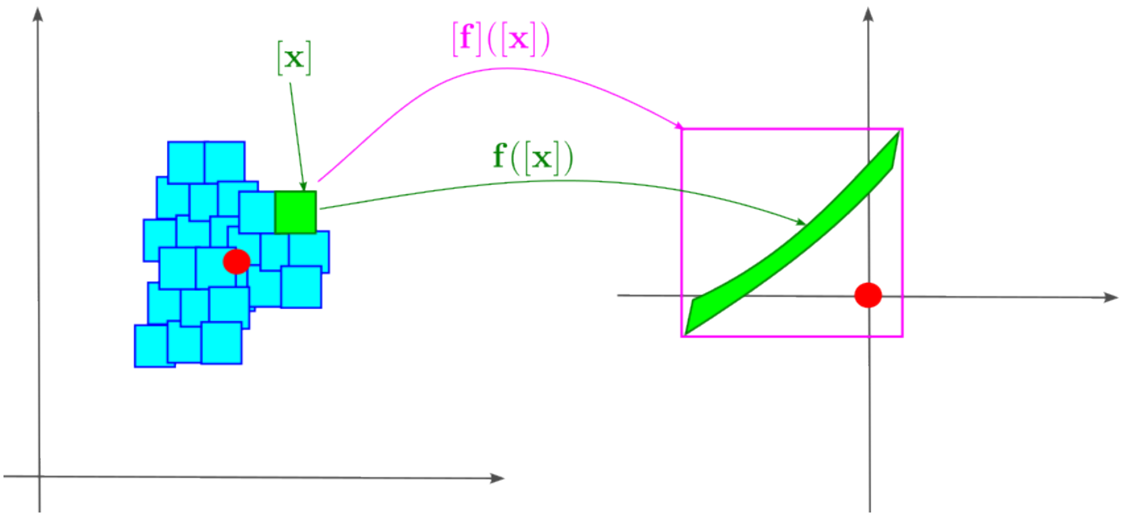
$\varepsilon = 2^{-8}$, Codac [6] generated 43173 boxes.

We still have a *Clustering effect*

2. Minimal contractors

Given a function $\mathbf{f}:\mathbb{R}^n \mapsto \mathbb{R}^p$. An inclusion function for $\mathbf{f}$ is minimal if

$$[\mathbf{f}]([\mathbf{x}]) = [\![\{\mathbf{y} = \mathbf{f}(\mathbf{x}) \mid \mathbf{x} \in [\mathbf{x}]\}]\!].$$

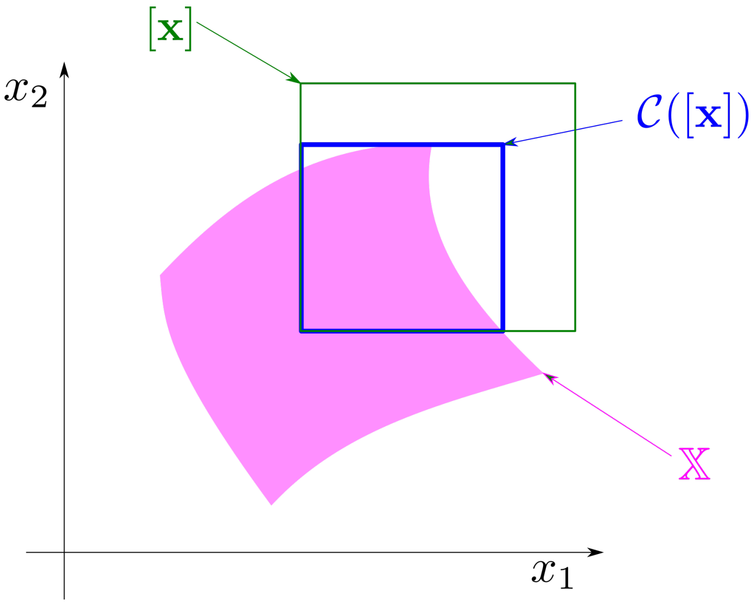


With a minimal inclusion, the clustering effect may exist, when solving $\mathbf{f}(\mathbf{x}) = \mathbf{0}$

A *contractor* associated to the set $\mathbb{X} \subset \mathbb{R}^n$ is a function $\mathscr{C} : \mathbb{IR}^n \mapsto \mathbb{IR}^n$ such that

$$\begin{aligned} \mathscr{C}([\mathbf{x}]) \subset [\mathbf{x}] &\qquad \text{(contraction)} \\ [\mathbf{x}] \cap \mathbb{X} \subset \mathscr{C}([\mathbf{x}]) &\qquad \text{(consistency)} \end{aligned}$$

It is *minimal* if $\mathscr{C}([\mathbf{x}]) = [\![[\mathbf{x}] \cap \mathbb{X}]\!]$.


$[\mathbf{x}]$
x_2
$\mathcal{C}([\mathbf{x}])$
$\mathbb{X}$
x_1

Tree matrices

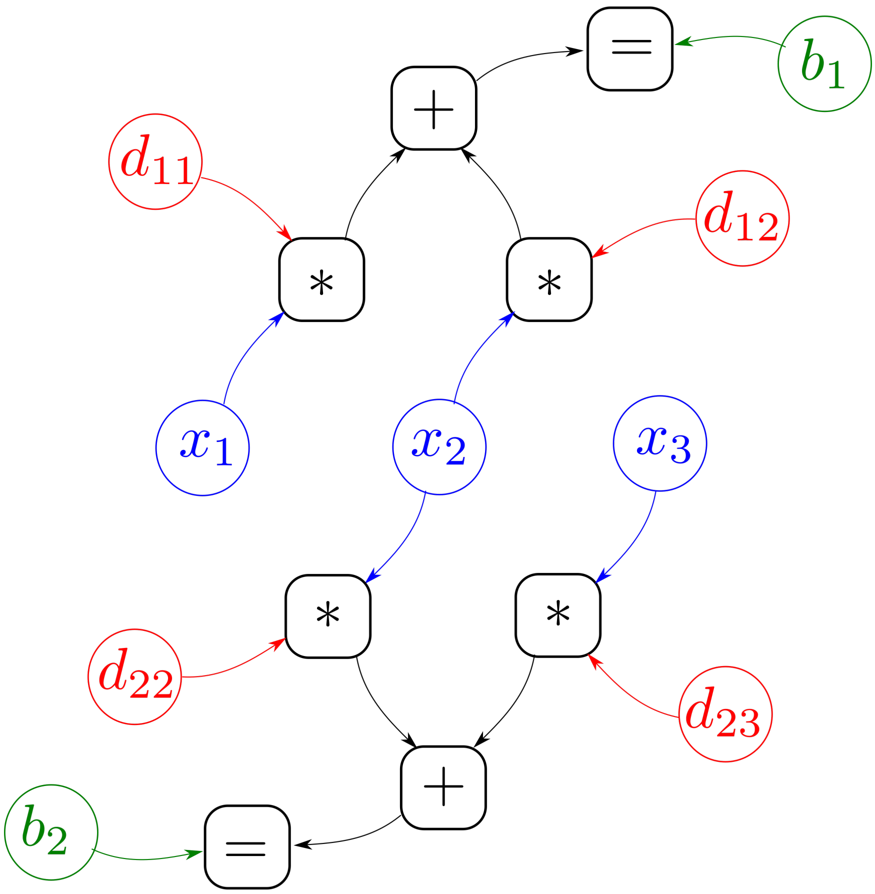
Consider the interval linear system:

