A time multiplexing technique to control the heading of an underwater robot using an inertia wheel

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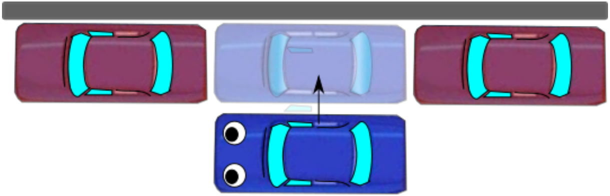
November 26, 2024, Faro, Palaiseau

1. Control with Lie brackets



To park, the blue car needs to move sideway

$$\begin{cases} \dot{x}_1 &= u_1 \cos x_3 \\ \dot{x}_2 &= u_1 \sin x_3 \\ \dot{x}_3 &= u_2 \end{cases}$$

$$\dot{\mathbf{x}} = \underbrace{\begin{pmatrix} \cos x_3 \\ \sin x_3 \\ 0 \end{pmatrix}}_{\mathbf{f}(\mathbf{x})} \cdot u_1 + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\mathbf{g}(\mathbf{x})} \cdot u_2$$

The Lie bracket between the two vector fields $\mathbf{f}$ and $\mathbf{g}$ is

$$[\mathbf{f},\mathbf{g}] = \frac{d\mathbf{g}}{d\mathbf{x}} \cdot \mathbf{f} - \frac{d\mathbf{f}}{d\mathbf{x}} \cdot \mathbf{g}.$$

The set of vector fields equipped with the Lie bracket is a Lie algebra. For instance

$$[\mathbf{f},[\mathbf{g},\mathbf{h}]]+[\mathbf{h},[\mathbf{f},\mathbf{g}]]+[\mathbf{g},[\mathbf{h},\mathbf{f}]]=\mathbf{0}$$

Example. For $\mathbf{f}(\mathbf{x}) = \mathbf{A} \cdot \mathbf{x}$, $\mathbf{g}(\mathbf{x}) = \mathbf{B} \cdot \mathbf{x}$, we have

$$\begin{aligned} [\mathbf{f},\mathbf{g}](\mathbf{x}) &= \frac{d\mathbf{g}}{d\mathbf{x}} \cdot \mathbf{f}(\mathbf{x}) - \frac{d\mathbf{f}}{d\mathbf{x}} \cdot \mathbf{g}(\mathbf{x}) \\ &= \mathbf{B} \cdot \mathbf{A} \cdot \mathbf{x} - \mathbf{A} \cdot \mathbf{B} \cdot \mathbf{x} \\ &= (\mathbf{BA} - \mathbf{AB}) \cdot \mathbf{x}. \end{aligned}$$

Consider the system

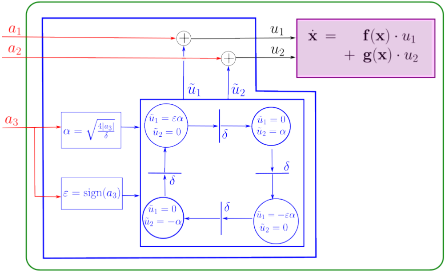
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \cdot u_1 + \mathbf{g}(\mathbf{x}) \cdot u_2.$$

Apply the following cyclic sequence:

$$\begin{array}{cccccc} t \in [0,\delta] & t \in [\delta,2\delta] & t \in [2\delta,3\delta] & t \in [3\delta,4\delta] & t \in [4\delta,5\delta] & \ldots \\ \mathbf{u} = (1,0) & \mathbf{u} = (0,1) & \mathbf{u} = (-1,0) & \mathbf{u} = (0,-1) & \mathbf{u} = (1,0) & \ldots \end{array}$$

where $\delta = o(1)$. We have

$$\mathbf{x}(t+2\delta) = \mathbf{x}(t-2\delta) + [\mathbf{f},\mathbf{g}]\,(\mathbf{x}(t))\,\delta^2 + o\left(\delta^2\right).$$



$$\Leftrightarrow$$

$$\begin{aligned}\dot{\mathbf{x}} = &\ \mathbf{f}(\mathbf{x}) \cdot a_1 \\ &+ \mathbf{g}(\mathbf{x}) \cdot a_2 \\ &+ [\mathbf{f}, \mathbf{g}](\mathbf{x}) \cdot a_3\end{aligned}$$

First order Dubins car:

$$\begin{cases} \dot{x}_1 & = & u_1 \cos x_3 \\ \dot{x}_2 & = & u_1 \sin x_3 \\ \dot{x}_3 & = & u_2 \end{cases}$$

or equivalently

$$\dot{\mathbf{x}} = \underbrace{\begin{pmatrix} \cos x_3 \\ \sin x_3 \\ 0 \end{pmatrix}}_{\mathbf{f}(\mathbf{x})} \cdot u_1 + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\mathbf{g}(\mathbf{x})} \cdot u_2$$

