DIRECTED ALGEBRAIC TOPOLOGY

AND

CONCURRENCY

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– Lecture 4 –

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ISOTHETIC REGIONS

Boolean structure

One-dimensional regions

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Let G be a finite graph, the collection $\mathcal{R}_1G$ of all finite unions of connected subsets of $|G|$ forms a Boolean subalgebra of $\mathrm{Pow}(|G|)$.

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Yet some infinite graphs may not enjoy the property e.g. when G is a graph with a single vertex and infinitely many arrows.

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- The collection of n-dimensional block coverings is denoted by $\mathrm{Cov}_n G$, it is preordered by

$$C \preccurlyeq C' \quad \equiv \quad \forall b \in C \ \exists b' \in C', \ b \subseteq b'$$

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- $\alpha_n(X) = \emptyset$ if and only if $X = \emptyset$.

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We have a Galois connection (γ_n, α_n) between $\mathrm{Cov}_n G$ and $\mathrm{Pow}(|G|^n)$ with $\gamma_n(D) = \bigcup D$ for all $D \in \mathrm{Cov}_n G$.

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Proof: any *connected* block is contained in a maximal *connected* block (by the Hausdorff maximal principle).

$$\bigcup_i{}^{\uparrow} \left(B_1^{(i)} \times \cdots \times B_n^{(i)}\right) \quad = \quad \left(\bigcup_i{}^{\uparrow} B_1^{(i)}\right) \times \cdots \times \left(\bigcup_i{}^{\uparrow} B_n^{(i)}\right)$$

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A subset $X \subseteq |G|^n$ is an isothetic region iff the collection of maximal subblocks of X is finite and covers X.

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Its maximal blocks are found among that of B^c therefore they have the form

$$D_1 \times \cdots \times D_{k-1} \times C_k \times D_{k+1} \times \cdots \times D_n$$

with $k \in \{1, \ldots, n\}$, C_k ranging through the connected components of B_k^c and D_j, for $j \neq k$, ranging through the connected components of $|G|$.

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Moreover if $\mathcal{B}$ and $\mathcal{B}'$ are block coverings of X and X' containing all their maximal blocks, then the union of the collections of maximal blocks of $B \cap B'$ for $B \in \mathcal{B}$ and $B' \in \mathcal{B}'$ is a block covering of $X \cap X'$ containing all its maximal blocks.

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where M_B is a maximal block of B^c.

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where M_B is a maximal block of B^c.
- The maximal blocks of X^c thus form a finite block covering of X^c.

A result from directed topology

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For all directed paths γ on $\upharpoonleft G \downharpoonleft^n$ and all $X \in \mathcal{R}_n G$, the inverse image of X by γ has finitely many connected components.

Additional operators

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The interior of a set is the difference between its closure and its boundary. It follows that the interior operator also preserves isothetic regions.

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- The future closure of A in X is $\overline{A}^{\mathrm{f}} := \mathrm{frw}(A, \overline{A})$ and
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Theorem: if A, B, and X are isothetic regions, then so are $\mathrm{frw}(A, B)$, $\mathrm{cone}^{\mathrm{f}} A$, $\overline{A}^{\mathrm{f}}$, and their duals.

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- A point $p \in \upharpoonleft G_i \downharpoonright$ is said to be terminal when $[\![\gamma]\!]$ is empty for all directed paths on $\upharpoonleft G_i \downharpoonright$ starting at p.