$$\begin{pmatrix} d_{11} & d_{12} & 0 \\ 0 & d_{22} & d_{23} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

where

$$d_{ij} \in [d_{ij}], x_j \in [x_j], b_i \in [b_i]$$

The optimal contraction can be obtained by a simple interval propagation [3].


=
b_1
+
d_{11}
*
*
d_{12}
x_1
x_2
x_3
*
*
d_{22}
d_{23}
+
b_2
=

No cycle for:

$$\begin{pmatrix} d_{11} & d_{12} & 0 & 0 \\ 0 & d_{22} & d_{23} & 0 \\ 0 & 0 & d_{33} & d_{34} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

A matrix $\mathbf{D}$ such that $\mathbf{D} \cdot \mathbf{x} = \mathbf{b}$ has no cycle is a *tree matrix*.

We a Gauss Jordan transformation:

$$\mathbf{A}\mathbf{x} = \mathbf{c} \Leftrightarrow \mathbf{Q} \cdot \mathbf{A} \cdot \mathbf{x} = \mathbf{Q} \cdot \mathbf{c}$$

we may get a tree matrix: $\mathbf{D} = \mathbf{Q} \cdot \mathbf{A}$.

Simplex contractor

For the linear system

$$\mathbf{Ax} = \mathbf{c}, \mathbf{x} \in [\mathbf{x}], \mathbf{c} \in [\mathbf{c}]$$

we can use the simplex algorithm to build the minimal contractor. Guarantee can be obtained with an inflation [5]

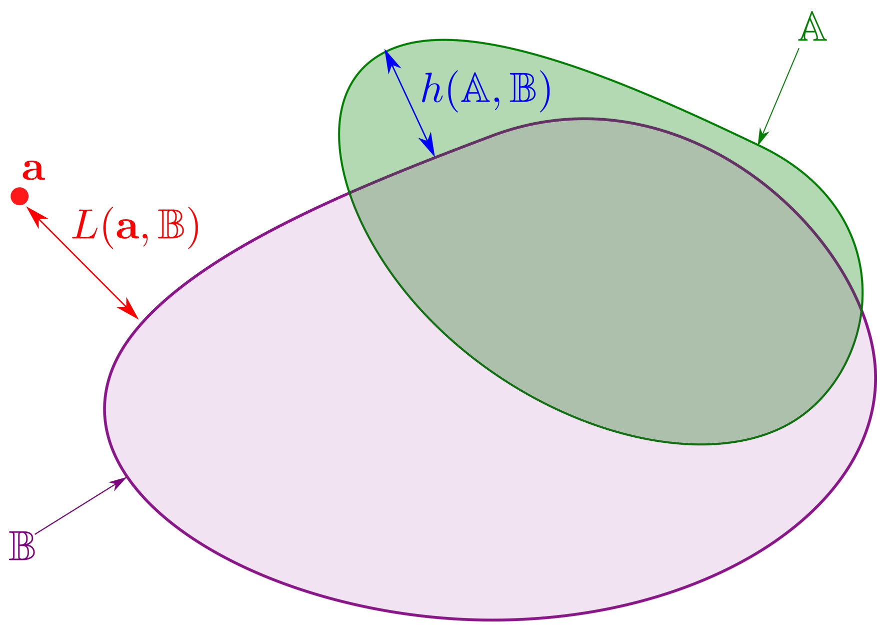
3. Asymptotic minimality

Proximity. Denote by $L(\mathbf{a},\mathbf{b})$ a distance between $\mathbf{a}$ and $\mathbf{b}$ of $\mathbb{R}^n$ induced by the L-norm (L_∞ or L_2).
The *proximity* of $\mathbb{A}$ to $\mathbb{B}$ is

$$h(\mathbb{A},\mathbb{B}) = \sup_{\mathbf{a}\in\mathbb{A}} L(\mathbf{a},\mathbb{B})$$

where

$$L(\mathbf{a},\mathbb{B}) = \inf_{\mathbf{b}\in\mathbb{B}} L(\mathbf{a},\mathbf{b}).$$



Proximity of $\mathbb{A}$ to $\mathbb{B}$

Definition. The pessimism of an inclusion function $[\mathbf{f}]$ is

$$\eta([\mathbf{x}]) = h([\mathbf{f}]([\mathbf{x}]), [\![\mathbf{f}([\mathbf{x}])]\!])$$