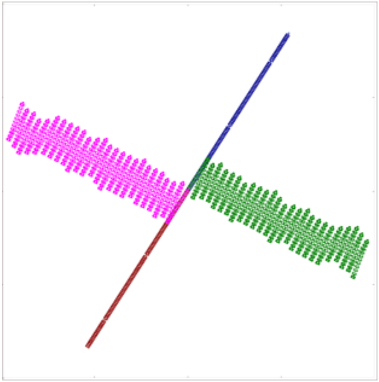
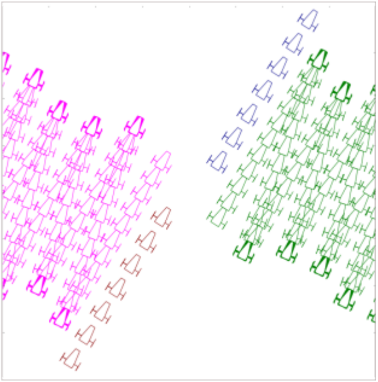
We have

$$
\begin{aligned}
[\mathbf{f},\mathbf{g}](\mathbf{x}) &= \underbrace{\frac{d\mathbf{g}}{d\mathbf{x}}(\mathbf{x})}_{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}} \cdot \underbrace{\mathbf{f}(\mathbf{x})}_{\begin{pmatrix} \cos x_3 \\ \sin x_3 \\ 0 \end{pmatrix}} - \underbrace{\frac{d\mathbf{f}}{d\mathbf{x}}(\mathbf{x})}_{\begin{pmatrix} 0 & 0 & -\sin x_3 \\ 0 & 0 & \cos x_3 \\ 0 & 0 & 0 \end{pmatrix}} \cdot \underbrace{\mathbf{g}(\mathbf{x})}_{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} \\
&= \begin{pmatrix} \sin x_3 \\ -\cos x_3 \\ 0 \end{pmatrix}
\end{aligned}
$$

We can now move the car laterally.

If we apply the cyclic sequence, we get

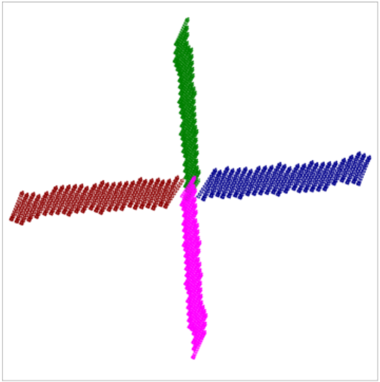
$$\dot{\mathbf{x}} = \underbrace{\begin{pmatrix} \cos x_3 \\ \sin x_3 \\ 0 \end{pmatrix}}_{\mathbf{f}(\mathbf{x})} \cdot a_1 + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\mathbf{g}(\mathbf{x})} \cdot a_2 + \underbrace{\begin{pmatrix} \sin x_3 \\ -\cos x_3 \\ 0 \end{pmatrix}}_{[\mathbf{f},\mathbf{g}](\mathbf{x})} \cdot a_3$$

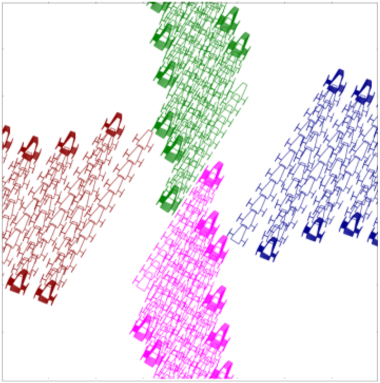



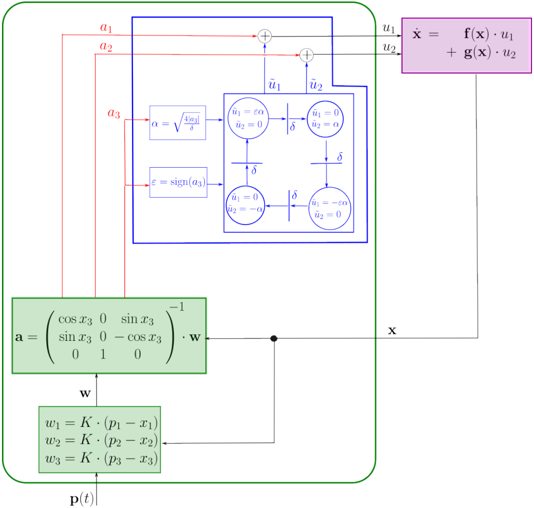
We have

$$\begin{array}{rcl}\dot{\mathbf{x}} & = & \mathbf{f}(\mathbf{x})\cdot a_1+\mathbf{g}(\mathbf{x})\cdot a_2+[\mathbf{f},\mathbf{g}](\mathbf{x})\cdot a_3 \\ & = & \mathbf{A}(\mathbf{x})\cdot\mathbf{a}\end{array}$$

We take $\mathbf{a}=\mathbf{A}^{-1}(\mathbf{x})\cdot\mathbf{w}$ to get $\dot{\mathbf{x}}=\mathbf{w}$, where $\mathbf{w}=\left(\dot{x}_d,\dot{y}_d,\dot{\theta}_d\right)$.







