DIRECTED ALGEBRAIC TOPOLOGY

AND

CONCURRENCY

Emmanuel Haucourt

emmanuel.haucourt@polytechnique.edu

MPRI : Concurrency (2.3.1)
– Lecture 5 –

2020 – 2021

THE FUNDAMENTAL CATEGORY

Abstract setting

Natural Transformations

morphisms of functors from $f : \mathcal{C} \to \mathcal{D}$ to $g : \mathcal{C} \to \mathcal{D}$

Natural Transformations

morphisms of functors from $f : \mathcal{C} \to \mathcal{D}$ to $g : \mathcal{C} \to \mathcal{D}$

A natural transformation $\eta : f \to g$ is a collection of morphisms $(\eta_x)_{x \in \mathrm{Ob}(\mathcal{C})}$ where $\eta_x \in \mathcal{D}[f(x), g(x)]$ and such that for all $\alpha \in \mathcal{C}[x, y]$ we have $\eta_y \circ f(\alpha) = g(\alpha) \circ \eta_x$ i.e. the following diagram commute

Natural Transformations

morphisms of functors from $f : \mathcal{C} \to \mathcal{D}$ to $g : \mathcal{C} \to \mathcal{D}$

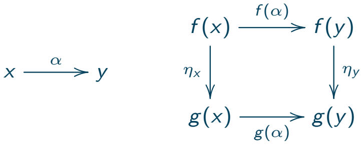
A natural transformation $\eta : f \to g$ is a collection of morphisms $(\eta_x)_{x \in \mathrm{Ob}(\mathcal{C})}$ where $\eta_x \in \mathcal{D}[f(x), g(x)]$ and such that for all $\alpha \in \mathcal{C}[x, y]$ we have $\eta_y \circ f(\alpha) = g(\alpha) \circ \eta_x$ i.e. the following diagram commute

$$x \xrightarrow{\alpha} y \qquad \begin{array}{ccc} f(x) & \xrightarrow{f(\alpha)} & f(y) \\ \downarrow \eta_x & & \downarrow \eta_y \\ g(x) & \xrightarrow[g(\alpha)]{} & g(y) \end{array}$$

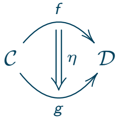
Natural Transformations

morphisms of functors from $f : \mathcal{C} \to \mathcal{D}$ to $g : \mathcal{C} \to \mathcal{D}$

A natural transformation $\eta : f \to g$ is a collection of morphisms $(\eta_x)_{x \in \mathrm{Ob}(\mathcal{C})}$ where $\eta_x \in \mathcal{D}[f(x), g(x)]$ and such that for all $\alpha \in \mathcal{C}[x, y]$ we have $\eta_y \circ f(\alpha) = g(\alpha) \circ \eta_x$ i.e. the following diagram commute



This description is summarized by the following diagram



Congruences on small categories

Congruences on small categories

A congruence on a small category $\mathcal{C}$ is an equivalence relation $\sim$ over $\mathrm{Mo}(\mathcal{C})$ such that:

Congruences on small categories

A congruence on a small category $\mathcal{C}$ is an equivalence relation $\sim$ over $\mathrm{Mo}(\mathcal{C})$ such that:

- $\gamma \sim \gamma' \quad \Rightarrow \quad \partial^{-}\gamma = \partial^{-}\gamma'$ and $\partial^{+}\gamma = \partial^{+}\gamma'$

Congruences on small categories

A congruence on a small category $\mathcal{C}$ is an equivalence relation $\sim$ over $\mathrm{Mo}(\mathcal{C})$ such that:

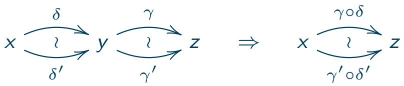
- $\gamma \sim \gamma' \quad \Rightarrow \quad \partial^{-}\gamma = \partial^{-}\gamma'$ and $\partial^{+}\gamma = \partial^{+}\gamma'$
- $\gamma \sim \gamma'$, $\delta \sim \delta'$ and $\partial^{-}\gamma = \partial^{+}\delta \quad \Rightarrow \quad \gamma \circ \delta \sim \gamma' \circ \delta'$

Congruences on small categories

A congruence on a small category $\mathcal{C}$ is an equivalence relation $\sim$ over $\mathrm{Mo}(\mathcal{C})$ such that:

- $\gamma \sim \gamma' \quad \Rightarrow \quad \partial^- \gamma = \partial^- \gamma'$ and $\partial^+ \gamma = \partial^+ \gamma'$
- $\gamma \sim \gamma'$, $\delta \sim \delta'$ and $\partial^- \gamma = \partial^+ \delta \quad \Rightarrow \quad \gamma \circ \delta \sim \gamma' \circ \delta'$

In diagrams we have

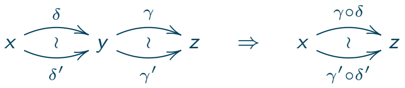


Congruences on small categories

A congruence on a small category $\mathcal{C}$ is an equivalence relation $\sim$ over $\mathrm{Mo}(\mathcal{C})$ such that:

- $\gamma \sim \gamma' \quad \Rightarrow \quad \partial^- \gamma = \partial^- \gamma'$ and $\partial^+ \gamma = \partial^+ \gamma'$
- $\gamma \sim \gamma'$, $\delta \sim \delta'$ and $\partial^- \gamma = \partial^+ \delta \quad \Rightarrow \quad \gamma \circ \delta \sim \gamma' \circ \delta'$

In diagrams we have



Hence the $\sim$-equivalence class of $\gamma \circ \delta$ only depends on the $\sim$-equivalence classes of γ and δ and we have a quotient category $\mathcal{C}/\sim$ in which the composition is given by

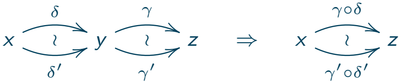
$$[\gamma] \circ [\delta] \quad = \quad [\gamma \circ \delta]$$

Congruences on small categories

A congruence on a small category $\mathcal{C}$ is an equivalence relation $\sim$ over $\mathrm{Mo}(\mathcal{C})$ such that:

- $\gamma \sim \gamma' \quad \Rightarrow \quad \partial^- \gamma = \partial^- \gamma'$ and $\partial^+ \gamma = \partial^+ \gamma'$
- $\gamma \sim \gamma'$, $\delta \sim \delta'$ and $\partial^- \gamma = \partial^+ \delta \quad \Rightarrow \quad \gamma \circ \delta \sim \gamma' \circ \delta'$

In diagrams we have



Hence the $\sim$-equivalence class of $\gamma \circ \delta$ only depends on the $\sim$-equivalence classes of γ and δ and we have a quotient category $\mathcal{C}/\sim$ in which the composition is given by

$$[\gamma] \circ [\delta] \quad = \quad [\gamma \circ \delta]$$

The quotient map $q : \gamma \in \mathrm{Mo}(\mathcal{C}) \mapsto [\gamma] \in \mathrm{Mo}(\mathcal{C})/\sim$ induces a functor $q : \mathcal{C} \to \mathcal{C}/\sim$

Natural congruences on a functor $P : \mathcal{C} \rightarrow \mathcal{C}at$

Natural congruences on a functor $P : \mathcal{C} \to \mathcal{C}at$

A natural congruence on a functor $P : \mathcal{C} \to \mathcal{C}at$ is a collection of congruences $\sim_X$ on PX, for X ranging through the objects of $\mathcal{C}$, such that for all morphisms $f : X \to Y$ of $\mathcal{C}$, for all $\alpha, \beta \in PX$,

$$\alpha \sim_X \beta \quad \Rightarrow \quad P(f)(\alpha) \sim_Y P(f)(\beta)$$

Natural congruences on a functor $P : \mathcal{C} \to \mathcal{C}at$

A natural congruence on a functor $P : \mathcal{C} \to \mathcal{C}at$ is a collection of congruences $\sim_X$ on PX, for X ranging through the objects of $\mathcal{C}$, such that for all morphisms $f : X \to Y$ of $\mathcal{C}$, for all $\alpha, \beta \in PX$,

$$\alpha \sim_X \beta \quad \Rightarrow \quad P(f)(\alpha) \sim_Y P(f)(\beta)$$

Then we can define the functor $\overrightarrow{\pi_1} \ : \ \mathcal{C} \to \mathcal{C}at$ as follows:

Natural congruences on a functor $P : \mathcal{C} \to \mathcal{C}at$

A natural congruence on a functor $P : \mathcal{C} \to \mathcal{C}at$ is a collection of congruences $\sim_X$ on PX, for X ranging through the objects of $\mathcal{C}$, such that for all morphisms $f : X \to Y$ of $\mathcal{C}$, for all $\alpha, \beta \in PX$,

$$\alpha \sim_X \beta \quad \Rightarrow \quad P(f)(\alpha) \sim_Y P(f)(\beta)$$

Then we can define the functor $\overrightarrow{\pi_1} : \mathcal{C} \to \mathcal{C}at$ as follows:

- for all $X \in \mathcal{C}$, $\pi_1(X) = P(X)/\sim_X$

Natural congruences on a functor $P : \mathcal{C} \to \mathcal{C}at$

A natural congruence on a functor $P : \mathcal{C} \to \mathcal{C}at$ is a collection of congruences $\sim_X$ on PX, for X ranging through the objects of $\mathcal{C}$, such that for all morphisms $f : X \to Y$ of $\mathcal{C}$, for all $\alpha, \beta \in PX$,

$$\alpha \sim_X \beta \quad \Rightarrow \quad P(f)(\alpha) \sim_Y P(f)(\beta)$$

Then we can define the functor $\overrightarrow{\pi_1} : \mathcal{C} \to \mathcal{C}at$ as follows:

- for all $X \in \mathcal{C}$, $\pi_1(X) = P(X)/\sim_X$
- for all $f : X \to Y$ in $\mathcal{C}$

Natural congruences on a functor $P : \mathcal{C} \to \mathcal{C}at$

A natural congruence on a functor $P : \mathcal{C} \to \mathcal{C}at$ is a collection of congruences $\sim_X$ on PX, for X ranging through the objects of $\mathcal{C}$, such that for all morphisms $f : X \to Y$ of $\mathcal{C}$, for all $\alpha, \beta \in PX$,

$$\alpha \sim_X \beta \quad \Rightarrow \quad P(f)(\alpha) \sim_Y P(f)(\beta)$$

Then we can define the functor $\overrightarrow{\pi_1} : \mathcal{C} \to \mathcal{C}at$ as follows:

- for all $X \in \mathcal{C}$, $\pi_1(X) = P(X)/\sim_X$
- for all $f : X \to Y$ in $\mathcal{C}$

$$X \xrightarrow{f} Y$$

$$\begin{array}{ccc} PX & \xrightarrow{Pf} & PY \\ {\scriptstyle q_X}\downarrow & & \downarrow{\scriptstyle q_Y} \\ \overrightarrow{\pi_1}X & \xrightarrow[\overrightarrow{\pi_1}f]{} & \overrightarrow{\pi_1}Y \end{array}$$

Natural congruences on a functor $P : \mathcal{C} \to \mathcal{C}at$

A natural congruence on a functor $P : \mathcal{C} \to \mathcal{C}at$ is a collection of congruences $\sim_X$ on PX, for X ranging through the objects of $\mathcal{C}$, such that for all morphisms $f : X \to Y$ of $\mathcal{C}$, for all $\alpha, \beta \in PX$,

$$\alpha \sim_X \beta \quad \Rightarrow \quad P(f)(\alpha) \sim_Y P(f)(\beta)$$

Then we can define the functor $\overrightarrow{\pi_1} : \mathcal{C} \to \mathcal{C}at$ as follows:

- for all $X \in \mathcal{C}$, $\pi_1(X) = P(X)/\sim_X$
- for all $f : X \to Y$ in $\mathcal{C}$

$$X \xrightarrow{f} Y$$

$$\begin{array}{ccc} PX & \xrightarrow{Pf} & PY \\ {\scriptstyle q_X}\downarrow & & \downarrow{\scriptstyle q_Y} \\ \overrightarrow{\pi_1}X & \xrightarrow[\overrightarrow{\pi_1}f]{} & \overrightarrow{\pi_1}Y \end{array}$$

The collection of quotient functors q_X, for X ranging through the objects of $\mathcal{C}$, provides a natural transformation from P to $\overrightarrow{\pi_1}$.

The directed path functor

Object part

Object part

Let X be a locally ordered space.

Object part

Let X be a locally ordered space.

- The objects of PX are the points of X.

Object part

Let X be a locally ordered space.

- The objects of PX are the points of X.
- The homset $PX(a, b)$ is

$$\bigcup_{r \geqslant 0} \{\gamma \in \mathcal{L}po([0, r], X) \mid \gamma(0) = a \text{ and } \gamma(r) = b\}$$

Object part

Let X be a locally ordered space.

- The objects of PX are the points of X.
- The homset $PX(a, b)$ is

$$\bigcup_{r \geqslant 0} \{\gamma \in \mathcal{L}po([0,r], X) \mid \gamma(0) = a \text{ and } \gamma(r) = b\}$$

- For $\delta : [0, r] \to X$ and $\gamma : [0, r'] \to X$ with $\delta(r) = \gamma(0)$, define the concatenation

$$\gamma \cdot \delta : \; [0, r + r'] \longrightarrow X$$

$$t \longmapsto \begin{cases} \delta(t) & \text{if } t \leqslant r \\ \gamma(t - r) & \text{if } t \geqslant r \end{cases}$$

Morphism part

Morphism part

The (Moore) path category construction gives rise to a functor P from $\mathcal{L}po$ to $\mathcal{C}at$ since for all $f \in \mathcal{L}po(X, Y)$ and all paths γ on X, the composite $f \circ \gamma$ is a path on Y.

Morphism part

The (Moore) path category construction gives rise to a functor P from $\mathcal{Lpo}$ to $\mathcal{Cat}$ since for all $f \in \mathcal{Lpo}(X, Y)$ and all paths γ on X, the composite $f \circ \gamma$ is a path on Y.

$$P : \mathcal{Lpo} \longrightarrow \mathcal{Cat}$$

$$\begin{array}{ccc} X & & PX \\ \downarrow f & \longmapsto & \downarrow Pf \\ Y & & PY \end{array}$$

Morphism part

The (Moore) path category construction gives rise to a functor P from $\mathcal{L}po$ to $\mathcal{C}at$ since for all $f \in \mathcal{L}po(X, Y)$ and all paths γ on X, the composite $f \circ \gamma$ is a path on Y.

$$P : \mathcal{L}po \longrightarrow \mathcal{C}at$$

$$\begin{array}{ccc} X & & PX \\ \downarrow f & \longmapsto & Pf \downarrow \\ Y & & PY \end{array}$$

with

$$Pf : PX \longrightarrow PY$$

$$\begin{array}{ccc} p & & f(p) \\ \downarrow \gamma & \longmapsto & f\circ\gamma \downarrow \\ q & & f(q) \end{array}$$

Natural congruences from directed homotopies

Equivalent directed paths on a local pospace X

Equivalent directed paths on a local pospace X

An elementary homotopy is a *finite* concatenation of directed and anti-directed homotopies.

Equivalent directed paths on a local pospace X

An elementary homotopy is a *finite* concatenation of directed and anti-directed homotopies.

If $\theta : [0, r] \to [0, r]$ is a reparametrization and $\gamma \in \mathcal{L}po([0, r], X)$, then γ and $\gamma \circ \theta$ are dihomotopic.

Equivalent directed paths on a local pospace X

An elementary homotopy is a *finite* concatenation of directed and anti-directed homotopies.

If $\theta : [0, r] \to [0, r]$ is a reparametrization and $\gamma \in \mathcal{L}po([0, r], X)$, then γ and $\gamma \circ \theta$ are dihomotopic.

Two directed paths $\gamma : [0, r'] \to X$ and $\delta : [0, r''] \to X$ on a local pospace are said to be equivalent (denoted by $\sim_X$) when there exists two reparametrizations $\theta : [0, r] \to [0, r']$ and $\psi : [0, r] \to [0, r'']$ such that there is an elementary homotopy between $\gamma \circ \theta$ and $\delta \circ \psi$.

Equivalent directed paths on a local pospace X

An elementary homotopy is a *finite* concatenation of directed and anti-directed homotopies.

If $\theta : [0, r] \to [0, r]$ is a reparametrization and $\gamma \in \mathcal{L}po([0, r], X)$, then γ and $\gamma \circ \theta$ are dihomotopic.

Two directed paths $\gamma : [0, r'] \to X$ and $\delta : [0, r''] \to X$ on a local pospace are said to be equivalent (denoted by $\sim_X$) when there exists two reparametrizations $\theta : [0, r] \to [0, r']$ and $\psi : [0, r] \to [0, r'']$ such that there is an elementary homotopy between $\gamma \circ \theta$ and $\delta \circ \psi$.

The relation $\sim_X$ is symmetric because ...

Equivalent directed paths on a local pospace X

An elementary homotopy is a *finite* concatenation of directed and anti-directed homotopies.

If $\theta : [0, r] \to [0, r]$ is a reparametrization and $\gamma \in \mathcal{L}po([0, r], X)$, then γ and $\gamma \circ \theta$ are dihomotopic.

Two directed paths $\gamma : [0, r'] \to X$ and $\delta : [0, r''] \to X$ on a local pospace are said to be equivalent (denoted by $\sim_X$) when there exists two reparametrizations $\theta : [0, r] \to [0, r']$ and $\psi : [0, r] \to [0, r'']$ such that there is an elementary homotopy between $\gamma \circ \theta$ and $\delta \circ \psi$.

The relation $\sim_X$ is symmetric because if $h(s, t)$ is an elementary homotopy, then so is the mapping $(s, t) \mapsto h(-s, t)$.

Equivalent directed paths on a local pospace X

An elementary homotopy is a *finite* concatenation of directed and anti-directed homotopies.

If $\theta : [0, r] \to [0, r]$ is a reparametrization and $\gamma \in \mathcal{L}po([0, r], X)$, then γ and $\gamma \circ \theta$ are dihomotopic.

Two directed paths $\gamma : [0, r'] \to X$ and $\delta : [0, r''] \to X$ on a local pospace are said to be equivalent (denoted by $\sim_X$) when there exists two reparametrizations $\theta : [0, r] \to [0, r']$ and $\psi : [0, r] \to [0, r'']$ such that there is an elementary homotopy between $\gamma \circ \theta$ and $\delta \circ \psi$.

The relation $\sim_X$ is symmetric because if $h(s, t)$ is an elementary homotopy, then so is the mapping $(s, t) \mapsto h(-s, t)$.

The relation $\sim_X$ is transitive because ...

Equivalent directed paths on a local pospace X

An elementary homotopy is a *finite* concatenation of directed and anti-directed homotopies.

If $\theta : [0, r] \to [0, r]$ is a reparametrization and $\gamma \in \mathcal{L}po([0, r], X)$, then γ and $\gamma \circ \theta$ are dihomotopic.