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Deadlock attractor of the Swiss Cross

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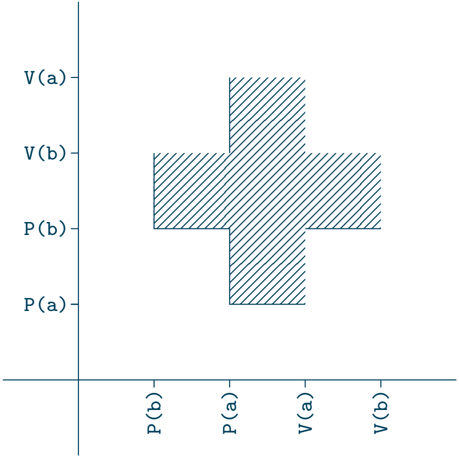
proc:
p = P(a).P(b).V(b).V(a)
q = P(b).P(a).V(a).V(b)
init:  p q
```



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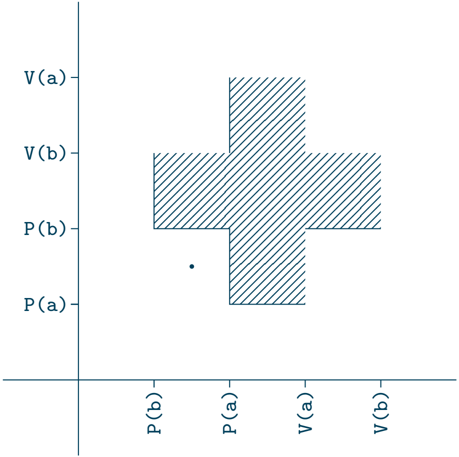

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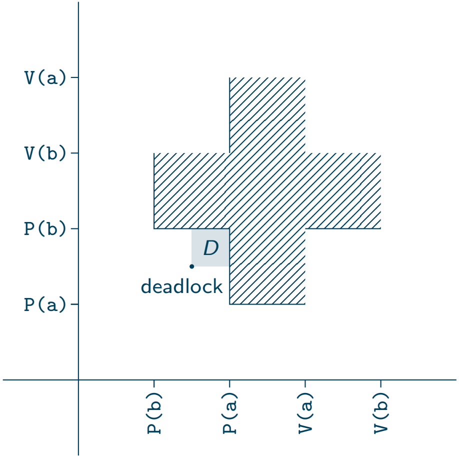

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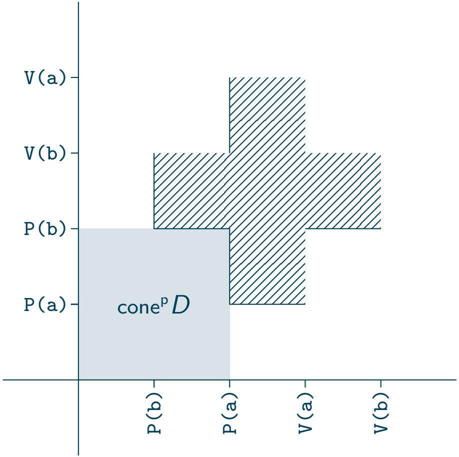

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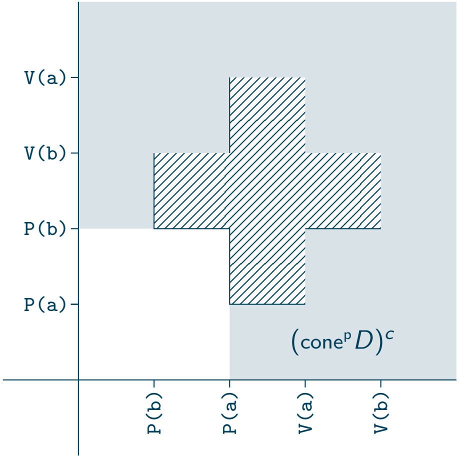

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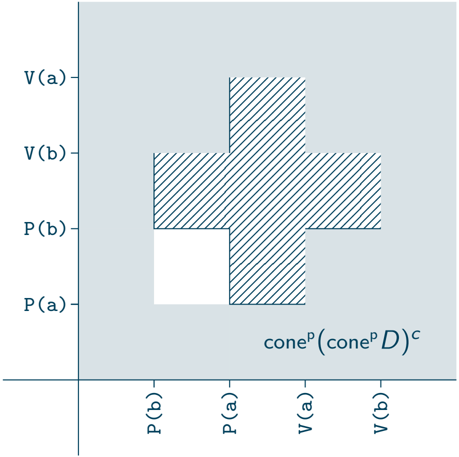

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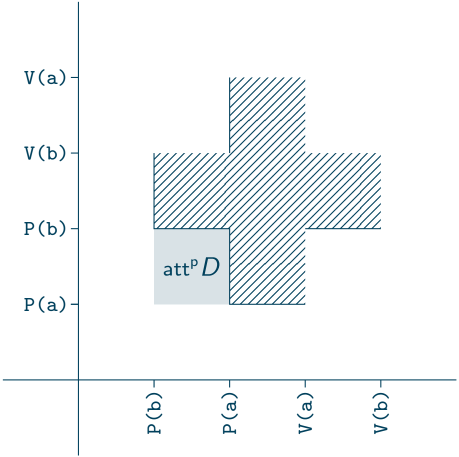
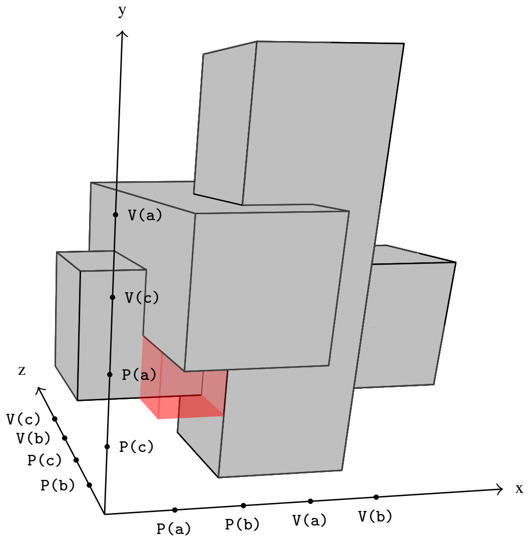


Three dining philosophers



FACTORING ISOTHETIC REGIONS

Free commutative monoids

Commutative monoids

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    - $(ab)c = a(bc)$
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  - In particular, if $f : X \to Y$ is a set map, then

$$M(f)(\phi) = \sum_{x \in X} \phi(x)f(x)$$

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- If M contains nontrivial units, then one can consider the quotient monoid $M/\sim$ where $x \sim y$ stands for: there exists a unit u s.t. $y = ux$

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  - Note that two free commutative monoids are isomorphic in $\mathcal{Cmon}$ iff their set of prime elements have the same cardinality e.g. $(\mathbb{N} \setminus \{0\}, \times, 1) \cong (\mathbb{Z}[X] \setminus \{0\}, \times, 1)$ in $\mathcal{Cmon}$