Definition [4]. An inclusion function $[\mathbf{f}]$ is of order j if

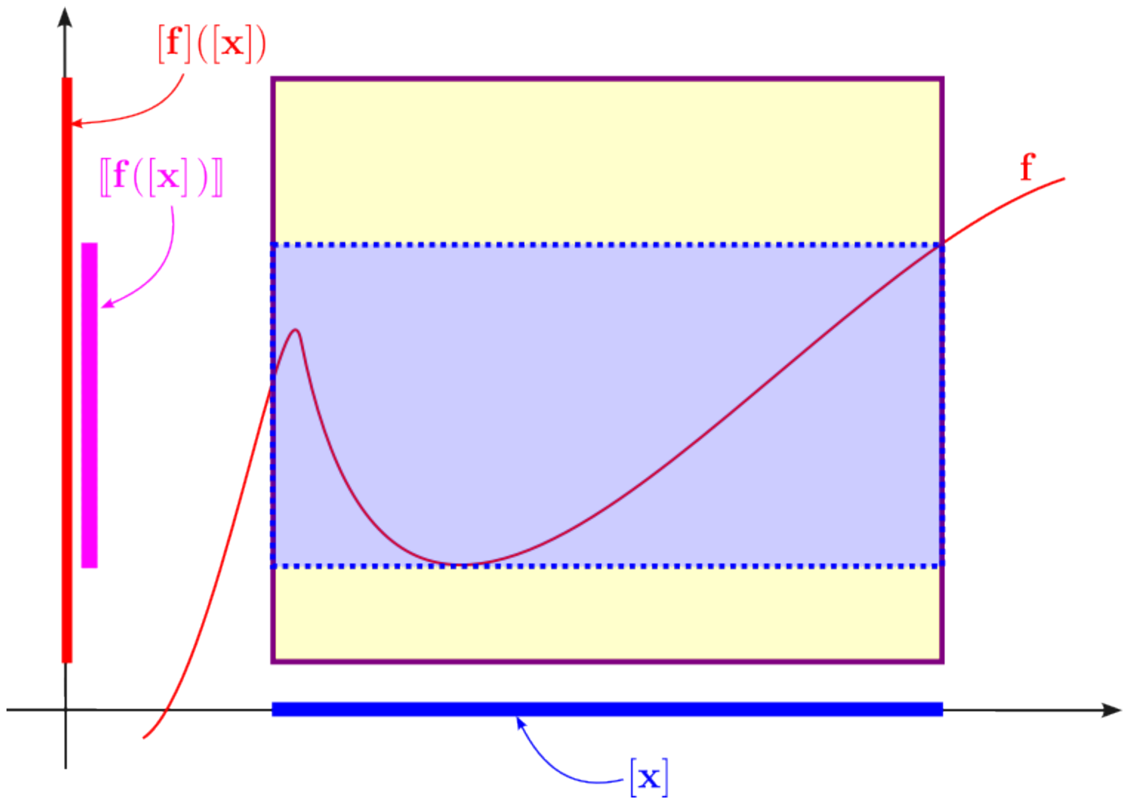
$$\eta([\mathbf{x}]) = o(w^j([\mathbf{x}]))$$

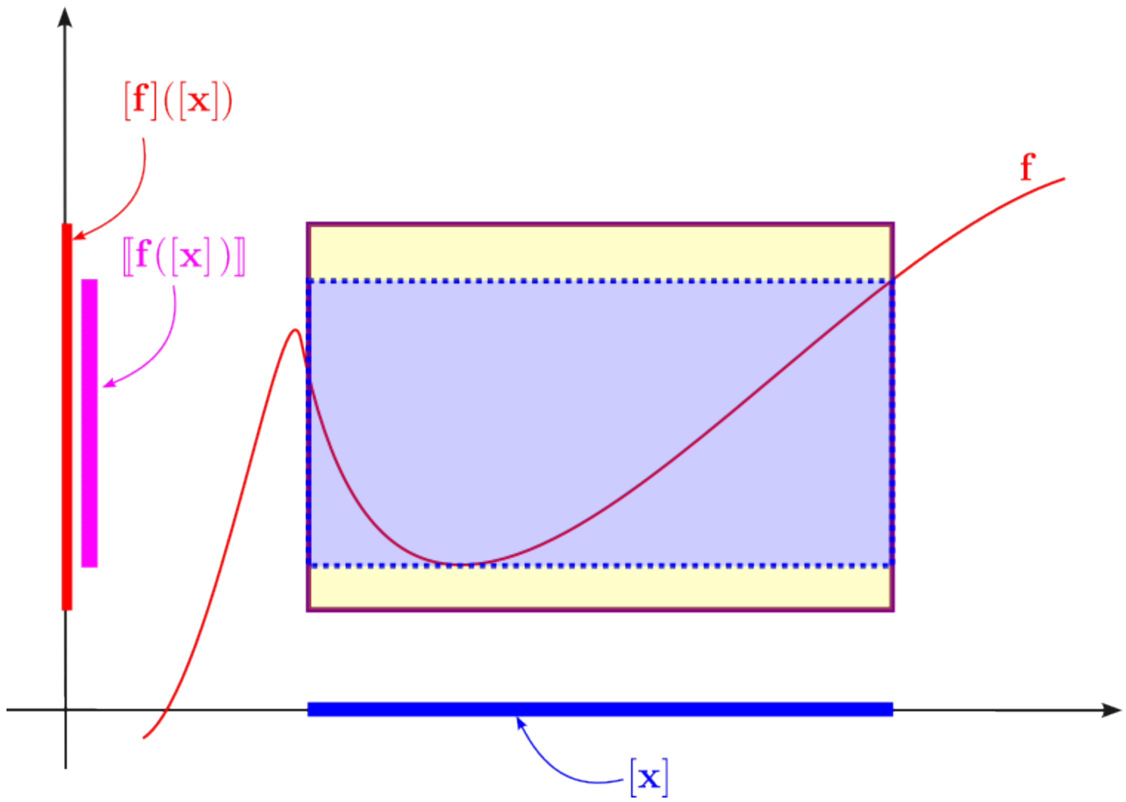
Definition. $[\mathbf{f}]$ is convergent if it is of order $j = 0$:

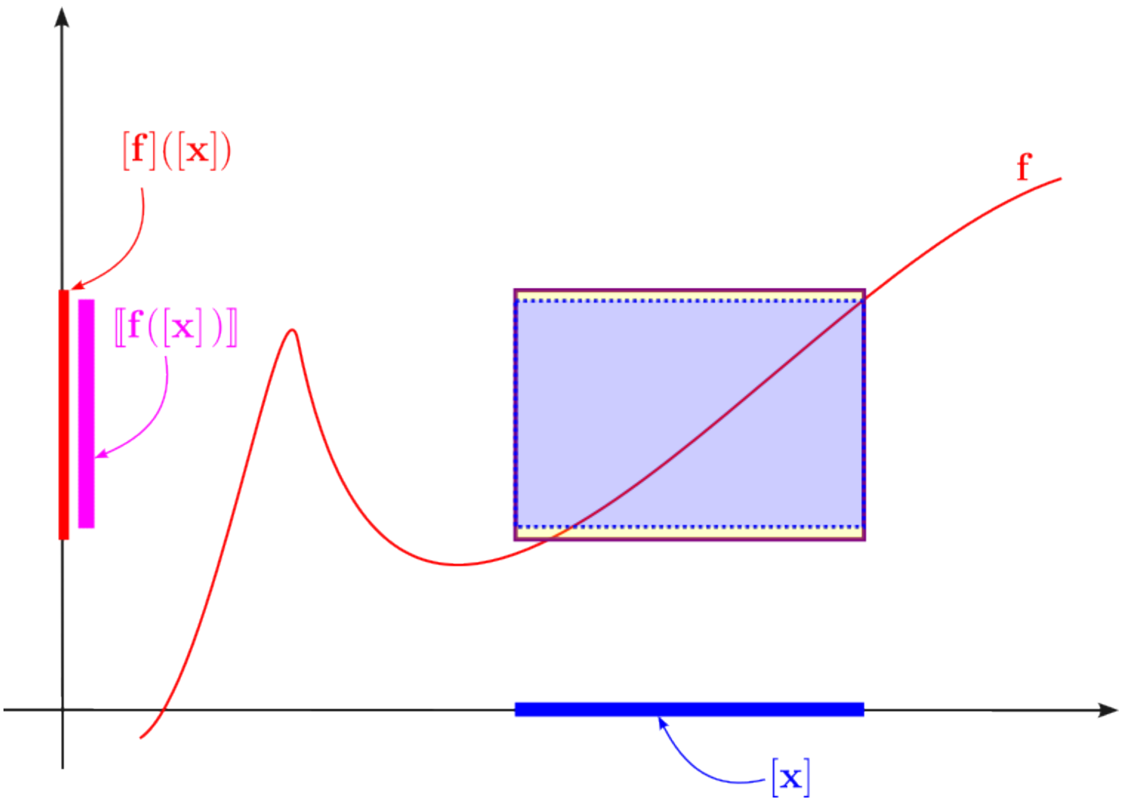
$$\eta([\mathbf{x}]) = o(w^0([\mathbf{x}])) = O(w([\mathbf{x}]))$$

Definition. $[\mathbf{f}]$ is asymptotically minimal if it is of order $j = 1$:

$$\eta([\mathbf{x}]) = o(w([\mathbf{x}]))$$


[f]([x])
[[f([x])]]
f
[x]


[f]([x])
[[f([x])]]
f
[x]


[f]([x])
[[f([x])]]
f
[x]

Proposition [4]. The centered form

$$[\mathbf{f}]([\mathbf{x}]) = \mathbf{f}(\mathbf{m}) + [\mathbf{f}'] ([\mathbf{x}]) \cdot ([\mathbf{x}] - \mathbf{m})$$

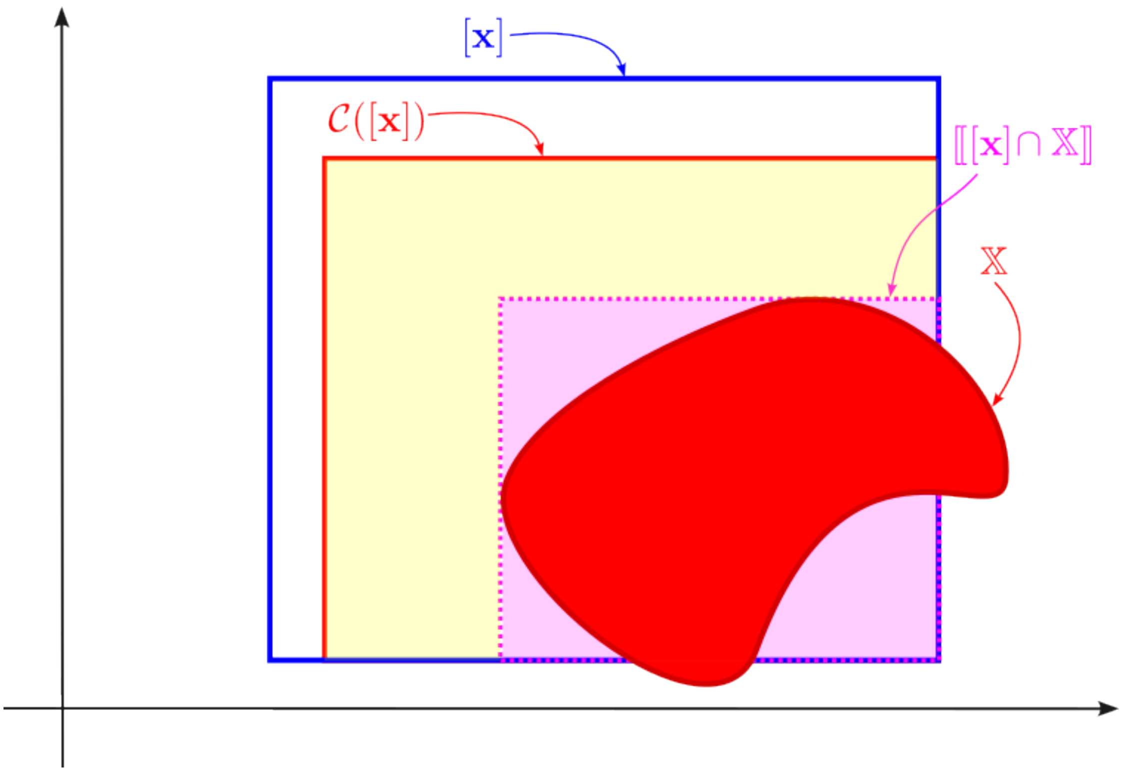
where $\mathbf{m} = \text{center}([\mathbf{x}])$ is asymptotically minimal.