Two directed paths $\gamma : [0, r'] \to X$ and $\delta : [0, r''] \to X$ on a local pospace are said to be equivalent (denoted by $\sim_X$) when there exists two reparametrizations $\theta : [0, r] \to [0, r']$ and $\psi : [0, r] \to [0, r'']$ such that there is an elementary homotopy between $\gamma \circ \theta$ and $\delta \circ \psi$.

The relation $\sim_X$ is symmetric because if $h(s, t)$ is an elementary homotopy, then so is the mapping $(s, t) \mapsto h(-s, t)$.

The relation $\sim_X$ is transitive because a concatenation of elementary homotopies is an elementary homotopy.

Equivalent directed paths on a local pospace X

An elementary homotopy is a *finite* concatenation of directed and anti-directed homotopies.

If $\theta : [0, r] \to [0, r]$ is a reparametrization and $\gamma \in \mathcal{L}po([0, r], X)$, then γ and $\gamma \circ \theta$ are dihomotopic.

Two directed paths $\gamma : [0, r'] \to X$ and $\delta : [0, r''] \to X$ on a local pospace are said to be equivalent (denoted by $\sim_X$) when there exists two reparametrizations $\theta : [0, r] \to [0, r']$ and $\psi : [0, r] \to [0, r'']$ such that there is an elementary homotopy between $\gamma \circ \theta$ and $\delta \circ \psi$.

The relation $\sim_X$ is symmetric because if $h(s, t)$ is an elementary homotopy, then so is the mapping $(s, t) \mapsto h(-s, t)$.

The relation $\sim_X$ is transitive because a concatenation of elementary homotopies is an elementary homotopy.

Given $x, y \in X$ and $r \in \mathbb{R}_+$, the relation $\sim_X$ is an equivalence relation on the set

$$\bigcup_{r \in \mathbb{R}_+} \{\gamma \in \mathcal{L}po([0, r], X) \mid \gamma(0) = x;\ \gamma(r) = y\}$$

Juxtaposition of homotopies

horizontal composition

Juxtaposition of homotopies

horizontal composition

Let $h : [0, r] \times [0, q] \to X$ and $h' : [0, r'] \times [0, q] \to X$ be homotopies from γ to δ and from γ' to δ' with $\partial^{+}\gamma = \partial^{-}\gamma'$.

Juxtaposition of homotopies

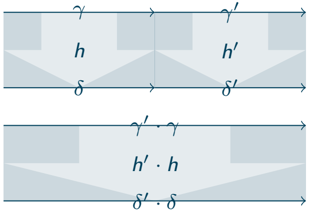
horizontal composition

Let $h : [0, r] \times [0, q] \to X$ and $h' : [0, r'] \times [0, q] \to X$ be homotopies from γ to δ and from γ' to δ' with $\partial^{+}\gamma = \partial^{-}\gamma'$.

The mapping $h' * h : [0, r + r'] \times [0, q] \to X$ defined by

$$h' * h(t, s) \quad = \quad \begin{cases} h(t, s) & \text{if } 0 \leqslant t \leqslant r \\ h'(t - r, s) & \text{if } r \leqslant t \leqslant r + r' \end{cases}$$

is a homotopy from γ to δ.



Juxtaposition of homotopies

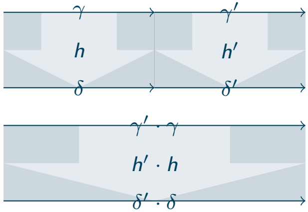
horizontal composition

Let $h : [0, r] \times [0, q] \to X$ and $h' : [0, r'] \times [0, q] \to X$ be homotopies from γ to δ and from γ' to δ' with $\partial^+\gamma = \partial^-\gamma'$.

The mapping $h' * h : [0, r + r'] \times [0, q] \to X$ defined by

$$h' * h(t, s) \quad = \quad \begin{cases} h(t, s) & \text{if } 0 \leqslant t \leqslant r \\ h'(t - r, s) & \text{if } r \leqslant t \leqslant r + r' \end{cases}$$

is a homotopy from γ to δ.

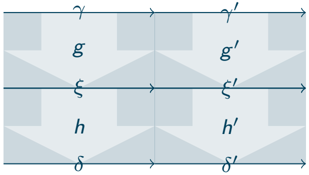


If h and h' are ((weakly) directed) homotopies, then so is their juxtaposition $h' \cdot h$.

Godement exchange law

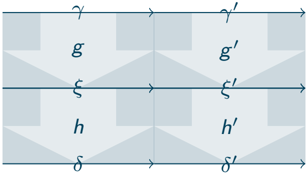
Godement exchange law

Suppose we have



Godement exchange law

Suppose we have

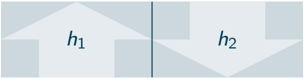


then it comes

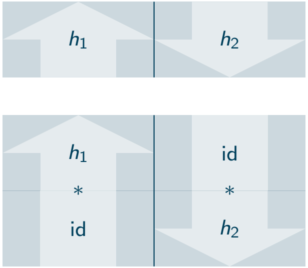
$$(g' * h') \cdot (g * h) = (g' \cdot g) * (h' \cdot h)$$

Applying Godement exchange law

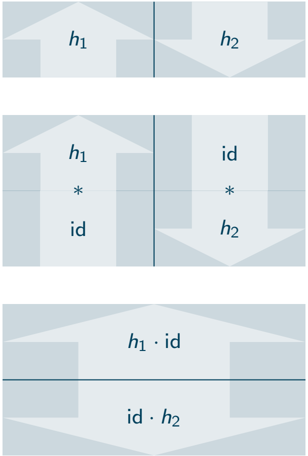
Applying Godement exchange law



Applying Godement exchange law



Applying Godement exchange law



Equivalences are congruences

Equivalences are congruences

If:

Equivalences are congruences

If:

- *h* is an elementary homotopy between $\gamma \circ \theta$ and $\delta \circ \psi$

Equivalences are congruences

If:

- *h* is an elementary homotopy between $\gamma \circ \theta$ and $\delta \circ \psi$
- *h*′ is an elementary homotopy between $\gamma' \circ \theta'$ and $\delta' \circ \psi'$

Equivalences are congruences

If:

- h is an elementary homotopy between $\gamma \circ \theta$ and $\delta \circ \psi$
- h' is an elementary homotopy between $\gamma' \circ \theta'$ and $\delta' \circ \psi'$
- the endpoint of γ is the starting point of γ'

Equivalences are congruences

If:

- h is an elementary homotopy between $\gamma \circ \theta$ and $\delta \circ \psi$
- h' is an elementary homotopy between $\gamma' \circ \theta'$ and $\delta' \circ \psi'$
- the endpoint of γ is the starting point of γ'

then $h \cdot h'$ is an elementary homotopy from ...

Equivalences are congruences

If:

- h is an elementary homotopy between $\gamma \circ \theta$ and $\delta \circ \psi$
- h' is an elementary homotopy between $\gamma' \circ \theta'$ and $\delta' \circ \psi'$
- the endpoint of γ is the starting point of γ'

then $h \cdot h'$ is an elementary homotopy from $(\gamma \cdot \gamma') \circ (\theta \cdot \theta')$ to $(\delta \cdot \delta') \circ (\psi \cdot \psi')$.

Equivalences are congruences

If:

- h is an elementary homotopy between $\gamma \circ \theta$ and $\delta \circ \psi$
- h' is an elementary homotopy between $\gamma' \circ \theta'$ and $\delta' \circ \psi'$
- the endpoint of γ is the starting point of γ'

then $h \cdot h'$ is an elementary homotopy from $(\gamma \cdot \gamma') \circ (\theta \cdot \theta')$ to $(\delta \cdot \delta') \circ (\psi \cdot \psi')$.

The relation $\sim_X$ is a congruence on $P(X)$

Naturality

Naturality

If h is a homotopy from γ to γ' on the topological space X and $f : X \to Y$ is a continuous map, then $f \circ h$ is a homotopy from $f \circ \gamma$ to $f \circ \gamma'$ on the topological space Y.

Naturality

If h is a homotopy from γ to γ' on the topological space X and $f : X \to Y$ is a continuous map, then $f \circ h$ is a homotopy from $f \circ \gamma$ to $f \circ \gamma'$ on the topological space Y.

If h is a (weakly) directed homotopy from γ to γ' on the local pospace space X and $f : X \to Y$ is a local pospace morphism, then $f \circ h$ is a (weakly) directed homotopy from $f \circ \gamma$ to $f \circ \gamma'$ on the local pospace space Y.

Naturality

If h is a homotopy from γ to γ' on the topological space X and $f : X \to Y$ is a continuous map, then $f \circ h$ is a homotopy from $f \circ \gamma$ to $f \circ \gamma'$ on the topological space Y.

If h is a (weakly) directed homotopy from γ to γ' on the local pospace space X and $f : X \to Y$ is a local pospace morphism, then $f \circ h$ is a (weakly) directed homotopy from $f \circ \gamma$ to $f \circ \gamma'$ on the local pospace space Y.

If $\gamma, \gamma' : [0, r] \to X$ are ((weakly) di)homotopic, then so are $f \circ \gamma, f \circ \gamma' : [0, r] \to Y$.

Conclusion

Conclusion

- The relations $\sim_X$ form a natural congruence on the directed path functor $P : \mathcal{Lpo} \to \mathcal{Cat}$.

Conclusion

- The relations $\sim_X$ form a natural congruence on the directed path functor $P\ :\ \mathcal{L}po \to \mathcal{C}at$.
- The fundamental category functor $\overrightarrow{\pi_1} : \mathcal{L}po \to \mathcal{C}at$ is defined accordingly.

Conclusion

- The relations $\sim_X$ form a natural congruence on the directed path functor $P : \mathcal{L}po \to \mathcal{C}at$.
- The fundamental category functor $\overrightarrow{\pi_1} : \mathcal{L}po \to \mathcal{C}at$ is defined accordingly.
- The fundamental groupoid functor $\Pi_1 : \mathcal{T}op \to \mathcal{G}rd$ is obtained by substituting "paths" and "homotopies" to "directed paths" and "elementary homotopies".

Basic properties and computations

- The fundamental category of locally ordered real line is the corresponding partial order.

- The fundamental category of locally ordered real line is the corresponding partial order.
- For all local pospaces X and Y we have

$$\overrightarrow{\pi_1}(X \times Y) \quad \cong \quad \overrightarrow{\pi_1}X \times \overrightarrow{\pi_1}Y$$

- The fundamental category of locally ordered real line is the corresponding partial order.
- For all local pospaces X and Y we have

$$\vec{\pi}_1(X \times Y) \quad \cong \quad \vec{\pi}_1 X \times \vec{\pi}_1 Y$$

- Given a pospace X, $\vec{\pi}_1 X$ is loop-free i.e.

$$\vec{\pi}_1 X(x,y) \neq \emptyset \text{ and } \vec{\pi}_1 X(y,x) \neq \emptyset \qquad \Rightarrow \qquad x = y \text{ and } \vec{\pi}_1 X(x,x) = \{\mathrm{id}_x\}$$

- The fundamental category of locally ordered real line is the corresponding partial order.
- For all local pospaces X and Y we have

$$\overrightarrow{\pi_1}(X \times Y) \quad \cong \quad \overrightarrow{\pi_1}X \times \overrightarrow{\pi_1}Y$$

- Given a pospace X, $\overrightarrow{\pi_1}X$ is loop-free i.e.

$$\overrightarrow{\pi_1}X(x,y) \neq \emptyset \text{ and } \overrightarrow{\pi_1}X(y,x) \neq \emptyset \qquad \Rightarrow \qquad x = y \text{ and } \overrightarrow{\pi_1}X(x,x) = \{\mathrm{id}_x\}$$

- The fundamental category of a local pospace has no nontrivial null homotopic directed paths i.e. any directed loop that is related to a constant path by an elementary homotopy is actually a constant.

- The fundamental category of locally ordered real line is the corresponding partial order.
- For all local pospaces X and Y we have

$$\overrightarrow{\pi_1}(X \times Y) \quad \cong \quad \overrightarrow{\pi_1}X \times \overrightarrow{\pi_1}Y$$

- Given a pospace X, $\overrightarrow{\pi_1}X$ is loop-free i.e.

$$\overrightarrow{\pi_1}X(x,y) \neq \emptyset \ \text{ and } \ \overrightarrow{\pi_1}X(y,x) \neq \emptyset \qquad \Rightarrow \qquad x = y \ \text{ and } \ \overrightarrow{\pi_1}X(x,x) = \{\mathrm{id}_x\}$$

- The fundamental category of a local pospace has no nontrivial null homotopic directed paths i.e. any directed loop that is related to a constant path by an elementary homotopy is actually a constant.
- In particular the fundamental category of a local pospace has no isomorphism but its identities.

The fundamental category of the locally ordered circle

The fundamental category of the locally ordered circle

- Given x, y, $\widehat{xy}$ is the anticlockwise arc from x to y.
  It is a singleton if $x = y$.

The fundamental category of the locally ordered circle

- Given x, y, $\widehat{xy}$ is the anticlockwise arc from x to y.
  It is a singleton if $x = y$.
- $\vec{\pi}_1\mathbb{S}^1(x, y) = \{x\} \times \mathbb{N} \times \{y\}$

The fundamental category of the locally ordered circle

- Given x, y, $\widehat{xy}$ is the anticlockwise arc from x to y.
  It is a singleton if $x = y$.
- $\overrightarrow{\pi_1}\mathbb{S}^1(x, y) = \{x\} \times \mathbb{N} \times \{y\}$
- the identities are the tuples $(x, 0, x)$

The fundamental category of the locally ordered circle

- Given x, y, $\widehat{xy}$ is the anticlockwise arc from x to y.
  It is a singleton if $x = y$.
- $\overrightarrow{\pi_1}\mathbb{S}^1(x, y) = \{x\} \times \mathbb{N} \times \{y\}$
- the identities are the tuples $(x, 0, x)$
- the composition is given by

The fundamental category of the locally ordered circle

- Given x, y, $\widehat{xy}$ is the anticlockwise arc from x to y.
  It is a singleton if $x = y$.
- $\vec{\pi}_1\mathbb{S}^1(x, y) = \{x\} \times \mathbb{N} \times \{y\}$
- the identities are the tuples $(x, 0, x)$
- the composition is given by
    - $(y, p, z) \circ (x, n, y) = (x, n + p, z)$ if $\widehat{xy} \cup \widehat{yz} \neq \mathbb{S}^1$

The fundamental category of the locally ordered circle

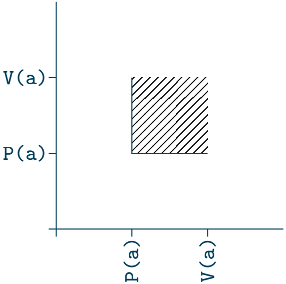
- Given x, y, $\widehat{xy}$ is the anticlockwise arc from x to y.
  It is a singleton if $x = y$.
- $\overrightarrow{\pi_1}\mathbb{S}^1(x, y) = \{x\} \times \mathbb{N} \times \{y\}$
- the identities are the tuples $(x, 0, x)$
- the composition is given by
  - $(y, p, z) \circ (x, n, y) = (x, n + p, z)$ if $\widehat{xy} \cup \widehat{yz} \neq \mathbb{S}^1$
  - $(y, p, z) \circ (x, n, y) = (x, n + p + 1, z)$ if $\widehat{xy} \cup \widehat{yz} = \mathbb{S}^1$

Plane without a square

$x = \mathbb{R}_+^2 \setminus]0, 1[^2$

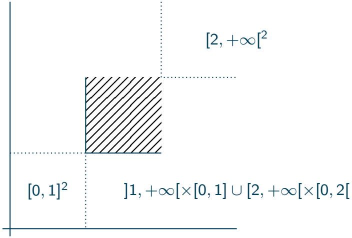
Plane without a square

$x = \mathbb{R}_+^2 \setminus]0, 1[^2$



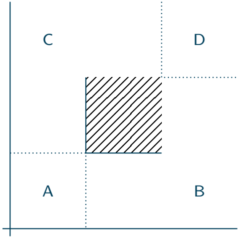
Plane without a square

$x = \mathbb{R}_+^2 \setminus]0,1[^2$



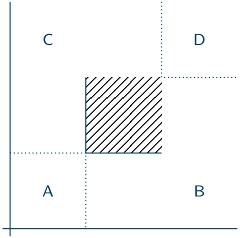
Plane without a square

$x = \mathbb{R}_+^2 \setminus]0,1[^2$



Plane without a square

$x = \mathbb{R}^2_+ \setminus]0,1[^2$



If $x \leqslant^2 y$, then $\overrightarrow{\pi_1}X(x,y)$ only depends on the elements of the partition x and y belong to.

$\rightarrow$	A	B	C	D
A	σ	β	γ	$\beta' \circ \beta$ $\alpha' \circ \alpha$
B		σ		β'
C			σ	γ'
D				σ

CATEGORY OF COMPONENTS

Motivation

Skeleta and equivalences of categories

Skeleta and equivalences of categories

- A skeleton of $\mathcal{C}$ is a full subcategory of $\mathcal{C}$ whose class of objects meets every isomorphism class of $\mathcal{C}$ exactly once.

Skeleta and equivalences of categories

- A skeleton of $\mathcal{C}$ is a full subcategory of $\mathcal{C}$ whose class of objects meets every isomorphism class of $\mathcal{C}$ exactly once.
- The skeleton of $\mathcal{C}$ is unique up to isomorphism, it is denoted by $\mathrm{sk}\mathcal{C}$.

Skeleta and equivalences of categories

- A skeleton of $\mathcal{C}$ is a full subcategory of $\mathcal{C}$ whose class of objects meets every isomorphism class of $\mathcal{C}$ exactly once.
- The skeleton of $\mathcal{C}$ is unique up to isomorphism, it is denoted by $\mathrm{sk}\mathcal{C}$.
- Two categories are equivalent (i.e. there exists an equivalence of categories between them) iff their skeleta are isomorphic.

Skeleta and equivalences of categories

- A skeleton of $\mathcal{C}$ is a full subcategory of $\mathcal{C}$ whose class of objects meets every isomorphism class of $\mathcal{C}$ exactly once.
- The skeleton of $\mathcal{C}$ is unique up to isomorphism, it is denoted by $\mathrm{sk}\mathcal{C}$.
- Two categories are equivalent (i.e. there exists an equivalence of categories between them) iff their skeleta are isomorphic.
- The skeleton of the category of finite sets is the full subcategory whose objects are $\{0, \ldots, n-1\}$ for $n \in \mathbb{N}$.

Skeleta and equivalences of categories

- A skeleton of $\mathcal{C}$ is a full subcategory of $\mathcal{C}$ whose class of objects meets every isomorphism class of $\mathcal{C}$ exactly once.
- The skeleton of $\mathcal{C}$ is unique up to isomorphism, it is denoted by sk$\mathcal{C}$.
- Two categories are equivalent (i.e. there exists an equivalence of categories between them) iff their skeleta are isomorphic.
- The skeleton of the category of finite sets is the full subcategory whose objects are $\{0, \ldots, n-1\}$ for $n \in \mathbb{N}$.
- The skeleton of the fundamental groupoid of a path-connected space is the fundamental group of this space.