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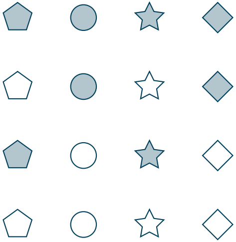
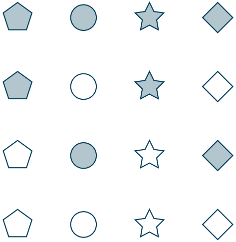
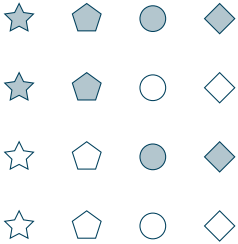
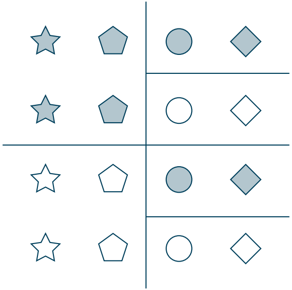
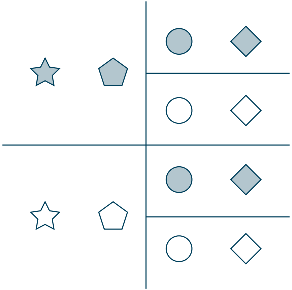
Monoids of homogeneous languages

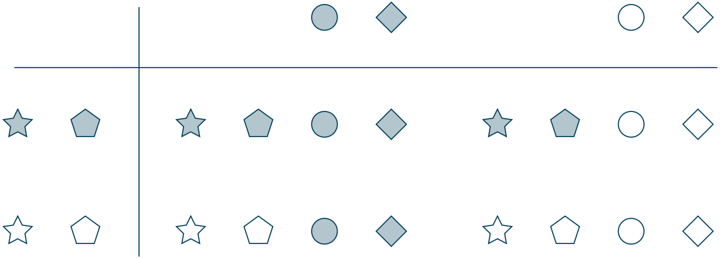












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$$\sigma \otimes \sigma'(k) := \left\{ \begin{array}{ccc} \sigma(k) & \text{if} & 1 \leqslant k \leqslant n \\ (\sigma'(k-n)) + n & \text{if} & n+1 \leqslant k \leqslant n+n' \end{array} \right.$$

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- A Godement exchange law is satisfied, which ensures that $\sim$ is actually a congruence:

$$(\sigma \cdot H) \cdot (\sigma' \cdot H') = (\sigma \otimes \sigma') \cdot (H \cdot H')$$

i.e. $H \sim K$ and $H' \sim K'$ implies $HH' \sim KK'$

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- For any homogeneous language H and $\sigma \in \mathfrak{S}_{\mathsf{dim}(H)}$, $\mathsf{card}(H) = \mathsf{card}(\sigma \cdot H)$ so we can define the cardinality of any element of $\mathcal{H}(\mathbb{A})$

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- The primality of H does not imply that of Card(H)
  e.g. $H = \{a, b, c, d\}$ is prime though card(H) $= 4$

The brute force algorithm for factoring in $\mathcal{H}_f(\mathbb{A})$

Theory

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Given $w \in \mathbb{A}^n$ and $I \subseteq \{1, \ldots, n\}$, we write $w_{|_I}$ for the subword of w consisting of letters with indices in I.