Right inverse of the first order Dubins car

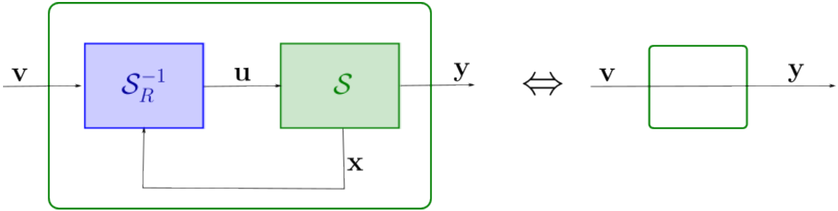
2. With drift

Second order Dubins car

$$\begin{cases} \dot{x}_1 & = & x_4 \cos x_3 \\ \dot{x}_2 & = & x_4 \sin x_3 \\ \dot{x}_3 & = & x_5 \\ \dot{x}_4 & = & u_1 \\ \dot{x}_5 & = & u_2 \end{cases}$$

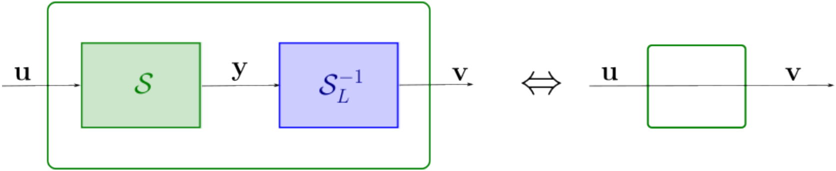
$$\dot{\mathbf{x}} = \underbrace{\begin{pmatrix} x_4 \cos x_3 \\ x_4 \sin x_3 \\ x_5 \\ 0 \\ 0 \end{pmatrix}}_{\mathbf{f}(\mathbf{x})} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{\mathbf{g}_1(\mathbf{x})} \cdot u_1 + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{\mathbf{g}_2(\mathbf{x})} \cdot u_2$$

To use a backstepping technique we decompose the system as a chain of right invertible systems.



The system $\mathscr{S}_R^{-1}$ is the right inverse of $\mathscr{S}$:

$$\mathbf{y} = \mathscr{S}(\mathbf{u}) = \mathscr{S} \circ \mathscr{S}_R^{-1}(\mathbf{v}) \simeq \mathbf{v}$$



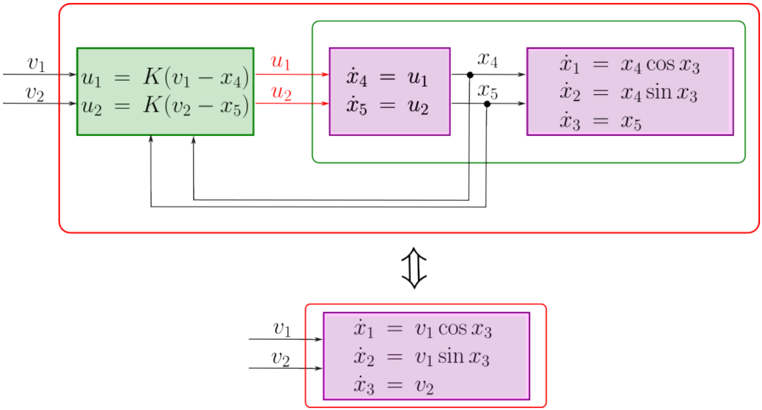
The system $\mathscr{S}_L^{-1}$ is the left inverse of $\mathscr{S}$: $\mathbf{v} = \mathscr{S}_L^{-1} \circ \mathscr{S}(\mathbf{u}) \simeq \mathbf{u}$

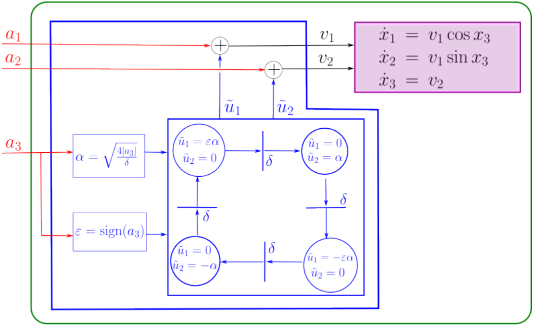
u_1, u_2

$$\begin{aligned} \dot{x}_4 &= u_1 \\ \dot{x}_5 &= u_2 \end{aligned}$$

x_4, x_5

$$\begin{aligned} \dot{x}_1 &= x_4 \cos x_3 \\ \dot{x}_2 &= x_4 \sin x_3 \\ \dot{x}_3 &= x_5 \end{aligned}$$


v_1
v_2
$u_1 = K(v_1 - x_4)$
$u_2 = K(v_2 - x_5)$
u_1
u_2
$\dot{x}_4 = u_1$
$\dot{x}_5 = u_2$
x_4
x_5
$\dot{x}_1 = x_4 \cos x_3$
$\dot{x}_2 = x_4 \sin x_3$
$\dot{x}_3 = x_5$
$\Updownarrow$
v_1
v_2
$\dot{x}_1 = v_1 \cos x_3$
$\dot{x}_2 = v_1 \sin x_3$
$\dot{x}_3 = v_2$