Skeleta and equivalences of categories

- A skeleton of $\mathcal{C}$ is a full subcategory of $\mathcal{C}$ whose class of objects meets every isomorphism class of $\mathcal{C}$ exactly once.
- The skeleton of $\mathcal{C}$ is unique up to isomorphism, it is denoted by $\mathrm{sk}\mathcal{C}$.
- Two categories are equivalent (i.e. there exists an equivalence of categories between them) iff their skeleta are isomorphic.
- The skeleton of the category of finite sets is the full subcategory whose objects are $\{0, \ldots, n-1\}$ for $n \in \mathbb{N}$.
- The skeleton of the fundamental groupoid of a path-connected space is the fundamental group of this space.
- Problem: The fundamental category of a local pospace has no isomorphisms but its identities, hence it is its own skeleton.

Loop-free and one-way categories

The categories *LfCat* and *OwCat*

The categories $\mathcal{LfCat}$ and $\mathcal{OwCat}$

- A category $\mathcal{C}$ is said to be one-way when all its endomorphisms are identities i.e. $\mathcal{C}(x,x) = \{\mathrm{id}_x\}$ for all x

  Every Grothendieck topos has a one-way site. C. MacLarty. Theor. Appl. of Cat. 16(5) pp 123-126 (2006).

The categories $\mathcal{LfCat}$ and $\mathcal{OwCat}$

- A category $\mathcal{C}$ is said to be one-way when all its endomorphisms are identities i.e. $\mathcal{C}(x,x) = \{\mathrm{id}_x\}$ for all x

  Every Grothendieck topos has a one-way site. C. MacLarty. Theor. Appl. of Cat. 16(5) pp 123-126 (2006).

- A one-way category $\mathcal{C}$ is said to be loop-free when for all x, y

$$\mathcal{C}(x,y) \neq \emptyset \text{ and } \mathcal{C}(y,x) \neq \emptyset \text{ implies } x = y$$

Complexes of groups and orbihedra *in* Group theory from a geometrical viewpoint.

A. Haefliger. World Scientific (1991).

The categories $\mathcal{LfCat}$ and $\mathcal{OwCat}$

- A category $\mathcal{C}$ is said to be one-way when all its endomorphisms are identities i.e. $\mathcal{C}(x,x) = \{\mathrm{id}_x\}$ for all x

  Every Grothendieck topos has a one-way site. C. MacLarty. Theor. Appl. of Cat. 16(5) pp 123-126 (2006).

- A one-way category $\mathcal{C}$ is said to be loop-free when for all x, y

$$\mathcal{C}(x,y) \neq \emptyset \text{ and } \mathcal{C}(y,x) \neq \emptyset \text{ implies } x = y$$

  Complexes of groups and orbihedra *in* Group theory from a geometrical viewpoint.
  A. Haefliger. World Scientific (1991).

- A loop-free category is its own skeleton

The categories $\mathcal{L}f\mathcal{C}at$ and $\mathcal{O}w\mathcal{C}at$

- A category $\mathcal{C}$ is said to be one-way when all its endomorphisms are identities i.e. $\mathcal{C}(x,x) = \{\mathrm{id}_x\}$ for all x

  Every Grothendieck topos has a one-way site. C. MacLarty. Theor. Appl. of Cat. 16(5) pp 123-126 (2006).

- A one-way category $\mathcal{C}$ is said to be loop-free when for all x, y

$$\mathcal{C}(x,y) \neq \emptyset \text{ and } \mathcal{C}(y,x) \neq \emptyset \text{ implies } x = y$$

  Complexes of groups and orbihedra *in* Group theory from a geometrical viewpoint.

  A. Haefliger. World Scientific (1991).

- A loop-free category is its own skeleton
- A category is one-way iff its skeleton is loop-free

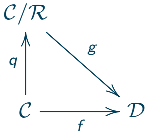
Generalized congruences

M. A. Bednarczyk, A. M. Borzyszkowski, W. Pawlowski. Theor. Appl. Cat. 5(11). 1999

Generalized congruences

M. A. Bednarczyk, A. M. Borzyszkowski, W. Pawlowski. Theor. Appl. Cat. 5(11). 1999

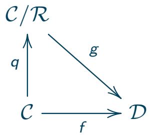
- Given a binary relation $\mathcal{R}$ on the set of morphisms of a category $\mathcal{C}$, there is a unique category $\mathcal{C}/\mathcal{R}$ and a unique functor $q : \mathcal{C} \to \mathcal{C}/\mathcal{R}$ such that for all functors $f : \mathcal{C} \to \mathcal{D}$, if $\alpha\mathcal{R}\beta \Rightarrow f(\alpha) = f(\beta)$, then there is a unique functor $g : \mathcal{C}/\mathcal{R} \to \mathcal{D}$ such that $f = g \circ q$



Generalized congruences

M. A. Bednarczyk, A. M. Borzyszkowski, W. Pawlowski. Theor. Appl. Cat. 5(11). 1999

- Given a binary relation $\mathcal{R}$ on the set of morphisms of a category $\mathcal{C}$, there is a unique category $\mathcal{C}/\mathcal{R}$ and a unique functor $q : \mathcal{C} \to \mathcal{C}/\mathcal{R}$ such that for all functors $f : \mathcal{C} \to \mathcal{D}$, if $\alpha\mathcal{R}\beta \Rightarrow f(\alpha) = f(\beta)$, then there is a unique functor $g : \mathcal{C}/\mathcal{R} \to \mathcal{D}$ such that $f = g \circ q$

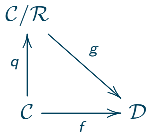


- Examples

Generalized congruences

M. A. Bednarczyk, A. M. Borzyszkowski, W. Pawlowski. Theor. Appl. Cat. 5(11). 1999

- Given a binary relation $\mathcal{R}$ on the set of morphisms of a category $\mathcal{C}$, there is a unique category $\mathcal{C}/\mathcal{R}$ and a unique functor $q : \mathcal{C} \to \mathcal{C}/\mathcal{R}$ such that for all functors $f : \mathcal{C} \to \mathcal{D}$, if $\alpha \mathcal{R} \beta \Rightarrow f(\alpha) = f(\beta)$, then there is a unique functor $g : \mathcal{C}/\mathcal{R} \to \mathcal{D}$ such that $f = g \circ q$

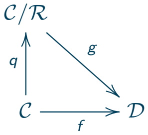


- Examples
  - any congruence is a generalized congruence.

Generalized congruences

M. A. Bednarczyk, A. M. Borzyszkowski, W. Pawlowski. Theor. Appl. Cat. 5(11). 1999

- Given a binary relation $\mathcal{R}$ on the set of morphisms of a category $\mathcal{C}$, there is a unique category $\mathcal{C}/\mathcal{R}$ and a unique functor $q : \mathcal{C} \to \mathcal{C}/\mathcal{R}$ such that for all functors $f : \mathcal{C} \to \mathcal{D}$, if $\alpha \mathcal{R} \beta \Rightarrow f(\alpha) = f(\beta)$, then there is a unique functor $g : \mathcal{C}/\mathcal{R} \to \mathcal{D}$ such that $f = g \circ q$

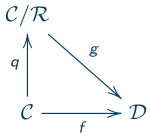


- Examples
  - any congruence is a generalized congruence.
  - $\mathcal{C}$ freely generated by $x \xrightarrow{\alpha} y$ with $\mathrm{id}_x \mathcal{R} \mathrm{id}_y$ (resp. with $\alpha \mathcal{R} \mathrm{id}_x$).

Generalized congruences

M. A. Bednarczyk, A. M. Borzyszkowski, W. Pawlowski. Theor. Appl. Cat. 5(11). 1999

- Given a binary relation $\mathcal{R}$ on the set of morphisms of a category $\mathcal{C}$, there is a unique category $\mathcal{C}/\mathcal{R}$ and a unique functor $q : \mathcal{C} \to \mathcal{C}/\mathcal{R}$ such that for all functors $f : \mathcal{C} \to \mathcal{D}$, if $\alpha \mathcal{R} \beta \Rightarrow f(\alpha) = f(\beta)$, then there is a unique functor $g : \mathcal{C}/\mathcal{R} \to \mathcal{D}$ such that $f = g \circ q$



- Examples
  - any congruence is a generalized congruence.
  - $\mathcal{C}$ freely generated by $x \xrightarrow{\alpha} y$ with $\mathrm{id}_x \mathcal{R} \mathrm{id}_y$ (resp. with $\alpha \mathcal{R} \mathrm{id}_x$).
  - $(\mathbb{N}, +, 0)$ with $0 \mathcal{R} n$ for some $n \in \mathbb{N}$.

Systems of weak isomorphisms

Goal

Goal

Let $\mathcal{C}$ be a one-way category:

Goal

Let $\mathcal{C}$ be a one-way category:

- Define a class Σ of morphisms of $\mathcal{C}$ so we can keep one representative in each class of Σ-related objects without loss of information

Goal

Let $\mathcal{C}$ be a one-way category:

- Define a class Σ of morphisms of $\mathcal{C}$ so we can keep one representative in each class of Σ-related objects without loss of information
- To do so, we are in search for a class that behaves much like the one of isomorphisms

Goal

Let $\mathcal{C}$ be a one-way category:

- Define a class Σ of morphisms of $\mathcal{C}$ so we can keep one representative in each class of Σ-related objects without loss of information
- To do so, we are in search for a class that behaves much like the one of isomorphisms
- From now on $\mathcal{C}$ denotes a one-way category

Potential weak isomorphisms

Let $\mathcal{C}$ be a one-way category

Potential weak isomorphisms

Let $\mathcal{C}$ be a one-way category

- For all morphisms σ and all objects z define

Potential weak isomorphisms

Let $\mathcal{C}$ be a one-way category

- For all morphisms σ and all objects z define
  - the σ, z-precomposition as $\gamma \in \mathcal{C}(\partial^{+}\sigma, z) \;\rightarrow\; \gamma \circ \sigma \in \mathcal{C}(\partial^{-}\sigma, z)$

Potential weak isomorphisms

Let $\mathcal{C}$ be a one-way category

- For all morphisms σ and all objects z define
  - the σ, z-precomposition as $\gamma \in \mathcal{C}(\partial^+\sigma, z) \;\to\; \gamma \circ \sigma \in \mathcal{C}(\partial^-\sigma, z)$
  - the z, σ-postcomposition as $\delta \in \mathcal{C}(z, \partial^-\sigma) \;\mapsto\; \sigma \circ \delta \in \mathcal{C}(z, \partial^+\sigma)$

Potential weak isomorphisms

Let $\mathcal{C}$ be a one-way category

- For all morphisms σ and all objects z define
  - the σ, z-precomposition as $\gamma \in \mathcal{C}(\partial^+\sigma, z) \;\to\; \gamma \circ \sigma \in \mathcal{C}(\partial^-\sigma, z)$
  - the z, σ-postcomposition as $\delta \in \mathcal{C}(z, \partial^-\sigma) \;\mapsto\; \sigma \circ \delta \in \mathcal{C}(z, \partial^+\sigma)$
- One may have $\mathcal{C}(\partial^+\sigma, z) = \emptyset$ or $\mathcal{C}(z, \partial^-\sigma) = \emptyset$

Potential weak isomorphisms

Let $\mathcal{C}$ be a one-way category

- For all morphisms σ and all objects z define
  - the σ, z-precomposition as $\gamma \in \mathcal{C}(\partial^+\sigma, z) \;\rightarrow\; \gamma \circ \sigma \in \mathcal{C}(\partial^-\sigma, z)$
  - the z, σ-postcomposition as $\delta \in \mathcal{C}(z, \partial^-\sigma) \;\mapsto\; \sigma \circ \delta \in \mathcal{C}(z, \partial^+\sigma)$
- One may have $\mathcal{C}(\partial^+\sigma, z) = \emptyset$ or $\mathcal{C}(z, \partial^-\sigma) = \emptyset$
- Note that σ is an isomorphism iff for all z both precomposition and postcomposition are bijective.

Potential weak isomorphisms

Let $\mathcal{C}$ be a one-way category

- For all morphisms σ and all objects z define
  - the σ, z-precomposition as $\gamma \in \mathcal{C}(\partial^+\sigma, z) \rightarrow \gamma \circ \sigma \in \mathcal{C}(\partial^-\sigma, z)$
  - the z, σ-postcomposition as $\delta \in \mathcal{C}(z, \partial^-\sigma) \mapsto \sigma \circ \delta \in \mathcal{C}(z, \partial^+\sigma)$
- One may have $\mathcal{C}(\partial^+\sigma, z) = \emptyset$ or $\mathcal{C}(z, \partial^-\sigma) = \emptyset$
- Note that σ is an isomorphism iff for all z both precomposition and postcomposition are bijective.
- The latter condition is weakened: σ is said to preserve the future cones (resp. past cones) when for all z if $\mathcal{C}(\partial^+\sigma, z) \neq \emptyset$ (resp. $\mathcal{C}(z, \partial^-\sigma) \neq \emptyset$) then the precomposition (resp. postcomposition) is bijective.

Potential weak isomorphisms

Let $\mathcal{C}$ be a one-way category

- For all morphisms σ and all objects z define
  - the σ, z-precomposition as $\gamma \in \mathcal{C}(\partial^+\sigma, z) \;\to\; \gamma \circ \sigma \in \mathcal{C}(\partial^-\sigma, z)$
  - the z, σ-postcomposition as $\delta \in \mathcal{C}(z, \partial^-\sigma) \;\mapsto\; \sigma \circ \delta \in \mathcal{C}(z, \partial^+\sigma)$
- One may have $\mathcal{C}(\partial^+\sigma, z) = \emptyset$ or $\mathcal{C}(z, \partial^-\sigma) = \emptyset$
- Note that σ is an isomorphism iff for all z both precomposition and postcomposition are bijective.
- The latter condition is weakened: σ is said to preserve the future cones (resp. past cones) when for all z if $\mathcal{C}(\partial^+\sigma, z) \neq \emptyset$ (resp. $\mathcal{C}(z, \partial^-\sigma) \neq \emptyset$) then the precomposition (resp. postcomposition) is bijective.
- Then σ is a potential weak isomorphism when it preserves both future cones and past cones. Potential weak isomorphisms compose.

Potential weak isomorphisms

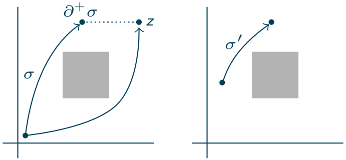
Let $\mathcal{C}$ be a one-way category

- For all morphisms σ and all objects z define
  - the σ, z-precomposition as $\gamma \in \mathcal{C}(\partial^+\sigma, z) \;\to\; \gamma\circ\sigma \in \mathcal{C}(\partial^-\sigma, z)$
  - the z, σ-postcomposition as $\delta \in \mathcal{C}(z, \partial^-\sigma) \;\mapsto\; \sigma\circ\delta \in \mathcal{C}(z, \partial^+\sigma)$
- One may have $\mathcal{C}(\partial^+\sigma, z) = \emptyset$ or $\mathcal{C}(z, \partial^-\sigma) = \emptyset$
- Note that σ is an isomorphism iff for all z both precomposition and postcomposition are bijective.
- The latter condition is weakened: σ is said to preserve the future cones (resp. past cones) when for all z if $\mathcal{C}(\partial^+\sigma, z) \neq \emptyset$ (resp. $\mathcal{C}(z, \partial^-\sigma) \neq \emptyset$) then the precomposition (resp. postcomposition) is bijective.
- Then σ is a potential weak isomorphism when it preserves both future cones and past cones. Potential weak isomorphisms compose.
- If $\mathcal{C}(x, y)$ contains a potential weak isomorphism, then it is a singleton

  Requires the assumption that $\mathcal{C}$ is one-way

An example of potential weak isomorphism

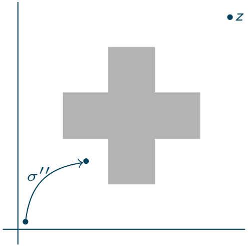
An example of potential weak isomorphism



Due to the lower dipath, the σ, z-precomposition is not bijective; yet σ' is a potential weak isomorphism.

An unwanted example of potential weak isomorphism

An unwanted example of potential weak isomorphism

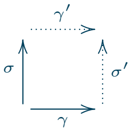


Note that σ'' is a potential weak isomorphism though there exists a morphism from $\partial^{-}\sigma''$ to z but none from $\partial^{+}\sigma''$ to z.