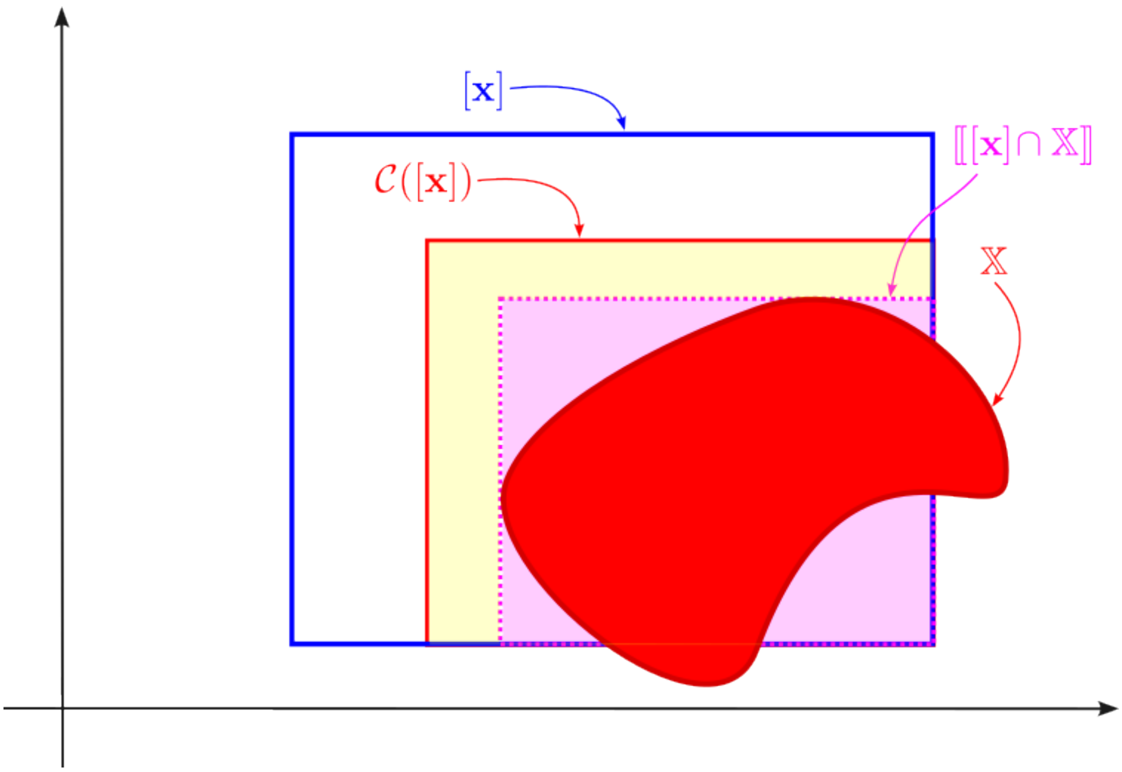
Definition. The *pessimism* of a contractor $\mathscr{C}$ for $\mathbb{X}$ at $[\mathbf{x}]$ is

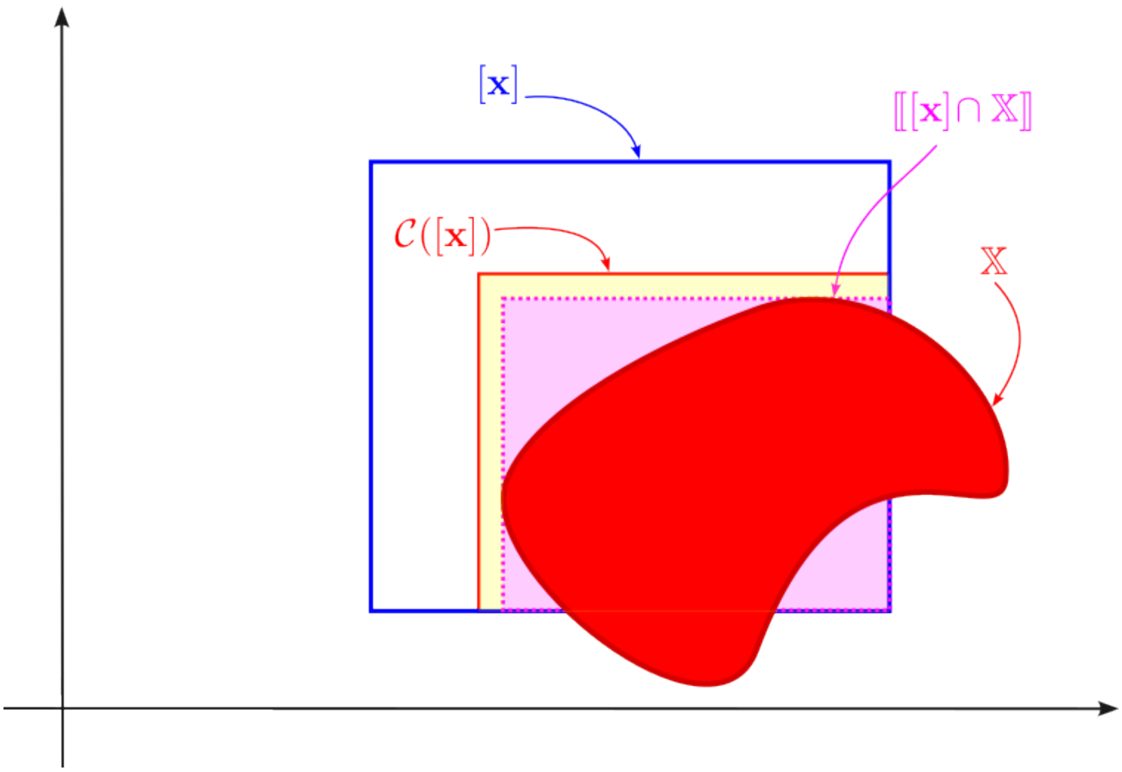
$$\eta([\mathbf{x}]) = h(\mathscr{C}([\mathbf{x}]), [\![[\mathbf{x}] \cap \mathbb{X}]\!])$$

Definition. A contractor $\mathscr{C}$ for $\mathbb{X}$ is of order j if

$$\eta([\mathbf{x}]) = o(w^j([\mathbf{x}]))$$


$[\mathbf{x}]$
$\mathcal{C}([\mathbf{x}])$
$[[\mathbf{x}] \cap \mathbb{X}]$
$\mathbb{X}$