$$\Updownarrow$$

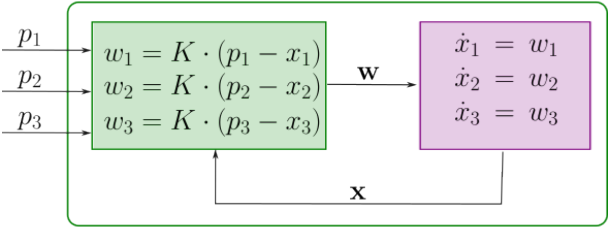
$$\xrightarrow{\mathbf{a}} \quad \boxed{\dot{\mathbf{x}} = \begin{pmatrix} \cos x_3 & 0 & \sin x_3 \\ \sin x_3 & 0 & -\cos x_3 \\ 0 & 1 & 0 \end{pmatrix} \cdot \mathbf{a}}$$

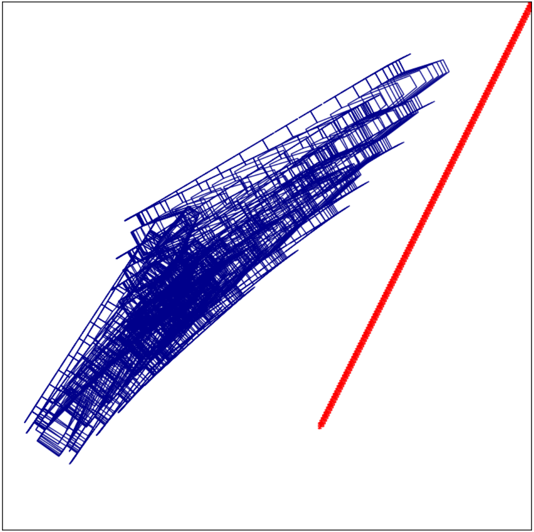
$$\mathbf{w} \longrightarrow \boxed{\mathbf{a} = \begin{pmatrix} \cos x_3 & 0 & \sin x_3 \\ \sin x_3 & 0 & -\cos x_3 \\ 0 & 1 & 0 \end{pmatrix}^{-1} \cdot \mathbf{w}} \xrightarrow{\mathbf{a}} \boxed{\dot{\mathbf{x}} = \begin{pmatrix} \cos x_3 & 0 & \sin x_3 \\ \sin x_3 & 0 & -\cos x_3 \\ 0 & 1 & 0 \end{pmatrix} \cdot \mathbf{a}}$$

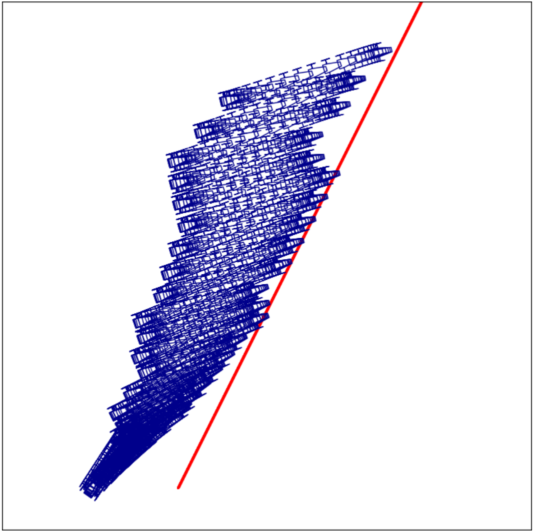
$$x_3$$

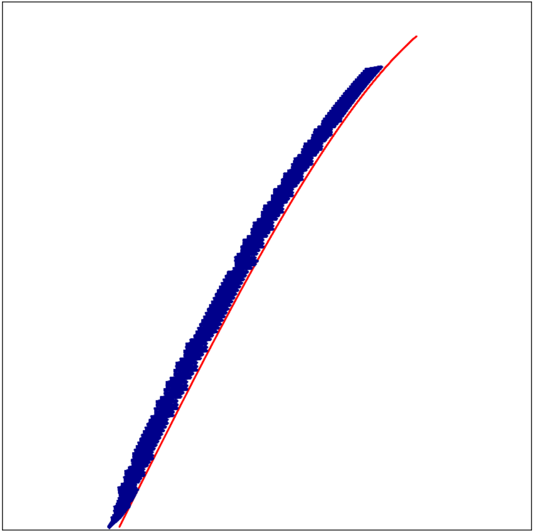
$$\Updownarrow$$

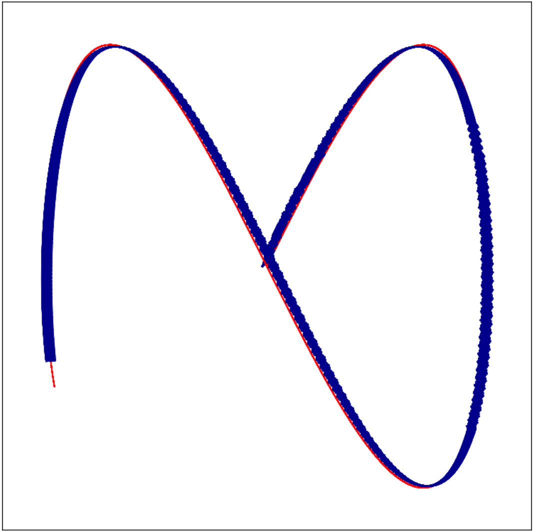
$$\mathbf{w} \longrightarrow \boxed{\begin{array}{rcl} \dot{x}_1 & = & w_1 \\ \dot{x}_2 & = & w_2 \\ \dot{x}_3 & = & w_3 \end{array}}$$