Stability under pushout and pullback

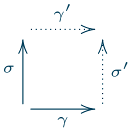
Stability under pushout and pullback

- A collection of morphisms Σ is said to be stable under pushout when for all $\sigma \in \Sigma$, for all γ with $\partial^- \gamma = \partial^- \sigma$, the pushout of σ along γ exists and belongs to Σ

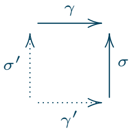


Stability under pushout and pullback

- A collection of morphisms Σ is said to be stable under pushout when for all $\sigma \in \Sigma$, for all γ with $\partial^- \gamma = \partial^- \sigma$, the pushout of σ along γ exists and belongs to Σ



- A collection of morphisms Σ is said to be stable under pullback when for all $\sigma \in \Sigma$, for all γ with $\partial^+ \gamma = \partial^+ \sigma$, the pullback of σ along γ exists and belongs to Σ



Greatest inner collection stable under pushout and pullback

Greatest inner collection stable under pushout and pullback

- Any collection Σ of morphisms of a category $\mathcal{C}$ admits a greatest subcollection that is stable under pushout and pullback

Greatest inner collection stable under pushout and pullback

- Any collection Σ of morphisms of a category $\mathcal{C}$ admits a greatest subcollection that is stable under pushout and pullback
- Construction:

Greatest inner collection stable under pushout and pullback

- Any collection Σ of morphisms of a category $\mathcal{C}$ admits a greatest subcollection that is stable under pushout and pullback
- Construction:
  - Start with $\Sigma_0 = \Sigma$

Greatest inner collection stable under pushout and pullback

- Any collection Σ of morphisms of a category $\mathcal{C}$ admits a greatest subcollection that is stable under pushout and pullback
- Construction:
  - Start with $\Sigma_0 = \Sigma$
  - For $n \in \mathbb{N}$ define Σ_{n+1} as the collection of morphisms $\sigma \in \Sigma_n$ s.t. the pushout and the pullback of σ along with all morphisms exist (when sources or targets match) and belong to Σ_n

$$\Sigma_0 \supseteq \cdots \Sigma_1 \supseteq \cdots \supseteq \Sigma_n \supseteq \Sigma_{n+1} \supseteq \cdots$$

Greatest inner collection stable under pushout and pullback

- Any collection Σ of morphisms of a category $\mathcal{C}$ admits a greatest subcollection that is stable under pushout and pullback
- Construction:
  - Start with $\Sigma_0 = \Sigma$
  - For $n \in \mathbb{N}$ define Σ_{n+1} as the collection of morphisms $\sigma \in \Sigma_n$ s.t. the pushout and the pullback of σ along with all morphisms exist (when sources or targets match) and belong to Σ_n

$$\Sigma_0 \supseteq \cdots \Sigma_1 \supseteq \cdots \supseteq \Sigma_n \supseteq \Sigma_{n+1} \supseteq \cdots$$

  - The expected subcollection is the decreasing intersection

$$\Sigma_\infty := \bigcap_{n\in\mathbb{N}}^{\downarrow} \Sigma_n$$

Greatest inner collection stable under pushout and pullback

- Any collection Σ of morphisms of a category $\mathcal{C}$ admits a greatest subcollection that is stable under pushout and pullback
- Construction:
  - Start with $\Sigma_0 = \Sigma$
  - For $n \in \mathbb{N}$ define Σ_{n+1} as the collection of morphisms $\sigma \in \Sigma_n$ s.t. the pushout and the pullback of σ along with all morphisms exist (when sources or targets match) and belong to Σ_n

$$\Sigma_0 \supseteq \cdots \Sigma_1 \supseteq \cdots \supseteq \Sigma_n \supseteq \Sigma_{n+1} \supseteq \cdots$$

  - The expected subcollection is the decreasing intersection

$$\Sigma_\infty := \bigcap_{n\in\mathbb{N}}^{\downarrow} \Sigma_n$$

- The collection Σ_∞ is stable under the action of $\mathrm{Aut}(\mathcal{C})$

Systems of weak isomorphisms

Systems of weak isomorphisms

- The class of isomorphisms of any category is stable under pushout and pullback

Systems of weak isomorphisms

- The class of isomorphisms of any category is stable under pushout and pullback
- A system of weak isomorphisms is a collection of potential weak isomorphisms that is stable under pushout and pullback

Systems of weak isomorphisms

- The class of isomorphisms of any category is stable under pushout and pullback
- A system of weak isomorphisms is a collection of potential weak isomorphisms that is stable under pushout and pullback
- The class of all isomorphisms of any category is a system of weak isomorphisms

Systems of weak isomorphisms

- The class of isomorphisms of any category is stable under pushout and pullback
- A system of weak isomorphisms is a collection of potential weak isomorphisms that is stable under pushout and pullback
- The class of all isomorphisms of any category is a system of weak isomorphisms
- If Σ is a system of weak isomorphisms, then so is its closure under composition

Systems of weak isomorphisms

- The class of isomorphisms of any category is stable under pushout and pullback
- A system of weak isomorphisms is a collection of potential weak isomorphisms that is stable under pushout and pullback
- The class of all isomorphisms of any category is a system of weak isomorphisms
- If Σ is a system of weak isomorphisms, then so is its closure under composition
- Hence we suppose the systems of weak isomorphisms are closed under composition

Examples of systems of weak ismorphisms

Examples of systems of weak ismorphisms

- Given a partition $\mathcal{P}$ of $\mathbb{R}$ into intervals, the following collection is a system of weak isomorphisms

$$\{(x, y) \mid x \leqslant y;\ \exists \boldsymbol{I} \in \mathcal{P},\ [x, y] \subseteq \boldsymbol{I}\}$$

Examples of systems of weak isomorphisms

- Given a partition $\mathcal{P}$ of $\mathbb{R}$ into intervals, the following collection is a system of weak isomorphisms

$$\{(x, y) \mid x \leqslant y;\ \exists \boldsymbol{I} \in \mathcal{P},\ [x, y] \subseteq \boldsymbol{I}\}$$

- In the preceding example, $\mathbb{R}$ can be replaced by any totally ordered set

Examples of systems of weak isomorphisms

- Given a partition $\mathcal{P}$ of $\mathbb{R}$ into intervals, the following collection is a system of weak isomorphisms

$$\{(x,y) \mid x \leqslant y;\ \exists I \in \mathcal{P},\ [x,y] \subseteq I\}$$

- In the preceding example, $\mathbb{R}$ can be replaced by any totally ordered set
- Let $\Sigma_i \subseteq \mathcal{C}_i$ be a family of collections of morphisms, then

$\prod_i \Sigma_i$ is a swi of $\prod_i \mathcal{C}_i$ iff each Σ_i is a swi of $\mathcal{C}_i$

Examples of systems of weak isomorphisms

- Given a partition $\mathcal{P}$ of $\mathbb{R}$ into intervals, the following collection is a system of weak isomorphisms

$$\{(x, y) \mid x \leqslant y;\ \exists \boldsymbol{I} \in \mathcal{P},\ [x, y] \subseteq \boldsymbol{I}\}$$

- In the preceding example, $\mathbb{R}$ can be replaced by any totally ordered set
- Let $\Sigma_i \subseteq \mathcal{C}_i$ be a family of collections of morphisms, then

$$\boxed{\prod_i \Sigma_i \text{ is a swi of } \prod_i \mathcal{C}_i \text{ iff each } \Sigma_i \text{ is a swi of } \mathcal{C}_i}$$

- The inverse image (resp. the direct image) of a system of weak isomorphisms by an equivalence of categories is a system of weak isomorphisms.

Pureness

Pureness

- A collection Σ of morphisms is said to be pure when

$$\gamma \circ \delta \in \mathbf{\Sigma} \;\Rightarrow\; \gamma, \delta \in \mathbf{\Sigma}$$

Pureness

- A collection Σ of morphisms is said to be pure when

$$\gamma \circ \delta \in \boldsymbol{\Sigma} \;\Rightarrow\; \gamma, \delta \in \boldsymbol{\Sigma}$$

- Given a one-way category $\mathcal{C}$ we have:

All the systems of weak isomorphisms of $\mathcal{C}$ are pure

The locale of systems of weak isomorphisms

The locale of systems of weak isomorphisms

- A locale is a complete lattice whose binary meet distributes over arbitrary join i.e.

$$x \wedge \left(\bigvee_i y_i \right) = \bigvee_i (x \wedge y_i)$$

The locale of systems of weak isomorphisms

- A locale is a complete lattice whose binary meet distributes over arbitrary join i.e.

$$x \wedge \left(\bigvee_i y_i \right) = \bigvee_i (x \wedge y_i)$$

- The collection ΩX open subsets of a topological space X form a locale and we have the functor $L : \mathcal{Top} \to \mathcal{Loc}$ (that admits a left adjoint) defined by

The locale of systems of weak isomorphisms

- A locale is a complete lattice whose binary meet distributes over arbitrary join i.e.

$$x \wedge \left(\bigvee_i y_i\right) = \bigvee_i (x \wedge y_i)$$

- The collection ΩX open subsets of a topological space X form a locale and we have the functor $L : \mathcal{Top} \to \mathcal{Loc}$ (that admits a left adjoint) defined by
  - $L(X) = \Omega X$

The locale of systems of weak isomorphisms

- A locale is a complete lattice whose binary meet distributes over arbitrary join i.e.

$$x \wedge \left(\bigvee_i y_i \right) = \bigvee_i (x \wedge y_i)$$

- The collection ΩX open subsets of a topological space X form a locale and we have the functor $L : \mathcal{Top} \to \mathcal{Loc}$ (that admits a left adjoint) defined by
    - $L(X) = \Omega X$
    - $L(f)(W) = f^{-1}(W)$ for all $f : X \to Y$ and $W \in \Omega Y$

The locale of systems of weak isomorphisms

- A locale is a complete lattice whose binary meet distributes over arbitrary join i.e.

$$x \wedge \left(\bigvee_i y_i \right) = \bigvee_i (x \wedge y_i)$$

- The collection ΩX open subsets of a topological space X form a locale and we have the functor $L : \mathcal{Top} \to \mathcal{Loc}$ (that admits a left adjoint) defined by
  - $L(X) = \Omega X$
  - $L(f)(W) = f^{-1}(W)$ for all $f : X \to Y$ and $W \in \Omega Y$
- The collection of systems of weak isomorphisms of a category has a greatest element

The locale of systems of weak isomorphisms

- A locale is a complete lattice whose binary meet distributes over arbitrary join i.e.

$$x \wedge \left(\bigvee_i y_i\right) = \bigvee_i (x \wedge y_i)$$

- The collection ΩX open subsets of a topological space X form a locale and we have the functor $L : \mathcal{Top} \to \mathcal{Loc}$ (that admits a left adjoint) defined by
  - $L(X) = \Omega X$
  - $L(f)(W) = f^{-1}(W)$ for all $f : X \to Y$ and $W \in \Omega Y$
- The collection of systems of weak isomorphisms of a category has a greatest element
- Given a one-way category $\mathcal{C}$ we have:

The locale of systems of weak isomorphisms

- A locale is a complete lattice whose binary meet distributes over arbitrary join i.e.

$$x \wedge \left(\bigvee_i y_i \right) = \bigvee_i (x \wedge y_i)$$

- The collection ΩX open subsets of a topological space X form a locale and we have the functor $L : \mathcal{Top} \to \mathcal{Loc}$ (that admits a left adjoint) defined by
  - $L(X) = \Omega X$
  - $L(f)(W) = f^{-1}(W)$ for all $f : X \to Y$ and $W \in \Omega Y$
- The collection of systems of weak isomorphisms of a category has a greatest element
- Given a one-way category $\mathcal{C}$ we have:
  - The collection of systems of weak isomorphisms of $\mathcal{C}$ forms a locale

The locale of systems of weak isomorphisms

- A locale is a complete lattice whose binary meet distributes over arbitrary join i.e.

$$x \wedge \Big(\bigvee_i y_i\Big) = \bigvee_i (x \wedge y_i)$$

- The collection ΩX open subsets of a topological space X form a locale and we have the functor $L : \mathcal{Top} \to \mathcal{Loc}$ (that admits a left adjoint) defined by
  - $L(X) = \Omega X$
  - $L(f)(W) = f^{-1}(W)$ for all $f : X \to Y$ and $W \in \Omega Y$
- The collection of systems of weak isomorphisms of a category has a greatest element
- Given a one-way category $\mathcal{C}$ we have:
  - The collection of systems of weak isomorphisms of $\mathcal{C}$ forms a locale
  - The greatest swi is invariant under the action of $\mathrm{Aut}(\mathcal{C})$

Components of a one-way category $\mathcal{C}$

Components of a one-way category $\mathcal{C}$

- From now on $\mathcal{C}$ is a one-way category and Σ is a system of weak isomorphisms on it

Components of a one-way category $\mathcal{C}$

- From now on $\mathcal{C}$ is a one-way category and Σ is a system of weak isomorphisms on it
- Recall that if $\mathcal{C}(x, y)$ meets Σ, then $\mathcal{C}(x, y)$ is a singleton, a fact that we represent on diagrams by: $x \xrightarrow{\Sigma} y$

Components of a one-way category $\mathcal{C}$

- From now on $\mathcal{C}$ is a one-way category and Σ is a system of weak isomorphisms on it
- Recall that if $\mathcal{C}(x, y)$ meets Σ, then $\mathcal{C}(x, y)$ is a singleton, a fact that we represent on diagrams by: $x \xrightarrow{\Sigma} y$
- Given two objects x and y of $\mathcal{C}$ t.f.a.e.:

Components of a one-way category $\mathcal{C}$

- From now on $\mathcal{C}$ is a one-way category and Σ is a system of weak isomorphisms on it
- Recall that if $\mathcal{C}(x, y)$ meets Σ, then $\mathcal{C}(x, y)$ is a singleton, a fact that we represent on diagrams by: $x \xrightarrow{\Sigma} y$
- Given two objects x and y of $\mathcal{C}$ t.f.a.e.:
  - there exists a Σ-zigzag between x and y

Components of a one-way category $\mathcal{C}$

- From now on $\mathcal{C}$ is a one-way category and Σ is a system of weak isomorphisms on it
- Recall that if $\mathcal{C}(x, y)$ meets Σ, then $\mathcal{C}(x, y)$ is a singleton, a fact that we represent on diagrams by: $x \xrightarrow{\Sigma} y$
- Given two objects x and y of $\mathcal{C}$ t.f.a.e.:
  - there exists a Σ-zigzag between x and y
  - there exists z such that $x \xleftarrow{\Sigma} z \xrightarrow{\Sigma} y$

Components of a one-way category $\mathcal{C}$

- From now on $\mathcal{C}$ is a one-way category and Σ is a system of weak isomorphisms on it
- Recall that if $\mathcal{C}(x, y)$ meets Σ, then $\mathcal{C}(x, y)$ is a singleton, a fact that we represent on diagrams by: $x \xrightarrow{\Sigma} y$
- Given two objects x and y of $\mathcal{C}$ t.f.a.e.:
    - there exists a Σ-zigzag between x and y
    - there exists z such that $x \xleftarrow{\Sigma} z \xrightarrow{\Sigma} y$
    - there exists z such that $x \xrightarrow{\Sigma} z \xleftarrow{\Sigma} y$

Components of a one-way category $\mathcal{C}$

- From now on $\mathcal{C}$ is a one-way category and Σ is a system of weak isomorphisms on it
- Recall that if $\mathcal{C}(x, y)$ meets Σ, then $\mathcal{C}(x, y)$ is a singleton, a fact that we represent on diagrams by: $x \xrightarrow{\Sigma} y$
- Given two objects x and y of $\mathcal{C}$ t.f.a.e.:
  - there exists a Σ-zigzag between x and y
  - there exists z such that $x \xleftarrow{\Sigma} z \xrightarrow{\Sigma} y$
  - there exists z such that $x \xrightarrow{\Sigma} z \xleftarrow{\Sigma} y$
- When any of the following property is satisfied x and y are said to be Σ-connected

Components of a one-way category $\mathcal{C}$

- From now on $\mathcal{C}$ is a one-way category and Σ is a system of weak isomorphisms on it
- Recall that if $\mathcal{C}(x,y)$ meets Σ, then $\mathcal{C}(x,y)$ is a singleton, a fact that we represent on diagrams by: $x \xrightarrow{\Sigma} y$
- Given two objects x and y of $\mathcal{C}$ t.f.a.e.:
  - there exists a Σ-zigzag between x and y
  - there exists z such that $x \xleftarrow{\Sigma} z \xrightarrow{\Sigma} y$
  - there exists z such that $x \xrightarrow{\Sigma} z \xleftarrow{\Sigma} y$
- When any of the following property is satisfied x and y are said to be Σ-connected
- Σ-connectedness is an equivalence relation on the objects of $\mathcal{C}$

Components of a one-way category $\mathcal{C}$

- From now on $\mathcal{C}$ is a one-way category and Σ is a system of weak isomorphisms on it
- Recall that if $\mathcal{C}(x, y)$ meets Σ, then $\mathcal{C}(x, y)$ is a singleton, a fact that we represent on diagrams by: $x \xrightarrow{\Sigma} y$
- Given two objects x and y of $\mathcal{C}$ t.f.a.e.:
  - there exists a Σ-zigzag between x and y
  - there exists z such that $x \xleftarrow{\Sigma} z \xrightarrow{\Sigma} y$
  - there exists z such that $x \xrightarrow{\Sigma} z \xleftarrow{\Sigma} y$
- When any of the following property is satisfied x and y are said to be Σ-connected
- Σ-connectedness is an equivalence relation on the objects of $\mathcal{C}$
- The equivalence classes are called a Σ-components

Structure of the Σ-components

Σ system of weak isomorphisms of $\mathcal{C}$ one-way category

Structure of the Σ-components

Σ system of weak isomorphisms of $\mathcal{C}$ one-way category

A prelattice is a preordered set in which $x \wedge y$ and $x \vee y$ exist for all x and y.
However they are defined only up to isomorphism

Structure of the Σ-components

Σ system of weak isomorphisms of $\mathcal{C}$ one-way category

A prelattice is a preordered set in which $x \wedge y$ and $x \vee y$ exist for all x and y.
However they are defined only up to isomorphism

Let K be a Σ-component of $\mathcal{C}$ and $\mathcal{K}$ be the full subcategory of $\mathcal{C}$ whose objects are the elements of K. The following properties are satisfied:

Structure of the Σ-components

Σ system of weak isomorphisms of $\mathcal{C}$ one-way category

A prelattice is a preordered set in which $x \wedge y$ and $x \vee y$ exist for all x and y.
However they are defined only up to isomorphism

Let K be a Σ-component of $\mathcal{C}$ and $\mathcal{K}$ be the full subcategory of $\mathcal{C}$ whose objects are the elements of K. The following properties are satisfied:

1. The category $\mathcal{K}$ is isomorphic with the preorder $(K, \preccurlyeq)$ where $x \preccurlyeq y$ stands for $\mathcal{C}[x, y] \neq \emptyset$. In particular, every diagram in $\mathcal{K}$ commutes.

Structure of the Σ-components

Σ system of weak isomorphisms of $\mathcal{C}$ one-way category

A prelattice is a preordered set in which $x \wedge y$ and $x \vee y$ exist for all x and y.
However they are defined only up to isomorphism

Let K be a Σ-component of $\mathcal{C}$ and $\mathcal{K}$ be the full subcategory of $\mathcal{C}$ whose objects are the elements of K. The following properties are satisfied:

1. The category $\mathcal{K}$ is isomorphic with the preorder $(K, \preccurlyeq)$ where $x \preccurlyeq y$ stands for $\mathcal{C}[x, y] \neq \emptyset$. In particular, every diagram in $\mathcal{K}$ commutes.
2. The preordered set $(K, \preccurlyeq)$ is a prelattice.

Structure of the Σ-components

Σ system of weak isomorphisms of $\mathcal{C}$ one-way category

A prelattice is a preordered set in which $x \wedge y$ and $x \vee y$ exist for all x and y.
However they are defined only up to isomorphism

Let K be a Σ-component of $\mathcal{C}$ and $\mathcal{K}$ be the full subcategory of $\mathcal{C}$ whose objects are the elements of K. The following properties are satisfied:

1. The category $\mathcal{K}$ is isomorphic with the preorder $(K, \preccurlyeq)$ where $x \preccurlyeq y$ stands for $\mathcal{C}[x, y] \neq \emptyset$. In particular, every diagram in $\mathcal{K}$ commutes.
2. The preordered set $(K, \preccurlyeq)$ is a prelattice.
3. If d and u are respectively a greatest lower bound and a least upper bound of the pair $\{x, y\}$, then Diagram 1 is both a pullback and a pushout in $\mathcal{C}$, and all the arrows appearing on the diagram belong to Σ.