[x]
$\mathcal{C}([\mathbf{x}])$
$[[\mathbf{x}] \cap \mathbb{X}]$
$\mathbb{X}$

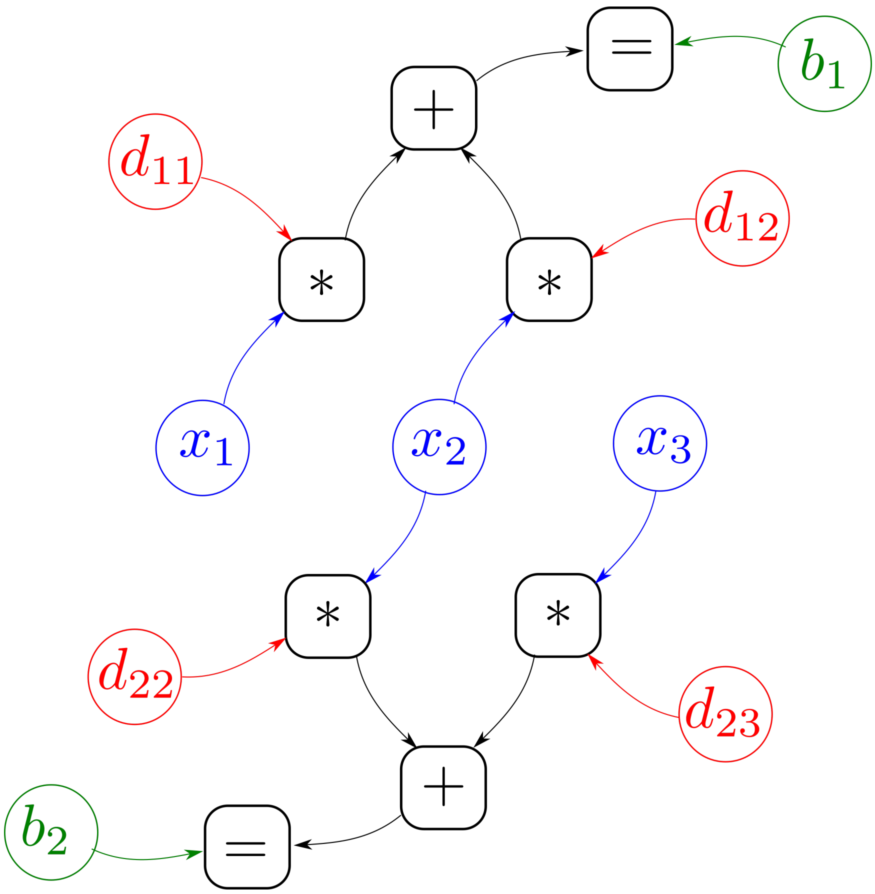

$[\mathbf{x}]$
$[[\mathbf{x}] \cap \mathbb{X}]$
$\mathcal{C}([\mathbf{x}])$
$\mathbb{X}$

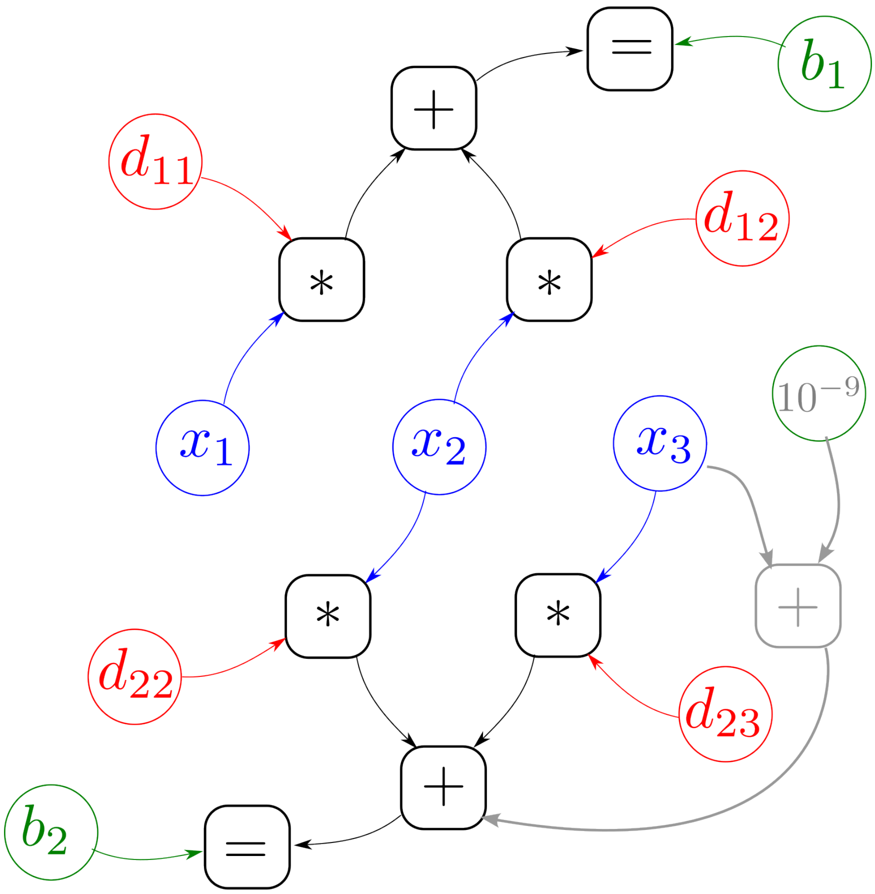
Proposition. Consider a set $\mathbb{X} = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{f}(\mathbf{x}) = \mathbf{0}\}$. Take $[\mathbf{x}]$ with center $\mathbf{m}$. Define $\mathbf{Q}$ s.t. $\mathbf{Q} \cdot \frac{d\mathbf{f}}{d\mathbf{x}}(\mathbf{m})$ is a tree matrix.
An interval propagation on;