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Denoting I^c for $\{1, \ldots, n\} \setminus I$, we have

$$[H] \quad = \quad [H_{|_I}] \cdot [H_{|_{I^c}}]$$

in $\mathcal{H}_f(\mathbb{A})$ if and only if for all words $u, v \in H$ there exists a word $w \in H$ such that

$$w_{|_I} \quad = \quad u_{|_I} \qquad \text{and} \qquad w_{|_{I^c}} \quad = \quad v_{|_{I^c}}$$

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Practice

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For $I \subseteq \{1, \ldots, n\}$ let $\pi_{|_I}$ be the “projection” that sends $w \in H$ to $w_{|_I} \in \mathbb{A}^{\mathrm{card(I)}}$.

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2. if $\pi_{|_{I^c}}(\pi^{-1}_{|_I}(u))$ does not depend on $u \in H_{|_I}$, then we have the factorization

$$[H] \quad = \quad [H_{|_I}] \cdot [H_{|_{I^c}}]$$

   and we are done

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2. if $\pi_{|_{I^c}}(\pi_{|_I}^{-1}(u))$ does not depend on $u \in H_{|_I}$, then we have the factorization

   $$[H] \quad = \quad [H_{|_I}] \cdot [H_{|_{I^c}}]$$

   and we are done
3. otherwise check whether there are still subsets of $\{1, \ldots, n\}$ to check:
   1. yes: go to step 1
   2. no: $[H]$ is prime

Homogeneous languages and isothetic regions

Factoring a program

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sem:  1 a b
sem:  2 c
```

```
proc:
  p = P(a);P(c);V(c);V(a)

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```

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init:  p q p q
```