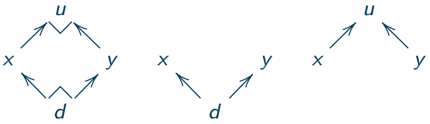


Diagram 1 Diagram 2 Diagram 3

Structure of the Σ-components

Σ system of weak isomorphisms of $\mathcal{C}$ one-way category

A prelattice is a preordered set in which $x \wedge y$ and $x \vee y$ exist for all x and y.
However they are defined only up to isomorphism

Let K be a Σ-component of $\mathcal{C}$ and $\mathcal{K}$ be the full subcategory of $\mathcal{C}$ whose objects are the elements of K. The following properties are satisfied:

1. The category $\mathcal{K}$ is isomorphic with the preorder $(K, \preccurlyeq)$ where $x \preccurlyeq y$ stands for $\mathcal{C}[x, y] \neq \emptyset$. In particular, every diagram in $\mathcal{K}$ commutes.
2. The preordered set $(K, \preccurlyeq)$ is a prelattice.
3. If d and u are respectively a greatest lower bound and a least upper bound of the pair $\{x, y\}$, then Diagram 1 is both a pullback and a pushout in $\mathcal{C}$, and all the arrows appearing on the diagram belong to Σ.
4. $\mathcal{C} = \mathcal{K}$ iff $\mathcal{C}$ is a prelattice, and Σ is the greatest system of weak isomorphisms of $\mathcal{C}$ i.e. all the morphisms in this case.

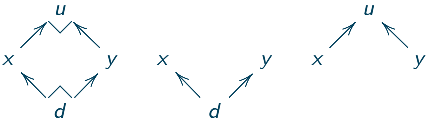


Diagram 1 Diagram 2 Diagram 3

Equivalent morphisms with respect to Σ

Equivalent morphisms with respect to Σ

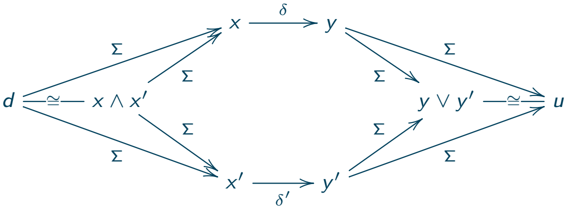
- Let $\delta \in \mathcal{C}(x, y)$ and $\delta' \in \mathcal{C}(x', y')$. Then write $\delta \sim \delta'$ when

Equivalent morphisms with respect to Σ

- Let $\delta \in \mathcal{C}(x, y)$ and $\delta' \in \mathcal{C}(x', y')$. Then write $\delta \sim \delta'$ when
  - $x \sim x'$ and $y \sim y'$, and

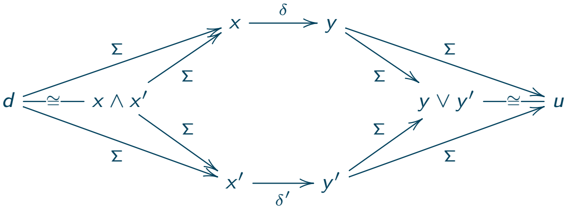
Equivalent morphisms with respect to Σ

- Let $\delta \in \mathcal{C}(x, y)$ and $\delta' \in \mathcal{C}(x', y')$. Then write $\delta \sim \delta'$ when
  - $x \sim x'$ and $y \sim y'$, and
  - the inner hexagon of the next diagram commutes



Equivalent morphisms with respect to Σ

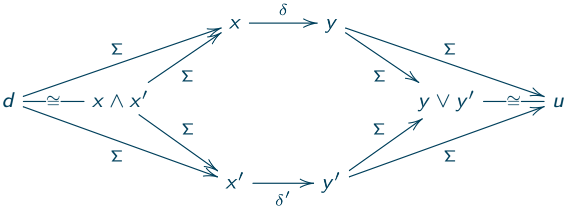
- Let $\delta \in \mathcal{C}(x, y)$ and $\delta' \in \mathcal{C}(x', y')$. Then write $\delta \sim \delta'$ when
  - $x \sim x'$ and $y \sim y'$, and
  - the inner hexagon of the next diagram commutes



  - Note that if $d \cong x \wedge x'$ and $u \cong y \vee y'$ then the outter hexagon also commutes, hence the relation $\sim$ is well defined.

Equivalent morphisms with respect to Σ

- Let $\delta \in \mathcal{C}(x, y)$ and $\delta' \in \mathcal{C}(x', y')$. Then write $\delta \sim \delta'$ when
  - $x \sim x'$ and $y \sim y'$, and
  - the inner hexagon of the next diagram commutes



  - Note that if $d \cong x \wedge x'$ and $u \cong y \vee y'$ then the outter hexagon also commutes, hence the relation $\sim$ is well defined.
- If $\gamma \sim \delta$ then $\partial^- \gamma \sim \partial^- \delta$ and $\partial^+ \gamma \sim \partial^+ \delta$

The relation $\sim$ is an equivalence

The relation $\sim$ is an equivalence

- The relation $\sim$ is:

The relation $\sim$ is an equivalence

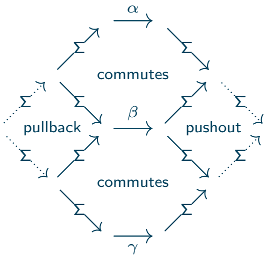
- The relation $\sim$ is:
  - reflexive since Σ contains all identities

The relation $\sim$ is an equivalence

- The relation $\sim$ is:
  - reflexive since Σ contains all identities
  - symmetric by definition

The relation $\sim$ is an equivalence

- The relation $\sim$ is:
  - reflexive since Σ contains all identities
  - symmetric by definition
  - transitive



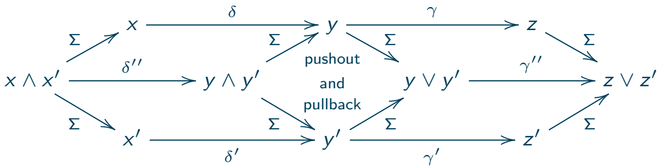
The relation $\sim$ fits with composition

The relation $\sim$ fits with composition

- Suppose $\partial^- \gamma = \partial^+ \delta$, $\partial^- \gamma' = \partial^+ \delta'$ and $\gamma \sim \gamma'$ and $\delta \sim \delta'$.

The relation $\sim$ fits with composition

- Suppose $\partial^- \gamma = \partial^+ \delta$, $\partial^- \gamma' = \partial^+ \delta'$ and $\gamma \sim \gamma'$ and $\delta \sim \delta'$.
- Then we have $\gamma \circ \delta \sim \gamma' \circ \delta'$



The category of components $\mathcal{C}/\Sigma$

The category of components $\mathcal{C}/\Sigma$

- The quotient category $\mathcal{C}/\Sigma$ (obtained by turning each morphism of Σ into an identity) can be defined as follows:

The category of components $\mathcal{C}/\Sigma$

- The quotient category $\mathcal{C}/\Sigma$ (obtained by turning each morphism of Σ into an identity) can be defined as follows:
  - The objects are the Σ-components

The category of components $\mathcal{C}/\Sigma$

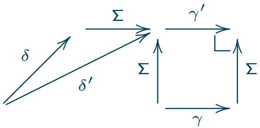
- The quotient category $\mathcal{C}/\Sigma$ (obtained by turning each morphism of Σ into an identity) can be defined as follows:
  - The objects are the Σ-components
  - The morphisms are the $\sim$-equivalence classes

The category of components $\mathcal{C}/\Sigma$

- The quotient category $\mathcal{C}/\Sigma$ (obtained by turning each morphism of Σ into an identity) can be defined as follows:
  - The objects are the Σ-components
  - The morphisms are the $\sim$-equivalence classes
- If $\partial^{-}\gamma \sim \partial^{+}\delta$ then

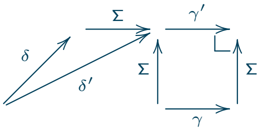
The category of components $\mathcal{C}/\Sigma$

- The quotient category $\mathcal{C}/\Sigma$ (obtained by turning each morphism of Σ into an identity) can be defined as follows:
  - The objects are the Σ-components
  - The morphisms are the $\sim$-equivalence classes
- If $\partial^- \gamma \sim \partial^+ \delta$ then
  - there exists γ' and δ' such that $\gamma' \sim \gamma$, $\delta' \sim \delta$, and $\partial^- \gamma' = \partial^+ \delta'$



The category of components $\mathcal{C}/\Sigma$

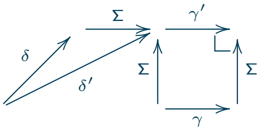
- The quotient category $\mathcal{C}/\Sigma$ (obtained by turning each morphism of Σ into an identity) can be defined as follows:
  - The objects are the Σ-components
  - The morphisms are the $\sim$-equivalence classes
- If $\partial^- \gamma \sim \partial^+ \delta$ then
  - there exists γ' and δ' such that $\gamma' \sim \gamma$, $\delta' \sim \delta$, and $\partial^- \gamma' = \partial^+ \delta'$



  - so we define $[\gamma] \circ [\delta] = [\gamma' \circ \delta']$

The category of components $\mathcal{C}/\Sigma$

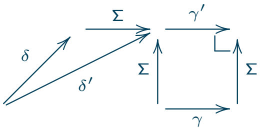
- The quotient category $\mathcal{C}/\Sigma$ (obtained by turning each morphism of Σ into an identity) can be defined as follows:
  - The objects are the Σ-components
  - The morphisms are the $\sim$-equivalence classes
- If $\partial^- \gamma \sim \partial^+ \delta$ then
  - there exists γ' and δ' such that $\gamma' \sim \gamma$, $\delta' \sim \delta$, and $\partial^- \gamma' = \partial^+ \delta'$



  - so we define $[\gamma] \circ [\delta] = [\gamma' \circ \delta']$
- We have the quotient functor $Q : \mathcal{C} \to \mathcal{C}/\Sigma$

The category of components $\mathcal{C}/\Sigma$

- The quotient category $\mathcal{C}/\Sigma$ (obtained by turning each morphism of Σ into an identity) can be defined as follows:
  - The objects are the Σ-components
  - The morphisms are the $\sim$-equivalence classes
- If $\partial^{-}\gamma \sim \partial^{+}\delta$ then
  - there exists γ' and δ' such that $\gamma' \sim \gamma$, $\delta' \sim \delta$, and $\partial^{-}\gamma' = \partial^{+}\delta'$



  - so we define $[\gamma] \circ [\delta] = [\gamma' \circ \delta']$
- We have the quotient functor $Q : \mathcal{C} \to \mathcal{C}/\Sigma$
- The category of components is $\mathcal{C}/\Sigma$ with Σ being the greatest swi of $\mathcal{C}$

Properties

Characterizing the identities of $\mathcal{C}/\Sigma$

Characterizing the identities of $\mathcal{C}/\Sigma$

For any morphism δ of $\mathcal{C}$ t.f.a.e.

- $\delta \in \Sigma$

Characterizing the identities of $\mathcal{C}/\Sigma$

For any morphism δ of $\mathcal{C}$ t.f.a.e.

- $\delta \in \Sigma$
- $[\delta] \subseteq \Sigma$

Characterizing the identities of $\mathcal{C}/\Sigma$

For any morphism δ of $\mathcal{C}$ t.f.a.e.

- $\delta \in \Sigma$
- $[\delta] \subseteq \Sigma$
- $[\delta]$ is an identity of $\mathcal{C}/\Sigma$

Characterizing the identities of $\mathcal{C}/\Sigma$

For any morphism δ of $\mathcal{C}$ t.f.a.e.

- $\delta \in \Sigma$
- $[\delta] \subseteq \Sigma$
- $[\delta]$ is an identity of $\mathcal{C}/\Sigma$

The quotient functor $Q : \mathcal{C} \to \mathcal{C}/\Sigma$ satisfies the following universal property:

Characterizing the identities of $\mathcal{C}/\Sigma$

For any morphism δ of $\mathcal{C}$ t.f.a.e.

- $\delta \in \Sigma$
- $[\delta] \subseteq \Sigma$
- $[\delta]$ is an identity of $\mathcal{C}/\Sigma$

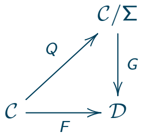
The quotient functor $Q : \mathcal{C} \to \mathcal{C}/\Sigma$ satisfies the following universal property:
for all functors $F : \mathcal{C} \to \mathcal{D}$ s.t. $F(\Sigma) \subseteq \{\text{identities of } \mathcal{D}\}$

Characterizing the identities of $\mathcal{C}/\Sigma$

For any morphism δ of $\mathcal{C}$ t.f.a.e.

- $\delta \in \Sigma$
- $[\delta] \subseteq \Sigma$
- $[\delta]$ is an identity of $\mathcal{C}/\Sigma$

The quotient functor $Q : \mathcal{C} \to \mathcal{C}/\Sigma$ satisfies the following universal property:
for all functors $F : \mathcal{C} \to \mathcal{D}$ s.t. $F(\Sigma) \subseteq \{\text{identities of } \mathcal{D}\}$
there exists a unique $G : \mathcal{C}/\Sigma \to \mathcal{D}$ s.t. $F = G \circ Q$



The fundamental properties of $\mathcal{C}/\Sigma$

with Σ being a system of weak isomorphisms of a one-way category $\mathcal{C}$

The fundamental properties of $\mathcal{C}/\Sigma$

with Σ being a system of weak isomorphisms of a one-way category $\mathcal{C}$

- The quotient functor $Q : \mathcal{C} \to \mathcal{C}/\Sigma$ is surjective on morphisms

The fundamental properties of $\mathcal{C}/\Sigma$

with Σ being a system of weak isomorphisms of a one-way category $\mathcal{C}$

- The quotient functor $Q : \mathcal{C} \to \mathcal{C}/\Sigma$ is surjective on morphisms
- The quotient category $\mathcal{C}/\Sigma$ is loop-free

The fundamental properties of $\mathcal{C}/\Sigma$

with Σ being a system of weak isomorphisms of a one-way category $\mathcal{C}$

- The quotient functor $Q : \mathcal{C} \to \mathcal{C}/\Sigma$ is surjective on morphisms
- The quotient category $\mathcal{C}/\Sigma$ is loop-free
- If $\mathcal{C}(x, y) \neq \emptyset$ then the following map is a bijection.

$$\delta \in \mathcal{C}(x, y) \ \mapsto \ Q(\delta) \in \mathcal{C}/\Sigma\big(Q(x), Q(y)\big)$$

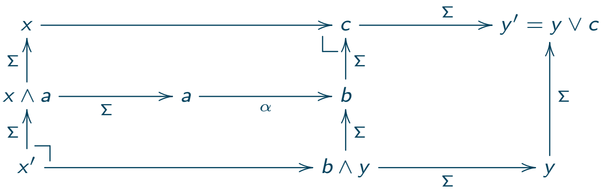
The fundamental properties of $\mathcal{C}/\Sigma$

with Σ being a system of weak isomorphisms of a one-way category $\mathcal{C}$

- The quotient functor $Q : \mathcal{C} \to \mathcal{C}/\Sigma$ is surjective on morphisms
- The quotient category $\mathcal{C}/\Sigma$ is loop-free
- If $\mathcal{C}(x, y) \neq \emptyset$ then the following map is a bijection.

$$\delta \in \mathcal{C}(x, y) \;\mapsto\; Q(\delta) \in \mathcal{C}/\Sigma\big(Q(x), Q(y)\big)$$

- If $\mathcal{C}/\Sigma\big(Q(x), Q(y)\big) \neq \emptyset$ then there exist x' and y' such that $\Sigma(x', x)$, $\Sigma(y, y')$, $\mathcal{C}(x', y)$, and $\mathcal{C}(x, y')$ are nonempty.



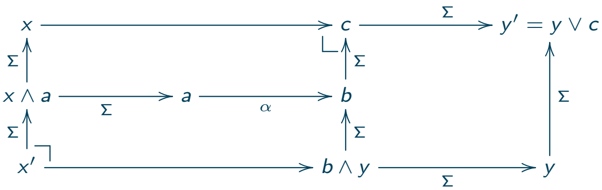
The fundamental properties of $\mathcal{C}/\Sigma$

with Σ being a system of weak isomorphisms of a one-way category $\mathcal{C}$

- The quotient functor $Q : \mathcal{C} \to \mathcal{C}/\Sigma$ is surjective on morphisms
- The quotient category $\mathcal{C}/\Sigma$ is loop-free
- If $\mathcal{C}(x, y) \neq \emptyset$ then the following map is a bijection.

$$\delta \in \mathcal{C}(x, y) \;\mapsto\; Q(\delta) \in \mathcal{C}/\Sigma\big(Q(x), Q(y)\big)$$

- If $\mathcal{C}/\Sigma\big(Q(x), Q(y)\big) \neq \emptyset$ then there exist x' and y' such that $\Sigma(x', x)$, $\Sigma(y, y')$, $\mathcal{C}(x', y)$, and $\mathcal{C}(x, y')$ are nonempty.



- The quotient functor Q preserves and reflects potential weak isomorphisms

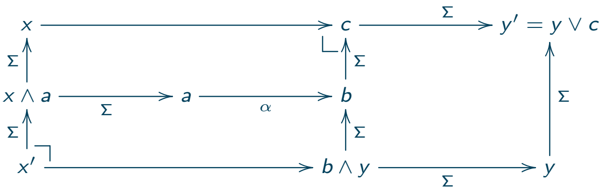
The fundamental properties of $\mathcal{C}/\Sigma$

with Σ being a system of weak isomorphisms of a one-way category $\mathcal{C}$

- The quotient functor $Q : \mathcal{C} \to \mathcal{C}/\Sigma$ is surjective on morphisms
- The quotient category $\mathcal{C}/\Sigma$ is loop-free
- If $\mathcal{C}(x, y) \neq \emptyset$ then the following map is a bijection.

$$\delta \in \mathcal{C}(x, y) \;\mapsto\; Q(\delta) \in \mathcal{C}/\Sigma\big(Q(x), Q(y)\big)$$

- If $\mathcal{C}/\Sigma\big(Q(x), Q(y)\big) \neq \emptyset$ then there exist x' and y' such that $\Sigma(x', x)$, $\Sigma(y, y')$, $\mathcal{C}(x', y)$, and $\mathcal{C}(x, y')$ are nonempty.



- The quotient functor Q preserves and reflects potential weak isomorphisms
- If $\mathcal{C}$ is finite then so is the quotient $\mathcal{C}/\Sigma$

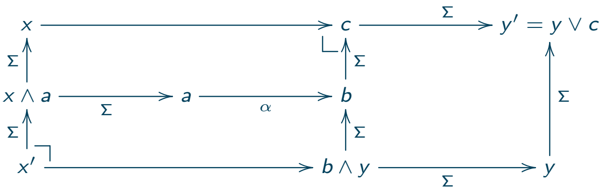
The fundamental properties of $\mathcal{C}/\Sigma$

with Σ being a system of weak isomorphisms of a one-way category $\mathcal{C}$

- The quotient functor $Q : \mathcal{C} \to \mathcal{C}/\Sigma$ is surjective on morphisms
- The quotient category $\mathcal{C}/\Sigma$ is loop-free
- If $\mathcal{C}(x, y) \neq \emptyset$ then the following map is a bijection.

$$\delta \in \mathcal{C}(x, y) \;\mapsto\; Q(\delta) \in \mathcal{C}/\Sigma\big(Q(x), Q(y)\big)$$

- If $\mathcal{C}/\Sigma\big(Q(x), Q(y)\big) \neq \emptyset$ then there exist x' and y' such that $\Sigma(x', x)$, $\Sigma(y, y')$, $\mathcal{C}(x', y)$, and $\mathcal{C}(x, y')$ are nonempty.