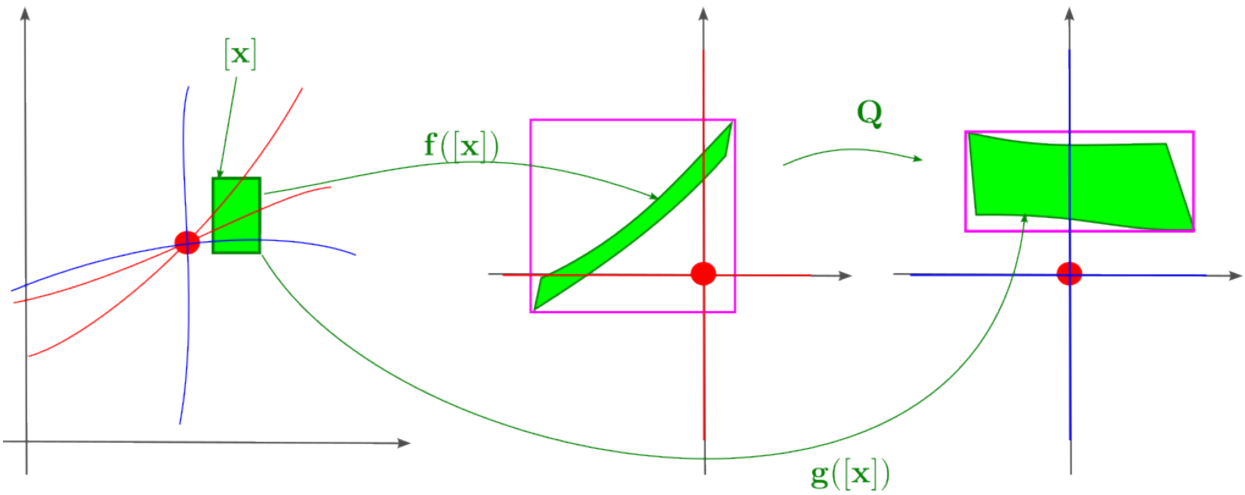
$$\begin{array}{c} \mathbf{Q} \cdot \mathbf{f}(\mathbf{m}) + \mathbf{Q} \cdot \mathbf{A} \cdot (\mathbf{x} - \mathbf{m}) = \mathbf{0} \\ \mathbf{A} \in [\frac{d\mathbf{f}}{d\mathbf{x}}]([\mathbf{x}]) \\ \mathbf{x} \in [\mathbf{x}] \end{array}$$

yields an asymptotically minimal contractor for $\mathbb{X}$.

Proof. ...


=
b_1
+
d_{11}
d_{12}
*
*
x_1
x_2
x_3
*
*
d_{22}
d_{23}
+
b_2
=


b_1
d_{11}
d_{12}
x_1
x_2
x_3
10^{-9}
d_{22}
d_{23}
b_2


[x]
f([x])
Q
g([x])

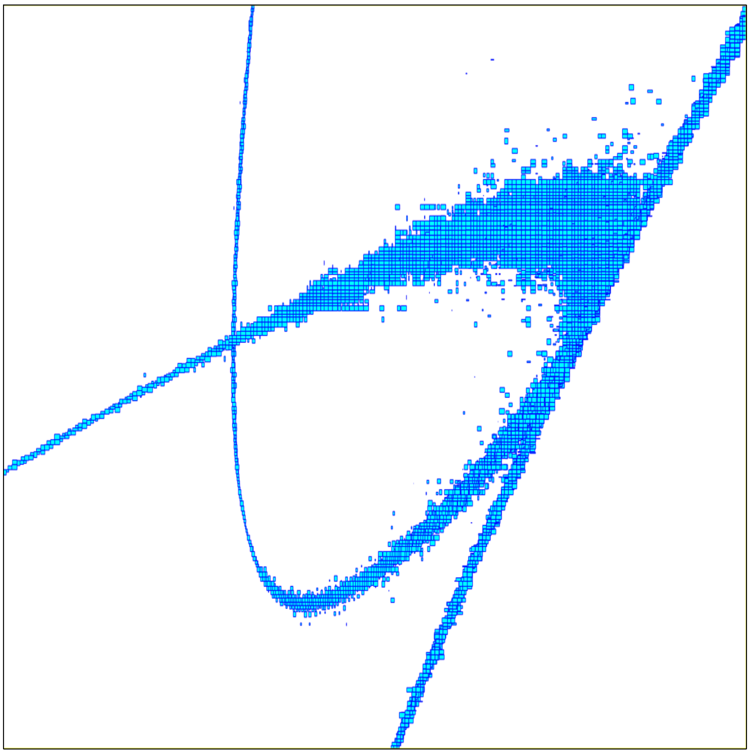
Centered contractor

Input: $\mathbf{f}, [\mathbf{x}]$

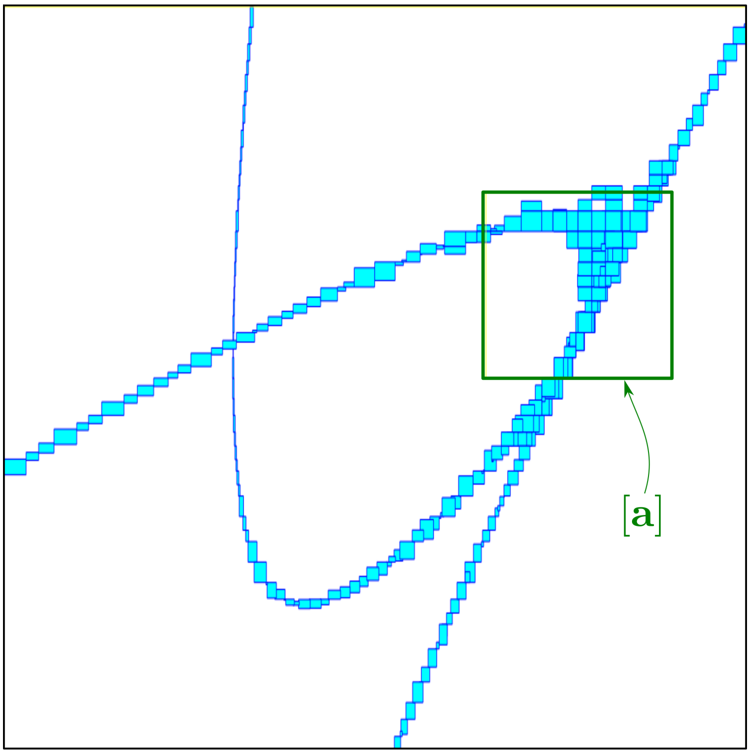
$\mathbf{m} = \text{center}([\mathbf{x}])$
Compute the Gauss-Jordan matrix $\mathbf{Q}$ for $\frac{d\mathbf{f}}{d\mathbf{x}}(\mathbf{m})$
Define $\mathbf{g}(\mathbf{x}) = \mathbf{Q} \cdot \mathbf{f}(\mathbf{x})$
For $i \in \{1, \ldots, p\}$
  For $j \in \{1, \ldots, n\}$
    $[\mathbf{a}] = [\frac{\partial g_i}{\partial \mathbf{x}}]([\mathbf{x}])$
    $[s] = \sum_{k \neq j} [a_k] \cdot ([x_k] - m_k)$
    $[x_j] = [x_j] \cap (-g_i(\mathbf{m}) - [s])$
Return $[\mathbf{x}]$

```
def GaussJordan(A):
    n=A.shape[0]
    m=A.shape[1]
    P,L,U = lu(A)
    Q=inv(P@L)
    for i in range(n-1, 0, -1):
        p=m-n
        K=U[i,i+p]*np.eye(n)
        K[0:i,i]=-U[0:i,i+p]
        Q=K@Q
        U=Q@A
    return Q
```