p_1
p_2
p_3
$w_1 = K \cdot (p_1 - x_1)$
$w_2 = K \cdot (p_2 - x_2)$
$w_3 = K \cdot (p_3 - x_3)$
$\mathbf{w}$
$\dot{x}_1 = w_1$
$\dot{x}_2 = w_2$
$\dot{x}_3 = w_3$
$\mathbf{x}$

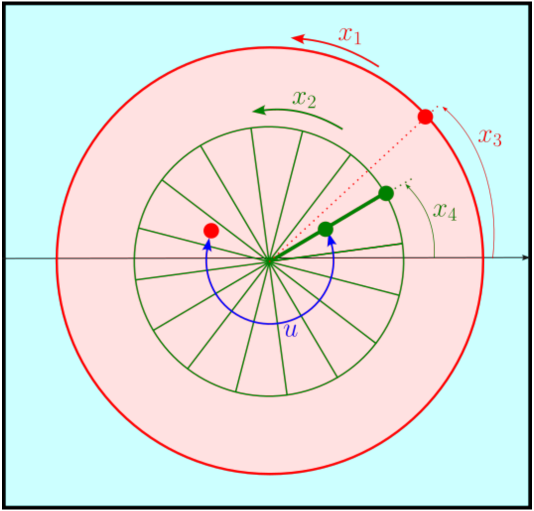








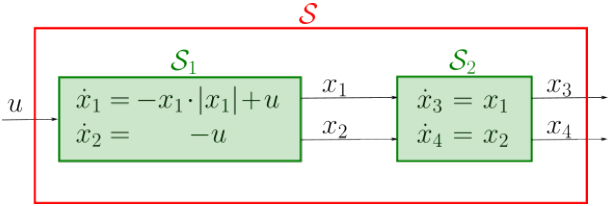
3. Swim disk


x_1
x_2
x_3
x_4
u

The state equations are

$$\mathscr{S}: \begin{cases} \dot{x}_1 = -x_1 \cdot |x_1| + u \\ \dot{x}_2 = -u \\ \dot{x}_3 = x_1 \\ \dot{x}_4 = x_2 \end{cases}$$

Can we control the two angles x_3, x_4 independently?


$\mathcal{S}$
$\mathcal{S}_1$
$\mathcal{S}_2$
u
$\dot{x}_1 = -x_1 \cdot |x_1| + u$
$\dot{x}_2 = -u$
x_1
x_2
$\dot{x}_3 = x_1$
$\dot{x}_4 = x_2$
x_3
x_4

Consider

$$\mathscr{S}_1 : \begin{cases} \dot{x}_1 &= -x_1 \cdot |x_1| + u \\ \dot{x}_2 &= -u \end{cases}$$

Note that the *small-time local controllability* can only be obtained for driftless states.
For $\mathscr{S}_1$, the driftless states have the form $\bar{\mathbf{x}} = (0, \bar{x}_2)$.
We want to control both x_1 and x_2.

Linearization approach

The linearized system around $\bar{\mathbf{x}}$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} u$$

does not satisfy the controllability criterion. Indeed, the rank of the controllability matrix is one.

With Lie brackets

Our system $\mathscr{S}_1$ has the form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} -x_1 \cdot |x_1| \\ 0 \end{pmatrix}}_{\mathbf{f}} + \underbrace{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{\mathbf{g}} \cdot u \cdot$$

If at a driftless state $\bar{\mathbf{x}}$, the *Lie ideal* $\text{Lie}(\mathbf{f}, \mathbf{g})$ spans all directions of $\mathbb{R}^n$, then we can *generally* conclude that the system is locally accessible.

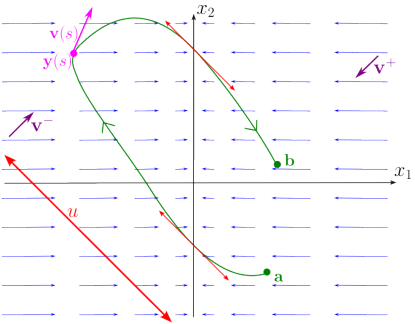
For our system, we generate Lie$(\mathbf{f}, \mathbf{g})$ as follows

$$
\begin{aligned}
[\mathbf{f}, \mathbf{g}](\mathbf{x}) &= \frac{d\mathbf{g}}{d\mathbf{x}} \cdot \mathbf{f} - \frac{d\mathbf{f}}{d\mathbf{x}} \cdot \mathbf{g} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -x_1 \cdot |x_1| \\ 0 \end{pmatrix} - \begin{pmatrix} -2|x_1| & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
&= \begin{pmatrix} 2|x_1| \\ 0 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
[\mathbf{f},[\mathbf{f},\mathbf{g}]](\mathbf{x}) &= \frac{d[\mathbf{f},\mathbf{g}]}{d\mathbf{x}}\cdot\mathbf{f} - \frac{d\mathbf{f}}{d\mathbf{x}}\cdot[\mathbf{f},\mathbf{g}] \\
&= \begin{pmatrix} -2\mathsf{sign}(x_1) & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -x_1\cdot|x_1| \\ 0 \end{pmatrix} \\
&\quad - \begin{pmatrix} -2|x_1| & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2|x_1| \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 4x_1^2 \\ 0 \end{pmatrix}
\end{aligned}
$$