- The quotient functor Q preserves and reflects potential weak isomorphisms
- If $\mathcal{C}$ is finite then so is the quotient $\mathcal{C}/\Sigma$
- $\mathcal{C}$ is a preorder iff $\mathcal{C}/\Sigma$ is a poset

Describing the localization of $\mathcal{C}$ by Σ

with Σ being a system of weak isomorphisms of a one-way category $\mathcal{C}$

Describing the localization of $\mathcal{C}$ by Σ

with Σ being a system of weak isomorphisms of a one-way category $\mathcal{C}$

- The objects of $\mathcal{C}[\Sigma^{-1}]$ are the objects of $\mathcal{C}$

Describing the localization of $\mathcal{C}$ by Σ

with Σ being a system of weak isomorphisms of a one-way category $\mathcal{C}$

- The objects of $\mathcal{C}[\Sigma^{-1}]$ are the objects of $\mathcal{C}$
- The morphisms are the equivalence classes of ordered pairs of coinitial morphisms (γ, σ) with $\sigma \in \Sigma$,

Describing the localization of $\mathcal{C}$ by Σ

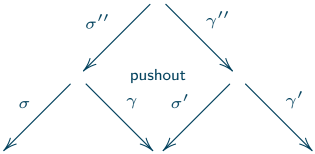
with Σ being a system of weak isomorphisms of a one-way category $\mathcal{C}$

- The objects of $\mathcal{C}[\Sigma^{-1}]$ are the objects of $\mathcal{C}$
- The morphisms are the equivalence classes of ordered pairs of coinitial morphisms (γ, σ) with $\sigma \in \Sigma$,
  - Two pairs (γ, σ) and (γ', σ') being equivalent when $\partial^{-}\sigma = \partial^{-}\sigma'$, $\partial^{-}\gamma = \partial^{-}\gamma'$, and $Q(\gamma) = Q(\gamma')$

Describing the localization of $\mathcal{C}$ by Σ

with Σ being a system of weak isomorphisms of a one-way category $\mathcal{C}$

- The objects of $\mathcal{C}[\Sigma^{-1}]$ are the objects of $\mathcal{C}$
- The morphisms are the equivalence classes of ordered pairs of coinitial morphisms (γ, σ) with $\sigma \in \Sigma$,
  - Two pairs (γ, σ) and (γ', σ') being equivalent when $\partial^{-}\sigma = \partial^{-}\sigma'$, $\partial^{-}\gamma = \partial^{-}\gamma'$, and $Q(\gamma) = Q(\gamma')$
  - In the diagram below we have $Q(\gamma' \circ \gamma'') = Q(\gamma') \circ Q(\gamma'') = Q(\gamma') \circ Q(\gamma)$ hence the composite $(\gamma' \circ \gamma'', \sigma \circ \sigma'')$ neither depend on the choice of the pushout nor on the representatives (γ, σ) and (γ', σ').



The canonical comparison $P : \mathcal{C}[\Sigma^{-1}] \to \mathcal{C}/\Sigma$

with Σ being a system of weak isomorphisms of a one-way category $\mathcal{C}$

The canonical comparison $P : \mathcal{C}[\Sigma^{-1}] \to \mathcal{C}/\Sigma$

with Σ being a system of weak isomorphisms of a one-way category $\mathcal{C}$

- Define I by $I(\gamma) := (\gamma, \mathrm{id}_{\partial^- \gamma})$ and the identity on objects

The canonical comparison $P : \mathcal{C}[\Sigma^{-1}] \to \mathcal{C}/\Sigma$

with Σ being a system of weak isomorphisms of a one-way category $\mathcal{C}$

- Define I by $I(\gamma) := (\gamma, \mathrm{id}_{\partial^- \gamma})$ and the identity on objects
- Given a functor $F : \mathcal{C} \to \mathcal{D}$ s.t. $F(\Sigma) \subseteq \{\text{isomorphisms of } \mathcal{D}\}$ define

The canonical comparison $P : \mathcal{C}[\Sigma^{-1}] \to \mathcal{C}/\Sigma$

with Σ being a system of weak isomorphisms of a one-way category $\mathcal{C}$

- Define I by $I(\gamma) := (\gamma, \mathrm{id}_{\partial^- \gamma})$ and the identity on objects
- Given a functor $F : \mathcal{C} \to \mathcal{D}$ s.t. $F(\Sigma) \subseteq \{\text{isomorphisms of } \mathcal{D}\}$ define
  - $G(x) := F(x)$ for all objects x of $\mathcal{C}[\Sigma^{-1}]$ and

The canonical comparison $P : \mathcal{C}[\Sigma^{-1}] \to \mathcal{C}/\Sigma$

with Σ being a system of weak isomorphisms of a one-way category $\mathcal{C}$

- Define I by $I(\gamma) := (\gamma, \mathrm{id}_{\partial^- \gamma})$ and the identity on objects
- Given a functor $F : \mathcal{C} \to \mathcal{D}$ s.t. $F(\Sigma) \subseteq \{\text{isomorphisms of } \mathcal{D}\}$ define
  - $G(x) := F(x)$ for all objects x of $\mathcal{C}[\Sigma^{-1}]$ and
  - $G(\gamma, \sigma) := F(\gamma) \circ (F(\sigma))^{-1}$ for any representative (γ, σ) of a morphism of $\mathcal{C}[\Sigma^{-1}]$

The canonical comparison $P : \mathcal{C}[\Sigma^{-1}] \to \mathcal{C}/\Sigma$

with Σ being a system of weak isomorphisms of a one-way category $\mathcal{C}$

- Define I by $I(\gamma) := (\gamma, \mathrm{id}_{\partial^- \gamma})$ and the identity on objects
- Given a functor $F : \mathcal{C} \to \mathcal{D}$ s.t. $F(\Sigma) \subseteq \{\text{isomorphisms of } \mathcal{D}\}$ define
  - $G(x) := F(x)$ for all objects x of $\mathcal{C}[\Sigma^{-1}]$ and
  - $G(\gamma, \sigma) := F(\gamma) \circ (F(\sigma))^{-1}$ for any representative (γ, σ) of a morphism of $\mathcal{C}[\Sigma^{-1}]$
- The functor $I : \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ then satisfies the universal property: for all functors $F : \mathcal{C} \to \mathcal{D}$ there exists a unique $G : \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ s.t. $F = G \circ I$

The canonical comparison $P : \mathcal{C}[\Sigma^{-1}] \to \mathcal{C}/\Sigma$

with Σ being a system of weak isomorphisms of a one-way category $\mathcal{C}$

- Define I by $I(\gamma) := (\gamma, \mathrm{id}_{\partial^- \gamma})$ and the identity on objects
- Given a functor $F : \mathcal{C} \to \mathcal{D}$ s.t. $F(\Sigma) \subseteq \{\text{isomorphisms of } \mathcal{D}\}$ define
  - $G(x) := F(x)$ for all objects x of $\mathcal{C}[\Sigma^{-1}]$ and
  - $G(\gamma, \sigma) := F(\gamma) \circ (F(\sigma))^{-1}$ for any representative (γ, σ) of a morphism of $\mathcal{C}[\Sigma^{-1}]$
- The functor $I : \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ then satisfies the universal property: for all functors $F : \mathcal{C} \to \mathcal{D}$ there exists a unique $G : \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ s.t. $F = G \circ I$
- In particular there is a unique functor P s.t. $Q = P \circ I$ with $Q : \mathcal{C} \to \mathcal{C}/\Sigma$ and we have

> The functor P is an equivalence of categories

The canonical comparison $P : \mathcal{C}[\Sigma^{-1}] \to \mathcal{C}/\Sigma$

with Σ being a system of weak isomorphisms of a one-way category $\mathcal{C}$

- Define I by $I(\gamma) := (\gamma, \mathrm{id}_{\partial^- \gamma})$ and the identity on objects
- Given a functor $F : \mathcal{C} \to \mathcal{D}$ s.t. $F(\Sigma) \subseteq \{\text{isomorphisms of } \mathcal{D}\}$ define
  - $G(x) := F(x)$ for all objects x of $\mathcal{C}[\Sigma^{-1}]$ and
  - $G(\gamma, \sigma) := F(\gamma) \circ (F(\sigma))^{-1}$ for any representative (γ, σ) of a morphism of $\mathcal{C}[\Sigma^{-1}]$
- The functor $I : \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ then satisfies the universal property: for all functors $F : \mathcal{C} \to \mathcal{D}$ there exists a unique $G : \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ s.t. $F = G \circ I$
- In particular there is a unique functor P s.t. $Q = P \circ I$ with $Q : \mathcal{C} \to \mathcal{C}/\Sigma$ and we have

The functor P is an equivalence of categories

- The skeleton of $\mathcal{C}[\Sigma^{-1}]$ is $\mathcal{C}/\Sigma$ and $\mathcal{C}[\Sigma^{-1}]$ is one-way.

Embeding $\mathcal{C}/\Sigma$ into $\mathcal{C}$

Embeding $\mathcal{C}/\Sigma$ into $\mathcal{C}$

- Let $\phi : \Sigma\text{-components of } \mathcal{C} \to \mathrm{Ob}(\mathcal{C})$ such that

Embeding $\mathcal{C}/\Sigma$ into $\mathcal{C}$

- Let $\phi : \Sigma$-components of $\mathcal{C} \to \mathrm{Ob}(\mathcal{C})$ such that
  - for all Σ-components K, K', if there exists $x \in K$ and $x' \in K'$ such that $\mathcal{C}(x, x') \neq \emptyset$, then $\mathcal{C}(\phi(K), \phi(K')) \neq \emptyset$

Embeding $\mathcal{C}/\Sigma$ into $\mathcal{C}$

- Let $\phi : \Sigma$-components of $\mathcal{C} \to \mathrm{Ob}(\mathcal{C})$ such that
  - for all Σ-components K, K', if there exists $x \in K$ and $x' \in K'$ such that $\mathcal{C}(x, x') \neq \emptyset$, then $\mathcal{C}(\phi(K), \phi(K')) \neq \emptyset$
  - in this case $\mathcal{C}/\Sigma$ is isomorphic with the full subcategory of $\mathcal{C}$ whose set of objects is $\mathrm{im}(\phi)$.

Embeding $\mathcal{C}/\Sigma$ into $\mathcal{C}$

- Let ϕ : Σ-components of $\mathcal{C} \to \mathrm{Ob}(\mathcal{C})$ such that
  - for all Σ-components K, K', if there exists $x \in K$ and $x' \in K'$ such that $\mathcal{C}(x, x') \neq \emptyset$, then $\mathcal{C}(\phi(K), \phi(K')) \neq \emptyset$
  - in this case $\mathcal{C}/\Sigma$ is isomorphic with the full subcategory of $\mathcal{C}$ whose set of objects is $\mathrm{im}(\phi)$.
  - the mapping ϕ is called an admissible choice (of canonical objects)

Embeding $\mathcal{C}/\Sigma$ into $\mathcal{C}$

- Let $\phi : \Sigma$-components of $\mathcal{C} \to \mathrm{Ob}(\mathcal{C})$ such that
  - for all Σ-components K, K', if there exists $x \in K$ and $x' \in K'$ such that $\mathcal{C}(x, x') \neq \emptyset$, then $\mathcal{C}(\phi(K), \phi(K')) \neq \emptyset$
  - in this case $\mathcal{C}/\Sigma$ is isomorphic with the full subcategory of $\mathcal{C}$ whose set of objects is $\mathrm{im}(\phi)$.
  - the mapping ϕ is called an admissible choice (of canonical objects)
- Write $\phi \preccurlyeq \phi'$ when $\mathcal{C}(\phi(K), \phi'(K)) \neq \emptyset$ for all Σ-components K

Embeding $\mathcal{C}/\Sigma$ into $\mathcal{C}$

- Let $\phi : \Sigma$-components of $\mathcal{C} \to \mathrm{Ob}(\mathcal{C})$ such that
  - for all Σ-components K, K', if there exists $x \in K$ and $x' \in K'$ such that $\mathcal{C}(x, x') \neq \emptyset$, then $\mathcal{C}(\phi(K), \phi(K')) \neq \emptyset$
  - in this case $\mathcal{C}/\Sigma$ is isomorphic with the full subcategory of $\mathcal{C}$ whose set of objects is $\mathrm{im}(\phi)$.
  - the mapping ϕ is called an admissible choice (of canonical objects)
- Write $\phi \preccurlyeq \phi'$ when $\mathcal{C}(\phi(K), \phi'(K)) \neq \emptyset$ for all Σ-components K
  - The collection of admissible choice then forms a (pre)lattice

Embeding $\mathcal{C}/\Sigma$ into $\mathcal{C}$

- Let ϕ : Σ-components of $\mathcal{C} \to \mathrm{Ob}(\mathcal{C})$ such that
  - for all Σ-components K, K', if there exists $x \in K$ and $x' \in K'$ such that $\mathcal{C}(x, x') \neq \emptyset$, then $\mathcal{C}(\phi(K), \phi(K')) \neq \emptyset$
  - in this case $\mathcal{C}/\Sigma$ is isomorphic with the full subcategory of $\mathcal{C}$ whose set of objects is $\mathrm{im}(\phi)$.
  - the mapping ϕ is called an admissible choice (of canonical objects)
- Write $\phi \preccurlyeq \phi'$ when $\mathcal{C}(\phi(K), \phi'(K)) \neq \emptyset$ for all Σ-components K
  - The collection of admissible choice then forms a (pre)lattice
  - If $\mathcal{C}/\Sigma$ is finite then there exists an admissible choice

Embeding $\mathcal{C}/\Sigma$ into $\mathcal{C}$

- Let ϕ : Σ-components of $\mathcal{C} \to \mathrm{Ob}(\mathcal{C})$ such that
  - for all Σ-components K, K', if there exists $x \in K$ and $x' \in K'$ such that $\mathcal{C}(x, x') \neq \emptyset$, then $\mathcal{C}(\phi(K), \phi(K')) \neq \emptyset$
  - in this case $\mathcal{C}/\Sigma$ is isomorphic with the full subcategory of $\mathcal{C}$ whose set of objects is $\mathrm{im}(\phi)$.
  - the mapping ϕ is called an admissible choice (of canonical objects)
- Write $\phi \preccurlyeq \phi'$ when $\mathcal{C}(\phi(K), \phi'(K)) \neq \emptyset$ for all Σ-components K
  - The collection of admissible choice then forms a (pre)lattice
  - If $\mathcal{C}/\Sigma$ is finite then there exists an admissible choice
  - If $\mathcal{C}/\Sigma$ is infinite the existence of an admissible choice is a open question

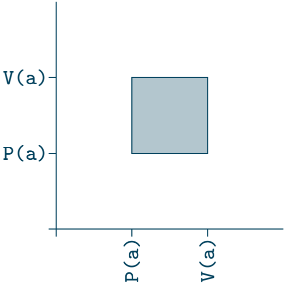
Examples

Plane without a square

$x = \mathbb{R}_+^2 \setminus]0,1[^2$

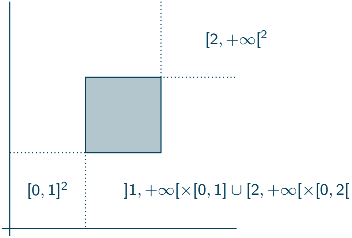
Plane without a square

$x = \mathbb{R}_+^2 \setminus]0,1[^2$



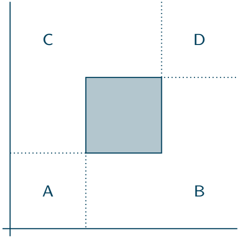
Plane without a square

$x = \mathbb{R}_+^2 \setminus]0, 1[^2$



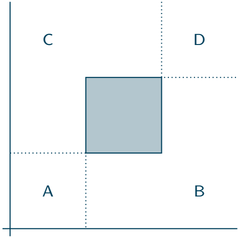
Plane without a square

$x = \mathbb{R}_+^2 \setminus]0,1[^2$



Plane without a square

$x = \mathbb{R}_+^2 \setminus]0,1[^2$

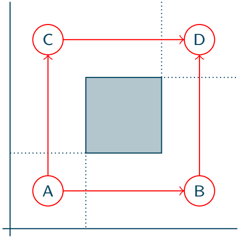


Let x, y such that $x \leqslant^2 y$, then $\overrightarrow{\pi_1}X(x,y)$ only depends on which elements of the partition x and y belong to

$\to$	A	B	C	D
A	σ	β	γ	$\beta' \circ \beta$ $\alpha' \circ \alpha$
B		σ		β'
C			σ	γ'
D				σ

Plane without a square

$x = \mathbb{R}^2_+ \setminus]0,1[^2$

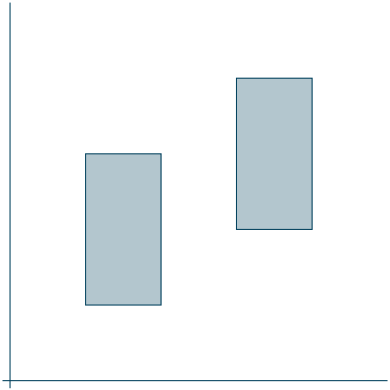


Let x, y such that $x \leqslant^2 y$, then $\overrightarrow{\pi_1}X(x,y)$ only depends on which elements of the partition x and y belong to

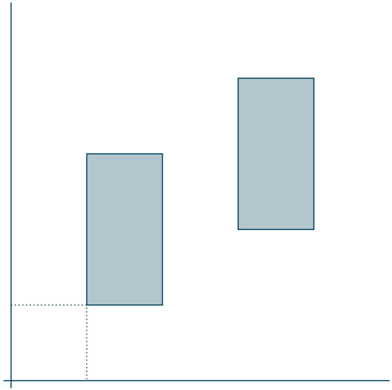
$\rightarrow$	A	B	C	D
A	σ	β	γ	$\beta' \circ \beta$ $\alpha' \circ \alpha$
B		σ		β'
C			σ	γ'
D				σ