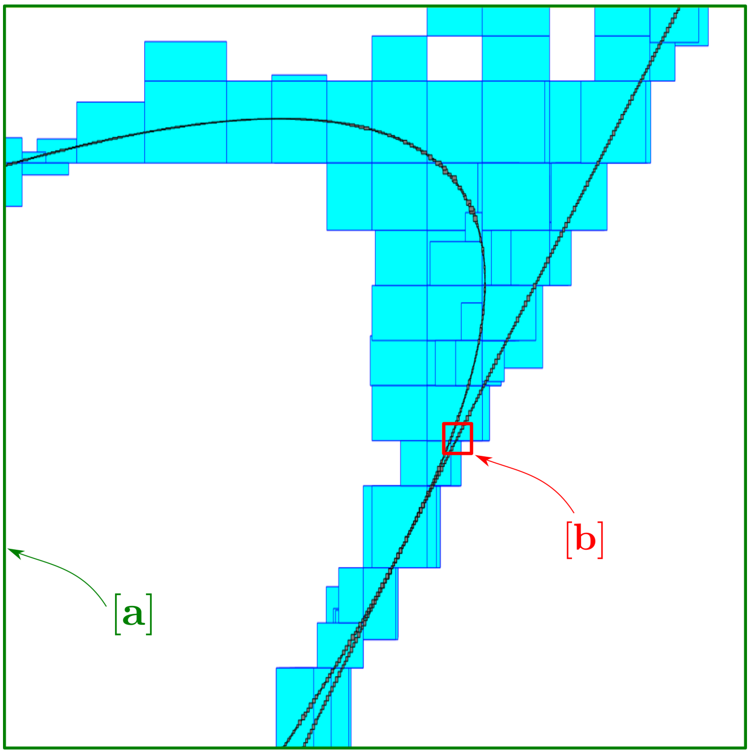
4. Results



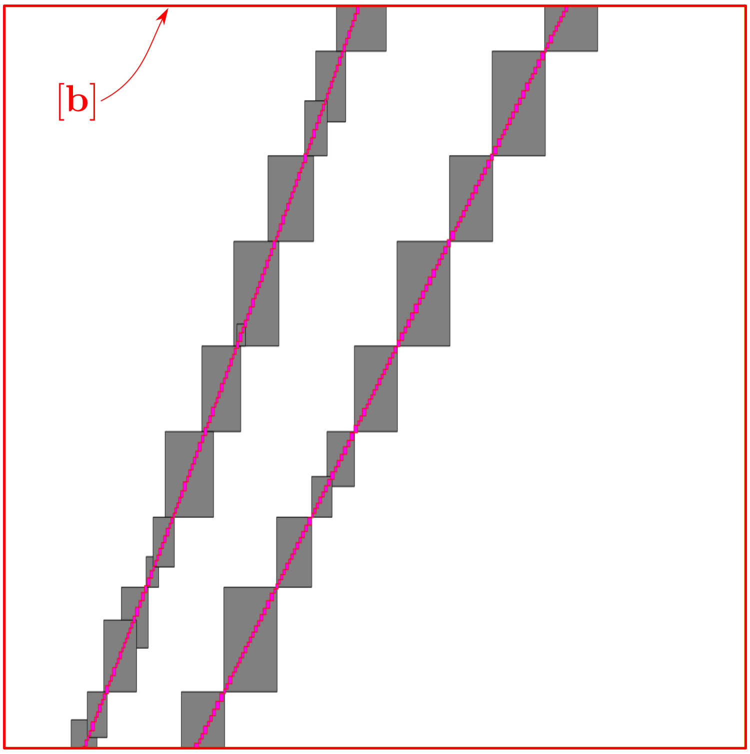
With a forward-backward contractor and $\varepsilon = 2^{-8}$



With the centered contractor $\varepsilon = 2^{-4}$



Blue: $\varepsilon = 2^{-4}$; Thin: $\varepsilon = 2^{-8}$



Gray: $\varepsilon = 2^{-8}$; Magenta: $\varepsilon = 2^{-12}$

Contributions

Notion of asymptotic minimal contractor
Link between the preconditioning and acyclic constraint networks
Better results than the basic affine arithmetic
No use of guaranteed linear programming

Perspectives

Compare with modern affine-arithmetic approaches
Improve the tree preconditioning
Use linear programming with an order 1 inflation
Implement in codac.io

L. Jaulin.
Solution globale et garantie de problèmes ensemblistes ; application à l'estimation non linéaire et à la commande robuste.
PhD dissertation, Université Paris-Sud, Orsay, France, 1994.

R. Malti, M. Rapaić, and V. Turkulov.
A unified framework for robust stability analysis of linear irrational systems in the parametric space.
Automatica, 2022.
Second version, under review (see also https://hal.archives-ouvertes.fr/hal-03646956).

U. Montanari and F. Rossi.
Constraint relaxation may be perfect.
Artificial Intelligence, 48(2):143–170, 1991.

R. Moore.

Methods and Applications of Interval Analysis.
Society for Industrial and Applied Mathematics, jan 1979.

A. Neumaier and O. Shcherbina.
Safe bounds in linear and mixed-integer linear programming.
Math. Program., 99(2):283–296, 2004.

S. Rohou.
Codac (Catalog Of Domains And Contractors), available at `http://codac.io/`.
Robex, Lab-STICC, ENSTA-Bretagne, 2021.

V. Turkulov, M. Rapaić, and R. Malti.
Stability analysis of time-delay systems in the parametric space.
Automatica, 2022.
Provisionally accepted. Third version submitted (see also https://arxiv.org/abs/2103.15629).