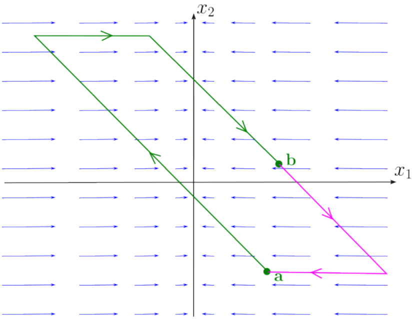
$$
\begin{aligned}
[[\mathbf{f},\mathbf{g}],\mathbf{g}] &= \frac{d\mathbf{g}}{d\mathbf{x}}\cdot[\mathbf{f},\mathbf{g}] - \frac{d[\mathbf{f},\mathbf{g}]}{d\mathbf{x}}\cdot\mathbf{g} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\cdot[\mathbf{f},\mathbf{g}] - \begin{pmatrix} -2\mathsf{sign}(x_1) & 0 \\ 0 & 0 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
&= \begin{pmatrix} -2\mathsf{sign}(x_1) \\ 0 \end{pmatrix}
\end{aligned}
$$

We observe that any element of Lie$(\mathbf{f},\mathbf{g})$ cancels at any driftless state $\bar{\mathbf{x}} = (0, \bar{x}_2)$
The criterion based on the Lie brackets fails.



A feasible path $\mathbf{y}(s)$ for $\mathscr{S}_1$ from $\mathbf{a}$ to $\mathbf{b}$

Proposition. Any state of the system $\mathscr{S}_1$ is accessible from any initial state.

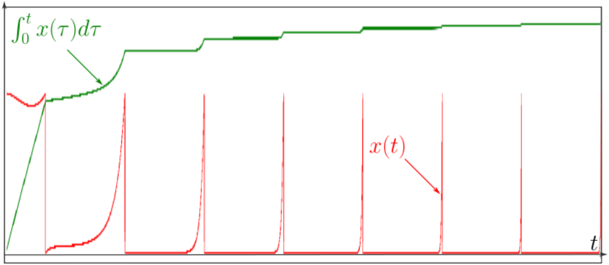


Average stability

The function $x(t) : \mathbb{R}^+ \to \mathbb{R}$ converges to zero on average, we will write $x \stackrel{a}{\to} 0$ if

$$\lim_{t\to\infty} \int_0^t x(\tau) d\tau \in \mathbb{R}$$

Example. The function $x(t) = (t\%1)^{t^2}$ satisfies $x \xrightarrow{a} 0$.



The system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is *stable on average* if,

$$\forall \mathbf{x}(0), x_i(t) \xrightarrow{a} 0$$

A system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$ is *stabilizable on average* if there exists a control $\mathbf{u}()$ such that the system is stable on average.

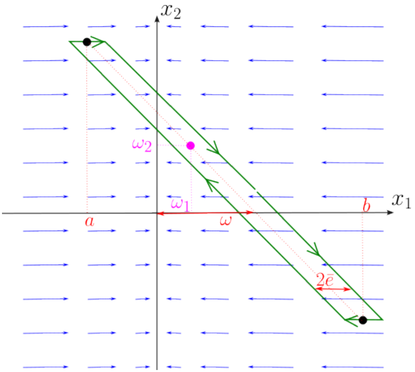
A system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$ is *right invertible* on average if for all $\omega \in \mathbb{R}^n$, there exists $\mathbf{u}(t)$ such that $\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z} + \omega, \mathbf{u})$ is stabilizable on average. It means that $\mathbf{z} = \mathbf{x} - \omega \stackrel{a}{\to} \mathbf{0}$.

Speed control of the swim disk

We want a controller for

$$\mathscr{S}_1 : \begin{cases} \dot{x}_1 = -x_1 \cdot |x_1| + u \\ \dot{x}_2 = -u \end{cases}$$

which stabilizes $\mathbf{x}$ at a given (ω_1, ω_2) on average.