Two rectangles

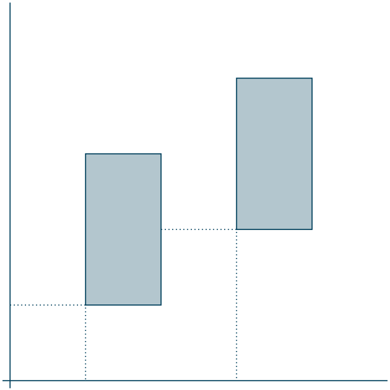
Two rectangles



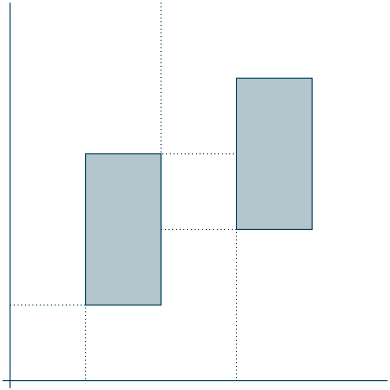
Two rectangles



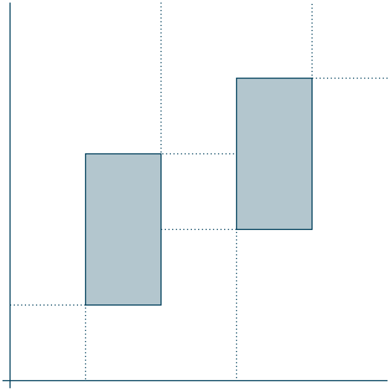
Two rectangles



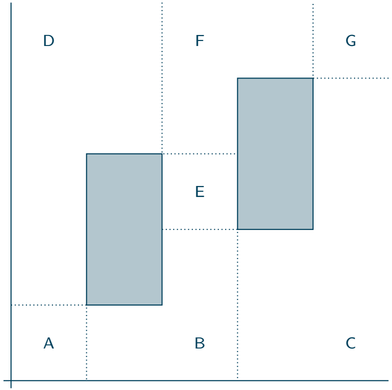
Two rectangles



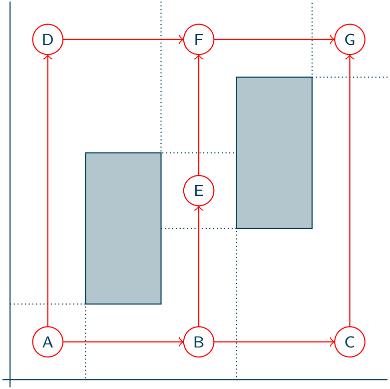
Two rectangles



Two rectangles

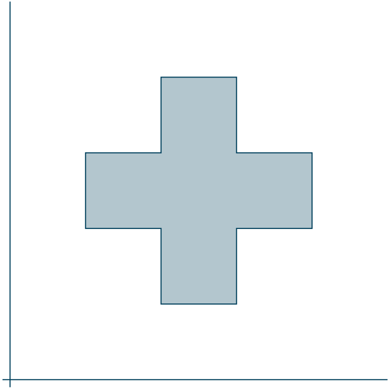


Two rectangles

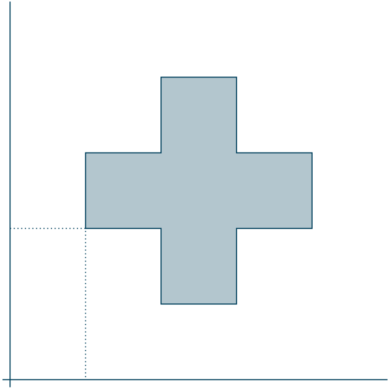


Swiss Flag

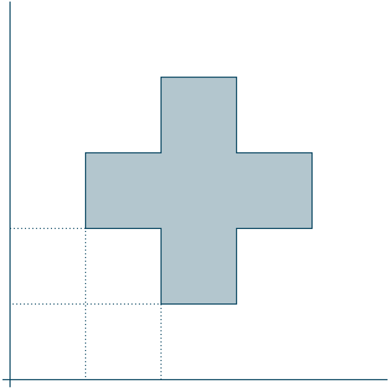
Swiss Flag



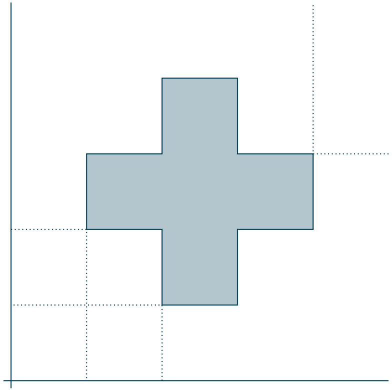
Swiss Flag



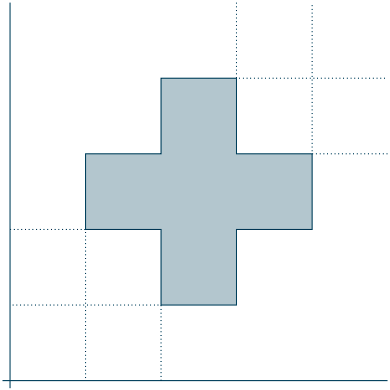
Swiss Flag



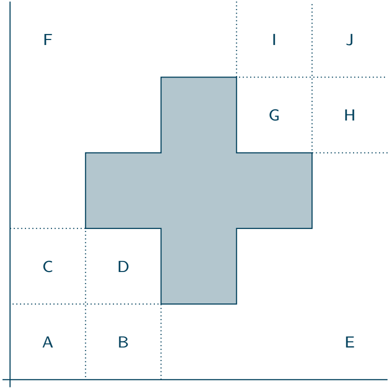
Swiss Flag



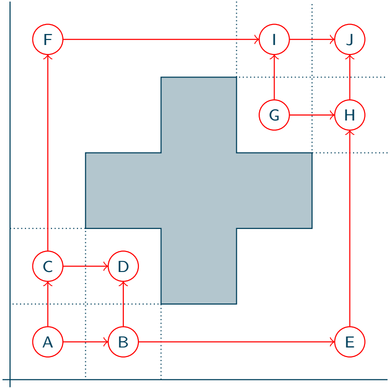
Swiss Flag



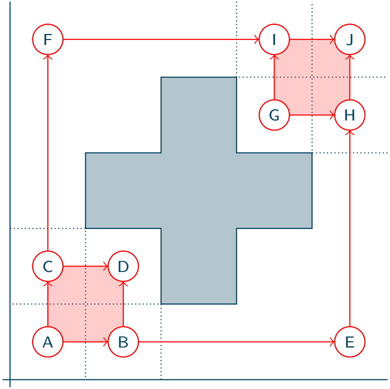
Swiss Flag



Swiss Flag

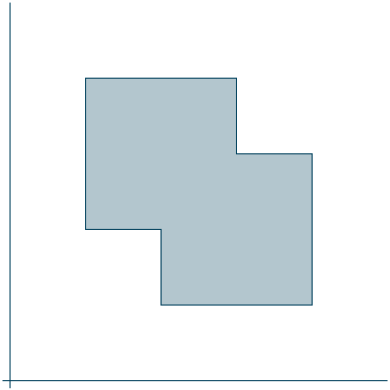


Swiss Flag

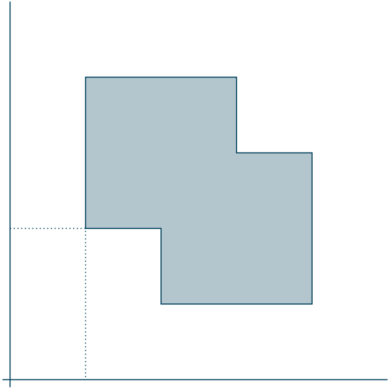


Achronal overlaping square

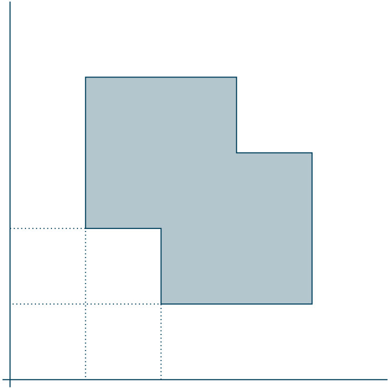
Achronal overlaping square



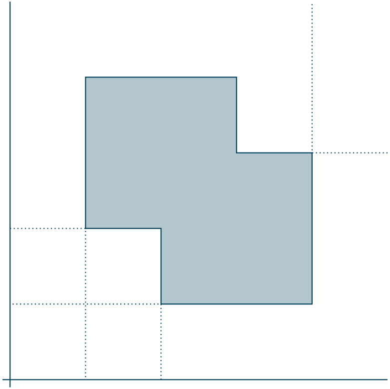
Achronal overlaping square



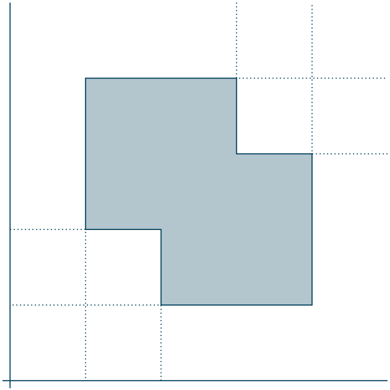
Achronal overlaping square



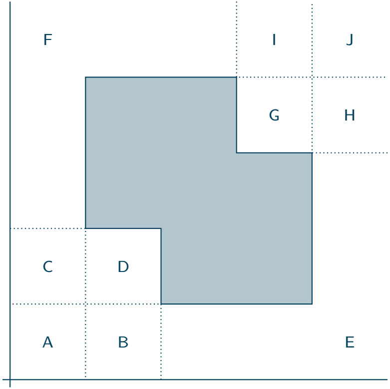
Achronal overlaping square



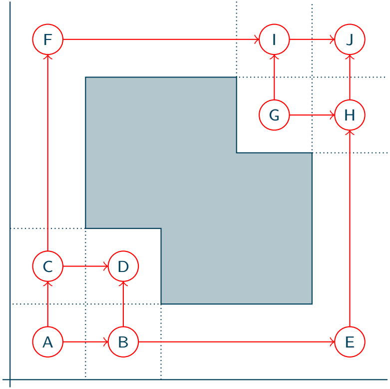
Achronal overlaping square



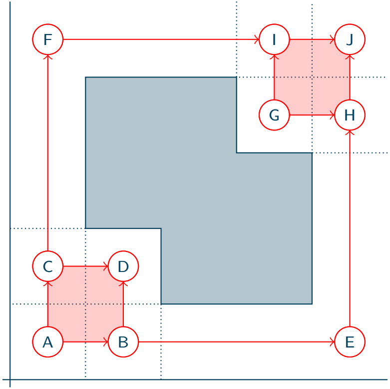
Achronal overlaping square



Achronal overlaping square

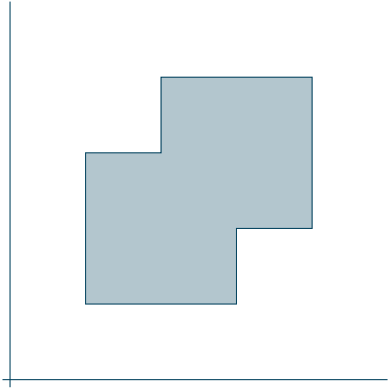


Achronal overlaping square

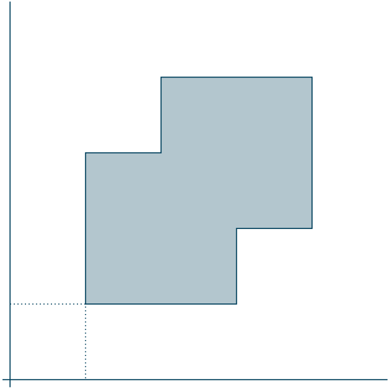


Diagonal overlaping squares

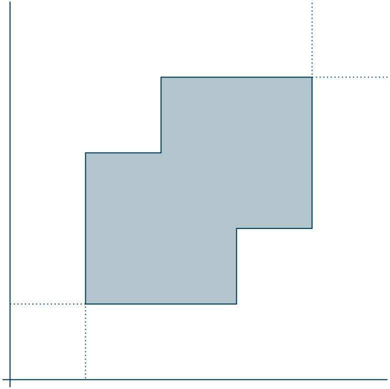
Diagonal overlaping squares



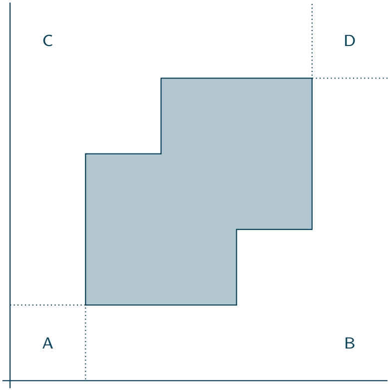
Diagonal overlaping squares



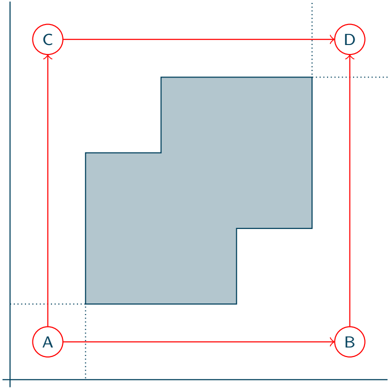
Diagonal overlaping squares



Diagonal overlaping squares



Diagonal overlaping squares

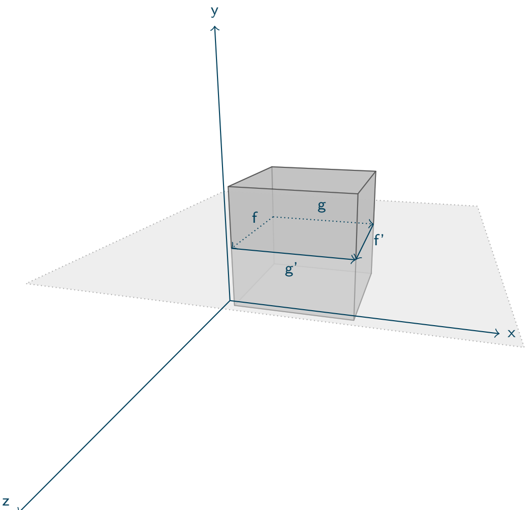


The floating cube

Non potential weak isomorphisms

The floating cube

Non potential weak isomorphisms

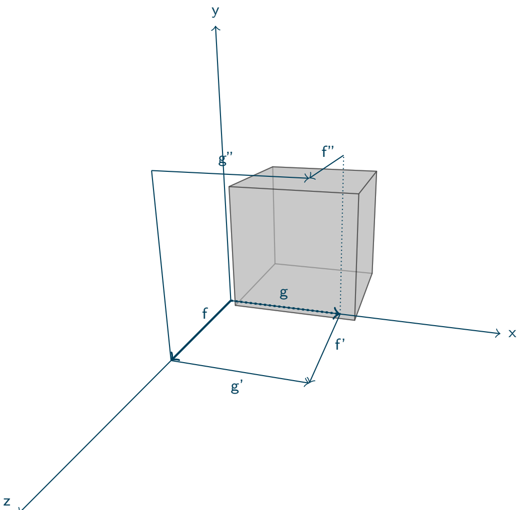


The floating cube

A "vee" that does not extend to a pushout

The floating cube

A “vee” that does not extend to a pushout

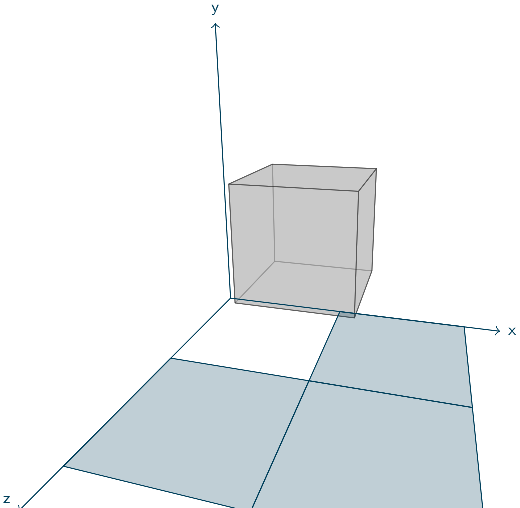


The floating cube

Some pushouts squares

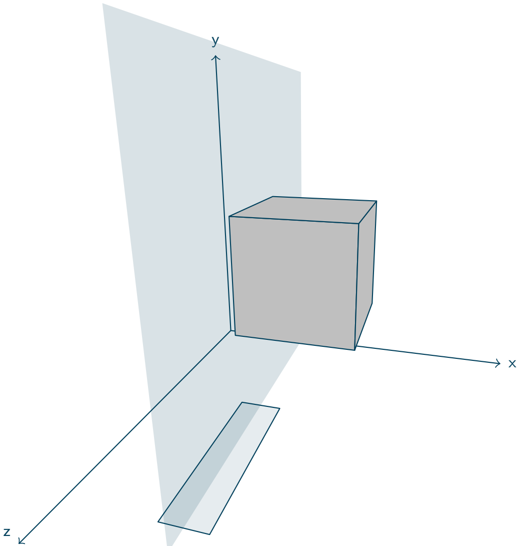
The floating cube

Some pushouts squares



The floating cube

The floating cube



The floating cube

The floating cube

- Since the pushout of f (resp. g) along g (resp. f) does not exist, $f, g \notin \Sigma$

The floating cube

- Since the pushout of f (resp. g) along g (resp. f) does not exist, $f, g \notin \Sigma$
- The commutative square f, g, f', and g' is a pullback:

The floating cube

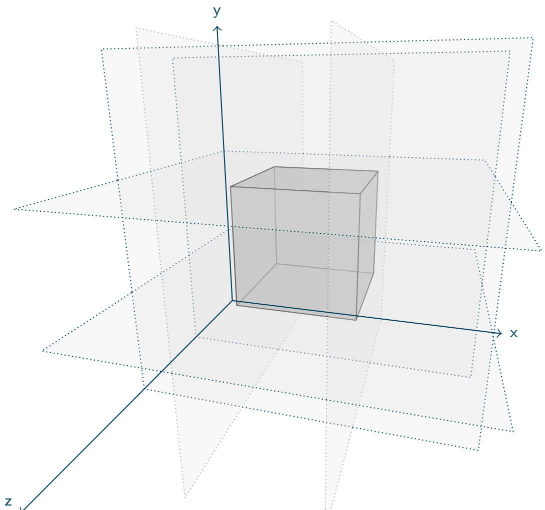
- Since the pushout of f (resp. g) along g (resp. f) does not exist, $f, g \notin \Sigma$
- The commutative square f, g, f', and g' is a pullback:
  - Therefore $f', g' \notin \Sigma$ (anyway they do not preserve the future cones)

The floating cube

boundaries of the components

The floating cube

boundaries of the components



Finite connected loop-free categories

Commutative monoid

of nonempty finite connected loop-free categories

Commutative monoid

of nonempty finite connected loop-free categories

- The Cartesian product of categories $\mathcal{A} \times \mathcal{B}$ is non-empty iff so are $\mathcal{A}$ and $\mathcal{B}$.
  If $\mathcal{A}$ and $\mathcal{B}$ are indeed nonempty then we also have

Commutative monoid

of nonempty finite connected loop-free categories

- The Cartesian product of categories $\mathcal{A} \times \mathcal{B}$ is non-empty iff so are $\mathcal{A}$ and $\mathcal{B}$.
  If $\mathcal{A}$ and $\mathcal{B}$ are indeed nonempty then we also have
  - $\mathcal{A} \times \mathcal{B}$ finite iff so are $\mathcal{A}$ and $\mathcal{B}$

Commutative monoid

of nonempty finite connected loop-free categories

- The Cartesian product of categories $\mathcal{A} \times \mathcal{B}$ is non-empty iff so are $\mathcal{A}$ and $\mathcal{B}$.