Factoring the space of states

brute force

$[0,1[$	$[0,1[$	$[0,+\infty[$	$[0,+\infty[$
$[0,1[$	$[4,+\infty[$	$[0,+\infty[$	$[0,+\infty[$
$[0,1[$	$[0,+\infty[$	$[0,+\infty[$	$[0,1[$
$[0,1[$	$[0,+\infty[$	$[0,+\infty[$	$[4,+\infty[$
$[4,+\infty[$	$[0,1[$	$[0,+\infty[$	$[0,+\infty[$
$[4,+\infty[$	$[4,+\infty[$	$[0,+\infty[$	$[0,+\infty[$
$[4,+\infty[$	$[0,+\infty[$	$[0,+\infty[$	$[0,1[$
$[4,+\infty[$	$[0,+\infty[$	$[0,+\infty[$	$[4,+\infty[$
$[0,+\infty[$	$[0,1[$	$[0,1[$	$[0,+\infty[$
$[0,+\infty[$	$[0,1[$	$[4,+\infty[$	$[0,+\infty[$
$[0,+\infty[$	$[4,+\infty[$	$[0,1[$	$[0,+\infty[$
$[0,+\infty[$	$[4,+\infty[$	$[4,+\infty[$	$[0,+\infty[$
$[0,+\infty[$	$[0,+\infty[$	$[0,1[$	$[0,1[$
$[0,+\infty[$	$[0,+\infty[$	$[0,1[$	$[4,+\infty[$
$[0,+\infty[$	$[0,+\infty[$	$[4,+\infty[$	$[0,1[$
$[0,+\infty[$	$[0,+\infty[$	$[4,+\infty[$	$[4,+\infty[$

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[0,1[	[0,1[	●	●
[0,1[	[4,+∞[	●	●
[0,1[	●	●	[0,1[
[0,1[	●	●	[4,+∞[
[4,+∞[	[0,1[	●	●
[4,+∞[	[4,+∞[	●	●
[4,+∞[	●	●	[0,1[
[4,+∞[	●	●	[4,+∞[
●	[0,1[	[0,1[	●
●	[0,1[	[4,+∞[	●
●	[4,+∞[	[0,1[	●
●	[4,+∞[	[4,+∞[	●
●	●	[0,1[	[0,1[
●	●	[0,1[	[4,+∞[
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●	●	[4,+∞[	[4,+∞[

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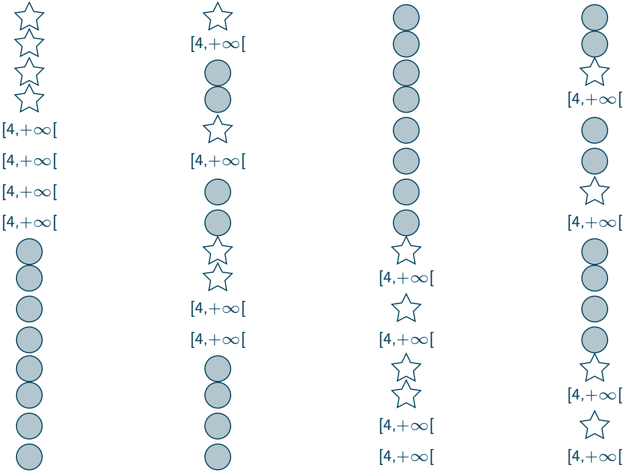

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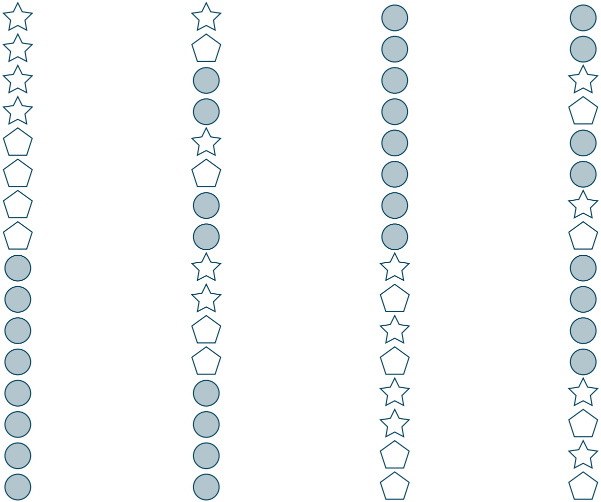