A swim cycle with parameters $\bar{e}$, a, b, ω

Proposition. The system:

$$\underbrace{\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{pmatrix} -x_1 \cdot |x_1| \\ 0 \end{pmatrix}}_{\mathbf{f}(\mathbf{x})} + \underbrace{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{\mathbf{g}(\mathbf{x})} \cdot u$$

can follow any swim cycle. Moreover, along the swim cycle, the period is

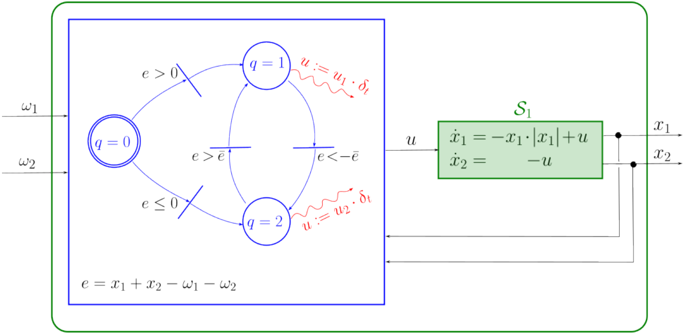
$$T = \frac{2\bar{e}}{a^2} + \frac{2\bar{e}}{b^2}$$

and the average is

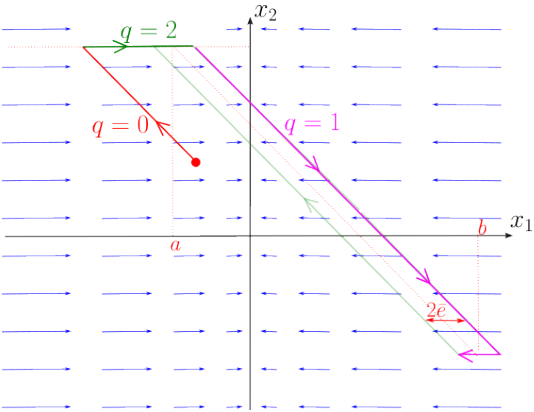
$$\begin{array}{ccccc} \omega_1 & = & \frac{1}{T}\int_0^T x_1(\tau) & = & \frac{a^2 b + b^2 a}{a^2+b^2} \\ \omega_2 & = & \frac{1}{T}\int_0^T x_2(\tau) & = & \frac{(\omega-a)b^2+(\omega-b)a^2}{a^2+b^2} \end{array}$$

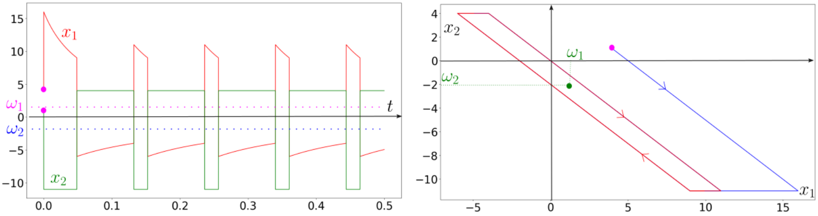
Proposition. The parameters $(a, \omega, \bar{e})$ of the swim cycle corresponding to ω_1, ω_2, T, b are

$$\omega = \omega_2 + \omega_1$$
$$a = \frac{-b^2 - b\sqrt{b^2 - 4(\omega_1 - b)\omega_1}}{2(b - \omega_1)}$$
$$\bar{e} = \frac{T}{\frac{2}{a^2} + \frac{2}{b^2}}$$



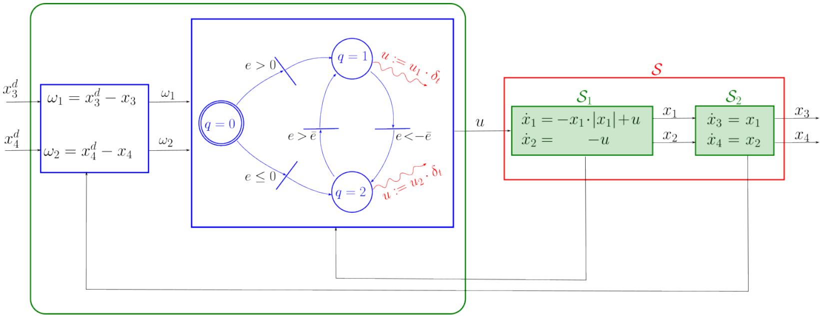
The blue controller is the right inverse of $\mathscr{S}_1$ in average