  If $\mathcal{A}$ and $\mathcal{B}$ are indeed nonempty then we also have
  - $\mathcal{A} \times \mathcal{B}$ finite iff so are $\mathcal{A}$ and $\mathcal{B}$
  - $\mathcal{A} \times \mathcal{B}$ connected iff so are $\mathcal{A}$ and $\mathcal{B}$

Commutative monoid

of nonempty finite connected loop-free categories

- The Cartesian product of categories $\mathcal{A} \times \mathcal{B}$ is non-empty iff so are $\mathcal{A}$ and $\mathcal{B}$.

  If $\mathcal{A}$ and $\mathcal{B}$ are indeed nonempty then we also have
  - $\mathcal{A} \times \mathcal{B}$ finite iff so are $\mathcal{A}$ and $\mathcal{B}$
  - $\mathcal{A} \times \mathcal{B}$ connected iff so are $\mathcal{A}$ and $\mathcal{B}$
  - $\mathcal{A} \times \mathcal{B}$ loop-free iff so are $\mathcal{A}$ and $\mathcal{B}$

Commutative monoid

of nonempty finite connected loop-free categories

- The Cartesian product of categories $\mathcal{A} \times \mathcal{B}$ is non-empty iff so are $\mathcal{A}$ and $\mathcal{B}$.

  If $\mathcal{A}$ and $\mathcal{B}$ are indeed nonempty then we also have
  - $\mathcal{A} \times \mathcal{B}$ finite iff so are $\mathcal{A}$ and $\mathcal{B}$
  - $\mathcal{A} \times \mathcal{B}$ connected iff so are $\mathcal{A}$ and $\mathcal{B}$
  - $\mathcal{A} \times \mathcal{B}$ loop-free iff so are $\mathcal{A}$ and $\mathcal{B}$
- $\mathcal{A} \cong \mathcal{A}'$ and $\mathcal{B} \cong \mathcal{B}'$ implies $\mathcal{A} \times \mathcal{A}' \cong \mathcal{B} \times \mathcal{B}'$

Commutative monoid

of nonempty finite connected loop-free categories

- The Cartesian product of categories $\mathcal{A} \times \mathcal{B}$ is non-empty iff so are $\mathcal{A}$ and $\mathcal{B}$.
  If $\mathcal{A}$ and $\mathcal{B}$ are indeed nonempty then we also have
  - $\mathcal{A} \times \mathcal{B}$ finite iff so are $\mathcal{A}$ and $\mathcal{B}$
  - $\mathcal{A} \times \mathcal{B}$ connected iff so are $\mathcal{A}$ and $\mathcal{B}$
  - $\mathcal{A} \times \mathcal{B}$ loop-free iff so are $\mathcal{A}$ and $\mathcal{B}$
- $\mathcal{A} \cong \mathcal{A}'$ and $\mathcal{B} \cong \mathcal{B}'$ implies $\mathcal{A} \times \mathcal{A}' \cong \mathcal{B} \times \mathcal{B}'$
- $(\mathcal{A} \times \mathcal{B}) \times \mathcal{C} \cong \mathcal{A} \times (\mathcal{B} \times \mathcal{C})$

Commutative monoid

of nonempty finite connected loop-free categories

- The Cartesian product of categories $\mathcal{A} \times \mathcal{B}$ is non-empty iff so are $\mathcal{A}$ and $\mathcal{B}$.
  If $\mathcal{A}$ and $\mathcal{B}$ are indeed nonempty then we also have
  - $\mathcal{A} \times \mathcal{B}$ finite iff so are $\mathcal{A}$ and $\mathcal{B}$
  - $\mathcal{A} \times \mathcal{B}$ connected iff so are $\mathcal{A}$ and $\mathcal{B}$
  - $\mathcal{A} \times \mathcal{B}$ loop-free iff so are $\mathcal{A}$ and $\mathcal{B}$
- $\mathcal{A} \cong \mathcal{A}'$ and $\mathcal{B} \cong \mathcal{B}'$ implies $\mathcal{A} \times \mathcal{A}' \cong \mathcal{B} \times \mathcal{B}'$
- $(\mathcal{A} \times \mathcal{B}) \times \mathcal{C} \cong \mathcal{A} \times (\mathcal{B} \times \mathcal{C})$
- $1 \times \mathcal{A} \cong \mathcal{A} \cong \mathcal{A} \times 1$

Commutative monoid

of nonempty finite connected loop-free categories

- The Cartesian product of categories $\mathcal{A} \times \mathcal{B}$ is non-empty iff so are $\mathcal{A}$ and $\mathcal{B}$.
  If $\mathcal{A}$ and $\mathcal{B}$ are indeed nonempty then we also have
  - $\mathcal{A} \times \mathcal{B}$ finite iff so are $\mathcal{A}$ and $\mathcal{B}$
  - $\mathcal{A} \times \mathcal{B}$ connected iff so are $\mathcal{A}$ and $\mathcal{B}$
  - $\mathcal{A} \times \mathcal{B}$ loop-free iff so are $\mathcal{A}$ and $\mathcal{B}$
- $\mathcal{A} \cong \mathcal{A}'$ and $\mathcal{B} \cong \mathcal{B}'$ implies $\mathcal{A} \times \mathcal{A}' \cong \mathcal{B} \times \mathcal{B}'$
- $(\mathcal{A} \times \mathcal{B}) \times \mathcal{C} \cong \mathcal{A} \times (\mathcal{B} \times \mathcal{C})$
- $1 \times \mathcal{A} \cong \mathcal{A} \cong \mathcal{A} \times 1$
- $\mathcal{A} \times \mathcal{B} \cong \mathcal{B} \times \mathcal{A}$

Commutative monoid

of nonempty finite connected loop-free categories

- The Cartesian product of categories $\mathcal{A} \times \mathcal{B}$ is non-empty iff so are $\mathcal{A}$ and $\mathcal{B}$.

  If $\mathcal{A}$ and $\mathcal{B}$ are indeed nonempty then we also have
  - $\mathcal{A} \times \mathcal{B}$ finite iff so are $\mathcal{A}$ and $\mathcal{B}$
  - $\mathcal{A} \times \mathcal{B}$ connected iff so are $\mathcal{A}$ and $\mathcal{B}$
  - $\mathcal{A} \times \mathcal{B}$ loop-free iff so are $\mathcal{A}$ and $\mathcal{B}$
- $\mathcal{A} \cong \mathcal{A}'$ and $\mathcal{B} \cong \mathcal{B}'$ implies $\mathcal{A} \times \mathcal{A}' \cong \mathcal{B} \times \mathcal{B}'$
- $(\mathcal{A} \times \mathcal{B}) \times \mathcal{C} \cong \mathcal{A} \times (\mathcal{B} \times \mathcal{C})$
- $1 \times \mathcal{A} \cong \mathcal{A} \cong \mathcal{A} \times 1$
- $\mathcal{A} \times \mathcal{B} \cong \mathcal{B} \times \mathcal{A}$
- The collection of isomorphism classes of nonempty finite connected loop-free categories is thus a commutative monoid $\mathcal{M}$

Commutative monoid

of nonempty finite connected loop-free categories

- The Cartesian product of categories $\mathcal{A} \times \mathcal{B}$ is non-empty iff so are $\mathcal{A}$ and $\mathcal{B}$.
  If $\mathcal{A}$ and $\mathcal{B}$ are indeed nonempty then we also have
  - $\mathcal{A} \times \mathcal{B}$ finite iff so are $\mathcal{A}$ and $\mathcal{B}$
  - $\mathcal{A} \times \mathcal{B}$ connected iff so are $\mathcal{A}$ and $\mathcal{B}$
  - $\mathcal{A} \times \mathcal{B}$ loop-free iff so are $\mathcal{A}$ and $\mathcal{B}$
- $\mathcal{A} \cong \mathcal{A}'$ and $\mathcal{B} \cong \mathcal{B}'$ implies $\mathcal{A} \times \mathcal{A}' \cong \mathcal{B} \times \mathcal{B}'$
- $(\mathcal{A} \times \mathcal{B}) \times \mathcal{C} \cong \mathcal{A} \times (\mathcal{B} \times \mathcal{C})$
- $1 \times \mathcal{A} \cong \mathcal{A} \cong \mathcal{A} \times 1$
- $\mathcal{A} \times \mathcal{B} \cong \mathcal{B} \times \mathcal{A}$
- The collection of isomorphism classes of nonempty finite connected loop-free categories is thus a commutative monoid $\mathcal{M}$

The commutative monoid $\mathcal{M}$ is free.

Criteria for primality

Criteria for primality

- The monoid $\mathcal{M}$ is graded by the following morphisms

Criteria for primality

- The monoid $\mathcal{M}$ is graded by the following morphisms
  - $\#\mathit{Ob} : \mathcal{C} \in \mathcal{M} \mapsto \mathrm{card}(\mathrm{Ob}(\mathcal{C})) \in (\mathbb{N} \setminus \{0\}, \times, 1)$

Criteria for primality

- The monoid $\mathcal{M}$ is graded by the following morphisms
  - $\#Ob : \mathcal{C} \in \mathcal{M} \mapsto \mathrm{card}(\mathrm{Ob}(\mathcal{C})) \in (\mathbb{N} \setminus \{0\}, \times, 1)$
  - $\#Mo : \mathcal{C} \in \mathcal{M} \mapsto \mathrm{card}(\mathrm{Mo}(\mathcal{C})) \in (\mathbb{N} \setminus \{0\}, \times, 1)$

Criteria for primality

- The monoid $\mathcal{M}$ is graded by the following morphisms
  - $\#Ob : \mathcal{C} \in \mathcal{M} \mapsto \text{card}(\text{Ob}(\mathcal{C})) \in (\mathbb{N} \setminus \{0\}, \times, 1)$
  - $\#Mo : \mathcal{C} \in \mathcal{M} \mapsto \text{card}(\text{Mo}(\mathcal{C})) \in (\mathbb{N} \setminus \{0\}, \times, 1)$
  - $\#Mo(\mathcal{C}) \geqslant 2 \times \#Ob(\mathcal{C}) - 1$, for all $\mathcal{C} \in \mathcal{M}$

Criteria for primality

- The monoid $\mathcal{M}$ is graded by the following morphisms
  - $\#Ob : \mathcal{C} \in \mathcal{M} \mapsto \mathrm{card}(\mathrm{Ob}(\mathcal{C})) \in (\mathbb{N} \setminus \{0\}, \times, 1)$
  - $\#Mo : \mathcal{C} \in \mathcal{M} \mapsto \mathrm{card}(\mathrm{Mo}(\mathcal{C})) \in (\mathbb{N} \setminus \{0\}, \times, 1)$
  - $\#Mo(\mathcal{C}) \geqslant 2 \times \#Ob(\mathcal{C}) - 1$, for all $\mathcal{C} \in \mathcal{M}$
- In particular if $\#Ob(\mathcal{C})$ or $\#Mo(\mathcal{C})$ is prime, then so is $\mathcal{C}$.
  The converse is false.

Criteria for primality

- The monoid $\mathcal{M}$ is graded by the following morphisms
  - $\#Ob : \mathcal{C} \in \mathcal{M} \mapsto \text{card}(\text{Ob}(\mathcal{C})) \in (\mathbb{N} \setminus \{0\}, \times, 1)$
  - $\#Mo : \mathcal{C} \in \mathcal{M} \mapsto \text{card}(\text{Mo}(\mathcal{C})) \in (\mathbb{N} \setminus \{0\}, \times, 1)$
  - $\#Mo(\mathcal{C}) \geqslant 2 \times \#Ob(\mathcal{C}) - 1$, for all $\mathcal{C} \in \mathcal{M}$
- In particular if $\#Ob(\mathcal{C})$ or $\#Mo(\mathcal{C})$ is prime, then so is $\mathcal{C}$.
  The converse is false.
- Any element of $\mathcal{M}$ freely generated by a graph, is prime

Comparing decompositions

Comparing decompositions

- The mapping $\mathcal{C} \in \mathcal{M} \mapsto \overrightarrow{\pi_0}(\mathcal{C}) \in \mathcal{M}$ is a morphism of monoids

Comparing decompositions

- The mapping $\mathcal{C} \in \mathcal{M} \mapsto \overrightarrow{\pi_0}(\mathcal{C}) \in \mathcal{M}$ is a morphism of monoids
- We would like to know which prime element of $\mathcal{M}$ are preserved by it

Comparing decompositions

- The mapping $\mathcal{C} \in \mathcal{M} \mapsto \overrightarrow{\pi_0}(\mathcal{C}) \in \mathcal{M}$ is a morphism of monoids
- We would like to know which prime element of $\mathcal{M}$ are preserved by it
- We known that $\overrightarrow{\pi_0}(\mathcal{C})$ is null iff $\mathcal{C}$ is a lattices (e.g. $\overrightarrow{\pi_0}(0 < 1) = \{0\}$ though $\{0 < 1\}$ is prime in $\mathcal{M}$)

Comparing decompositions

- The mapping $\mathcal{C} \in \mathcal{M} \mapsto \overrightarrow{\pi_0}(\mathcal{C}) \in \mathcal{M}$ is a morphism of monoids
- We would like to know which prime element of $\mathcal{M}$ are preserved by it
- We known that $\overrightarrow{\pi_0}(\mathcal{C})$ is null iff $\mathcal{C}$ is a lattices (e.g. $\overrightarrow{\pi_0}(0 < 1) = \{0\}$ though $\{0 < 1\}$ is prime in $\mathcal{M}$)
- For all d-spaces X and Y, $\overrightarrow{\pi_1}(X \times Y) \cong \overrightarrow{\pi_1}X \times \overrightarrow{\pi_1}Y$

Comparing decompositions

- The mapping $\mathcal{C} \in \mathcal{M} \mapsto \overrightarrow{\pi_0}(\mathcal{C}) \in \mathcal{M}$ is a morphism of monoids
- We would like to know which prime element of $\mathcal{M}$ are preserved by it
- We known that $\overrightarrow{\pi_0}(\mathcal{C})$ is null iff $\mathcal{C}$ is a lattices (e.g. $\overrightarrow{\pi_0}(0 < 1) = \{0\}$ though $\{0 < 1\}$ is prime in $\mathcal{M}$)
- For all d-spaces X and Y, $\overrightarrow{\pi_1}(X \times Y) \cong \overrightarrow{\pi_1}X \times \overrightarrow{\pi_1}Y$
- Hence $\mathcal{N}' := \{X \in \mathcal{H}_f\lceil G\rfloor \mid \overrightarrow{\pi_1}X \text{ is nonempty, connected, and loop-free}\}$
  is a pure submonoid of $\mathcal{H}_f\lceil G\rfloor$

Comparing decompositions

- The mapping $\mathcal{C} \in \mathcal{M} \mapsto \overrightarrow{\pi_0}(\mathcal{C}) \in \mathcal{M}$ is a morphism of monoids
- We would like to know which prime element of $\mathcal{M}$ are preserved by it
- We known that $\overrightarrow{\pi_0}(\mathcal{C})$ is null iff $\mathcal{C}$ is a lattices (e.g. $\overrightarrow{\pi_0}(0 < 1) = \{0\}$ though $\{0 < 1\}$ is prime in $\mathcal{M}$)
- For all d-spaces X and Y, $\overrightarrow{\pi_1}(X \times Y) \cong \overrightarrow{\pi_1} X \times \overrightarrow{\pi_1} Y$
- Hence $\mathcal{N}' := \{X \in \mathcal{H}_f \upharpoonleft G \downharpoonright \mid \overrightarrow{\pi_1} X$ is nonempty, connected, and loop-free$\}$
  is a pure submonoid of $\mathcal{H}_f \upharpoonleft G \downharpoonright$
- Then $\mathcal{N} := \{X \in \mathcal{N}' \mid \overrightarrow{\pi_0}(\overrightarrow{\pi_1} X)$ is finite$\}$ is a pure submonoid of $\mathcal{N}'$

Comparing decompositions

- The mapping $\mathcal{C} \in \mathcal{M} \mapsto \overrightarrow{\pi_0}(\mathcal{C}) \in \mathcal{M}$ is a morphism of monoids
- We would like to know which prime element of $\mathcal{M}$ are preserved by it
- We known that $\overrightarrow{\pi_0}(\mathcal{C})$ is null iff $\mathcal{C}$ is a lattices (e.g. $\overrightarrow{\pi_0}(0 < 1) = \{0\}$ though $\{0 < 1\}$ is prime in $\mathcal{M}$)
- For all d-spaces X and Y, $\overrightarrow{\pi_1}(X \times Y) \cong \overrightarrow{\pi_1} X \times \overrightarrow{\pi_1} Y$
- Hence $\mathcal{N}' := \{X \in \mathcal{H}_f \upharpoonleft G \downharpoonleft \mid \overrightarrow{\pi_1} X$ is nonempty, connected, and loop-free$\}$ is a pure submonoid of $\mathcal{H}_f \upharpoonleft G \downharpoonleft$
- Then $\mathcal{N} := \{X \in \mathcal{N}' \mid \overrightarrow{\pi_0}(\overrightarrow{\pi_1} X)$ is finite$\}$ is a pure submonoid of $\mathcal{N}'$
- Therefore it is free commutative and we would like to know which prime elements are preserved by $X \in \mathcal{N} \mapsto \overrightarrow{\pi_0}(\overrightarrow{\pi_1} X) \in \mathcal{M}$

Comparing decompositions

- The mapping $\mathcal{C} \in \mathcal{M} \mapsto \overrightarrow{\pi_0}(\mathcal{C}) \in \mathcal{M}$ is a morphism of monoids
- We would like to know which prime element of $\mathcal{M}$ are preserved by it
- We known that $\overrightarrow{\pi_0}(\mathcal{C})$ is null iff $\mathcal{C}$ is a lattices (e.g. $\overrightarrow{\pi_0}(0 < 1) = \{0\}$ though $\{0 < 1\}$ is prime in $\mathcal{M}$)
- For all d-spaces X and Y, $\overrightarrow{\pi_1}(X \times Y) \cong \overrightarrow{\pi_1}X \times \overrightarrow{\pi_1}Y$
- Hence $\mathcal{N}' := \{X \in \mathcal{H}_f\lceil G\rfloor \mid \overrightarrow{\pi_1}X$ is nonempty, connected, and loop-free$\}$ is a pure submonoid of $\mathcal{H}_f\lceil G\rfloor$
- Then $\mathcal{N} := \{X \in \mathcal{N}' \mid \overrightarrow{\pi_0}(\overrightarrow{\pi_1}X)$ is finite$\}$ is a pure submonoid of $\mathcal{N}'$
- Therefore it is free commutative and we would like to know which prime elements are preserved by $X \in \mathcal{N} \mapsto \overrightarrow{\pi_0}(\overrightarrow{\pi_1}X) \in \mathcal{M}$
- Conjecture

> If $P \in \mathcal{N}$ is prime and $\overrightarrow{\pi_1}(P)$ is not a lattice, then $\overrightarrow{\pi_0}(\overrightarrow{\pi_1}(P))$ is prime