Factoring the space of states

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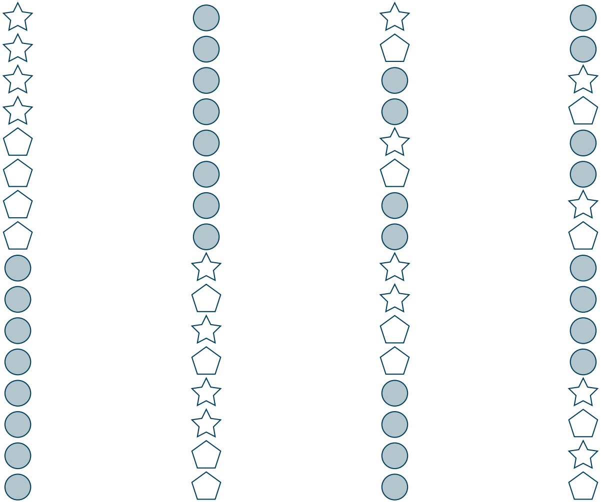

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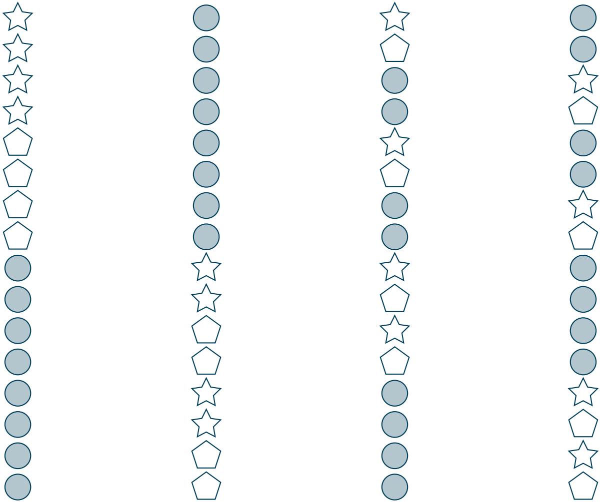

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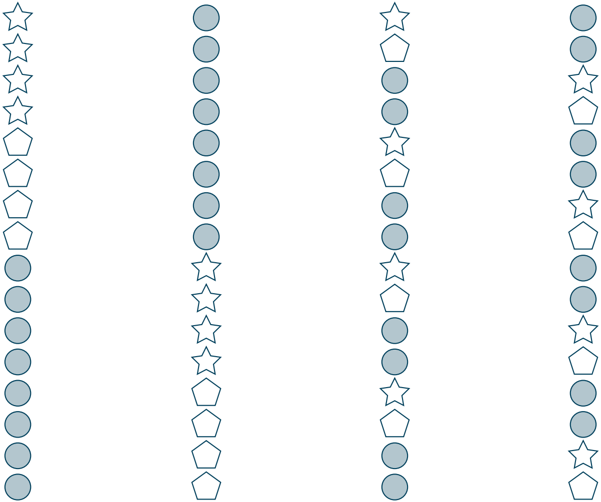

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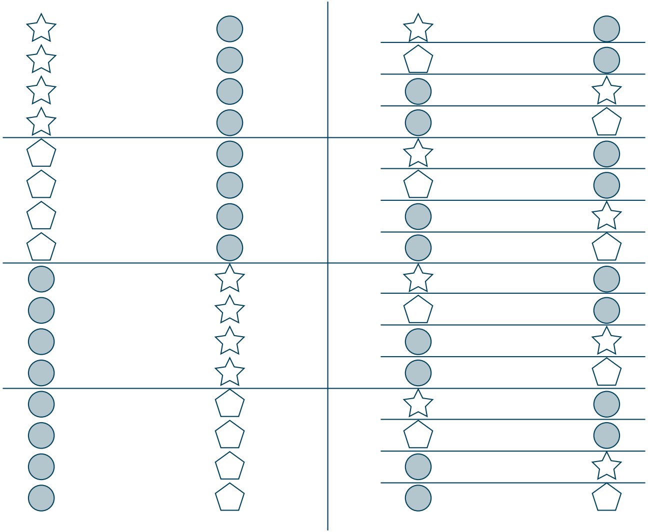