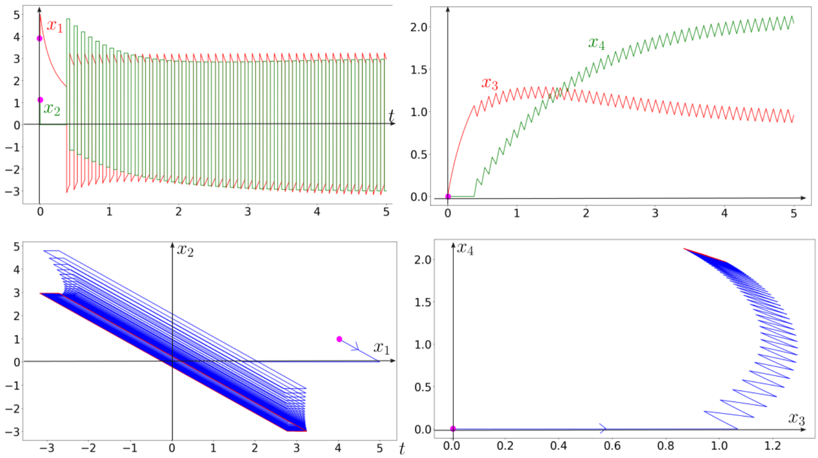

x_2
$q = 2$
$q = 0$
$q = 1$
x_1
a
b
$2\bar{e}$



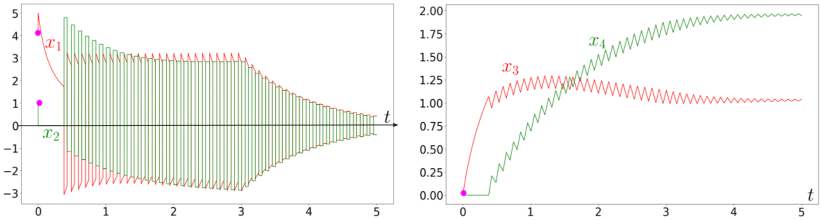
The controller leads (x_1, x_2) to the desired speeds $(\omega_1, \omega_2) = (-2, 1)$

Position control of the swim disk


x_3^d
x_4^d
$\omega_1 = x_3^d - x_3$
$\omega_2 = x_4^d - x_4$
ω_1
ω_2
$e > 0$
$e \leq 0$
$q = 0$
$q = 1$
$q = 2$
$e > \bar{e}$
$e < -\bar{e}$
$u := u_1 \cdot \delta_t$
$u := u_2 \cdot \delta_t$
u
S
S_1
S_2
$\dot{x}_1 = -x_1 \cdot |x_1| + u$
$\dot{x}_2 = -u$
x_1
x_2
$\dot{x}_3 = x_1$
$\dot{x}_4 = x_2$
x_3
x_4



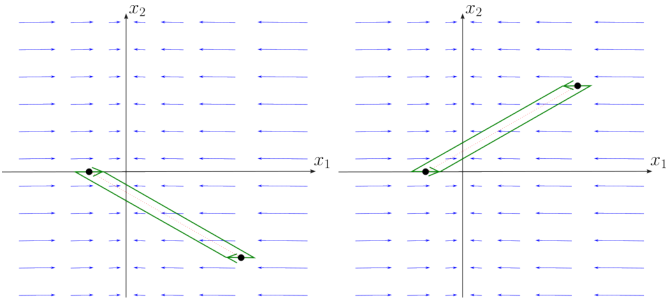
The controller leads the output (x_3, x_4) to the desired position $(1, 2)$

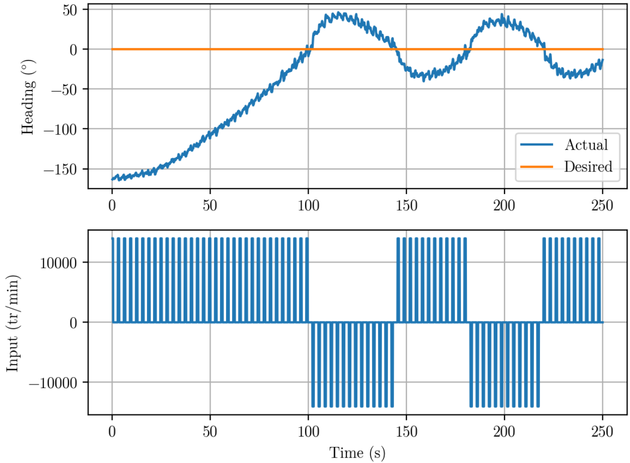


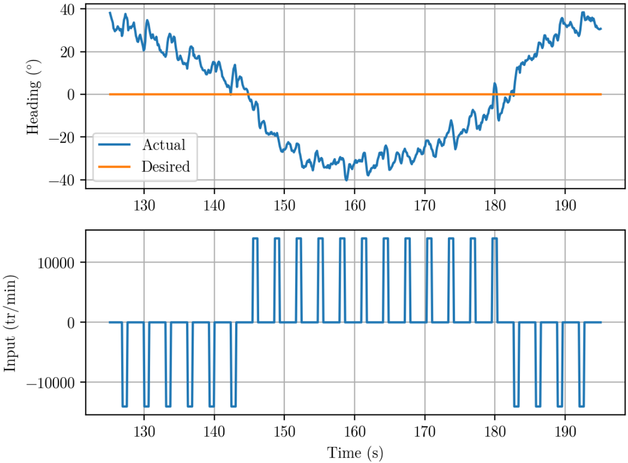
A damping is added at time $t = 3$

Experiment

https://youtu.be/IB_A_1ePN34


x_2
x_1
x_2
x_1


50
0
−50
−100
−150
Heading (°)
Actual
Desired
0
50
100
150
200
250
10000
0
−10000
Input (tr/min)
0
50
100
150
200
250
Time (s)


40
20
0
−20
−40
Heading (°)
Actual
Desired
130
140
150
160
170
180
190
10000
0
−10000
Input (tr/min)
Time (s)

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2. Small-time local controllability [6], section 15.1.3
3. Left invertibility [3] and [2], section 3.3.3
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