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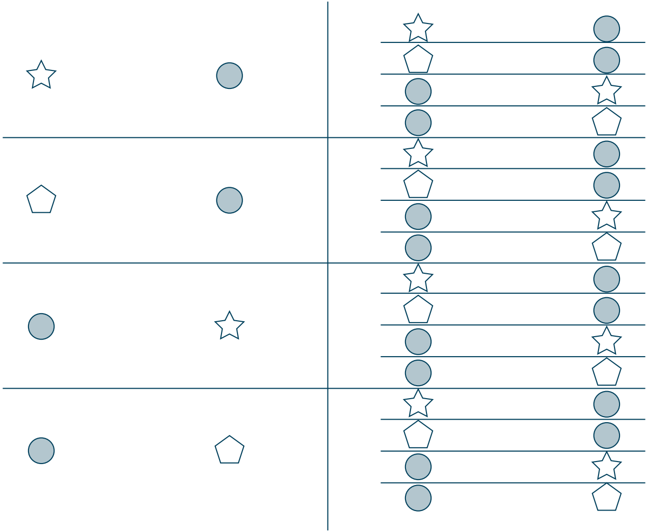

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	☆ ●	⬠ ●	● ☆	● ⬠
☆ ●	☆ ● ☆ ●	☆ ● ⬠ ●	☆ ● ● ☆	☆ ● ● ⬠
⬠ ●	⬠ ● ☆ ●	⬠ ● ⬠ ●	⬠ ● ● ☆	⬠ ● ● ⬠
● ☆	● ☆ ☆ ●	● ☆ ⬠ ●	● ☆ ● ☆	● ☆ ● ⬠
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init:  2q
```

Factoring a program

`sem: 1 a`	`sem: 1 b`
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`init: 2p`	`init: 2q`

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- If $\preccurlyeq$ is the equality relation, then $X \preccurlyeq Y$ amounts to $H_X \subseteq H_Y$ for some representatives H_X and H_Y of X and Y.

Homogeneous languages

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  - the concatenation of words on $\mathbb{A}$ corresponds to the cartesian product of blocks

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- The induced morphism γ does not preserve the prime elements e.g. consider a covering of $[0,1]^2$ with 3 disctinct rectangles

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$$\boxed{\text{If } \mathcal{F} = \alpha(X^c) \text{ then we obtain the prime factorization of } X}$$

Factoring a program

```
sem:  1 a b
sem:  2 c
```

```
proc:
  p = P(a);P(c);V(c);V(a)

  q = P(b);P(c);V(c);V(b)
```

```
init:  p q p q
```

Factoring the space of states

subtle

[2,3[	[2,3[	[2,3[	$[0,+\infty[$
[2,3[	[2,3[	$[0,+\infty[$	[2,3[
[1,4[	$[0,+\infty[$	[1,4[	$[0,+\infty[$
[2,3[	$[0,+\infty[$	[2,3[	[2,3[
$[0,+\infty[$	[1,4[	$[0,+\infty[$	[1,4[
$[0,+\infty[$	[2,3[	[2,3[	[2,3[

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[2,3[	[2,3[	[2,3[	[0,+∞[
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[2,3[	[0,+∞[	[2,3[	[2,3[
[0,+∞[	[1,4[	[0,+∞[	[1,4[
[0,+∞[	[2,3[	[2,3[	[2,3[

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[1,4[	[0,$+\infty$[	[1,4[	[0,$+\infty$[
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