DIRECTED ALGEBRAIC TOPOLOGY

AND

CONCURRENCY

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MPRI : Concurrency (2.3.1)
– Lecture 2 –

2020 – 2021

AN ALGEBRAIC TOPOLOGY TEASER

Categories

Category $\mathcal{C}$

Definition (the "underlying graph" part)

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- We define the homset $\mathcal{C}(x,y) := \left\{\gamma \in \mathrm{Mo}(\mathcal{C}) \;\middle|\; \partial^{-}\gamma = x \text{ and } \partial^{+}\gamma = y\right\}$

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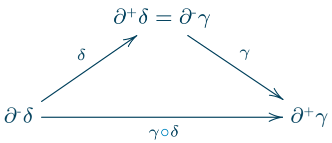
$$\mathrm{Mo}(\mathcal{C}) \underset{t}{\overset{s}{\rightrightarrows}} \mathrm{Ob}(\mathcal{C}), \qquad \mathrm{id} : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Mo}(\mathcal{C})$$

- The binary composition is a partially defined and often denoted by $\circ$

$$\Big\{(\gamma,\delta) \mid \gamma,\delta \text{ morphisms of } \mathcal{C} \text{ s.t. } \partial^{-}\gamma = \partial^{+}\delta\Big\} \xrightarrow{\text{composition}} \mathrm{Mo}(\mathcal{C})$$



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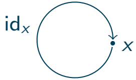

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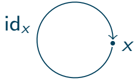

- For all morphisms γ one has $\mathrm{id}_{\partial^+\gamma} \circ \gamma = \gamma = \gamma \circ \mathrm{id}_{\partial^-\gamma}$

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- The opposite of a category is obtained by reversing all its arrows (i.e. by swapping the roles of the source and the target)

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- if $r \circ s = \text{id}$ then r is called a retract/split epimorphism and s is called a section/split monomorphism.

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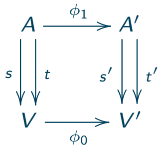

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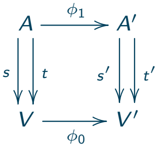

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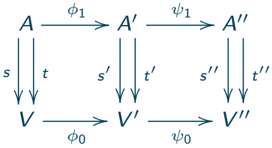

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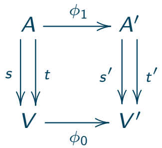

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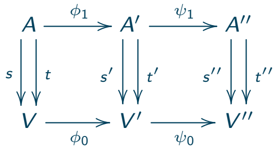

with $s'(\phi_1(\alpha)) = \phi_0(\partial^- \alpha)$ and $t'(\phi_1(\alpha)) = \phi_0(\partial^+ \alpha)$

Topology

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Topological spaces and continuous maps form the category $\mathcal{Top}$

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Every subset of a Hausdorff space is saturated.

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A Hausdorff space is locally compact iff each of its points admits a compact neighbourhood.

Functors

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$$\begin{array}{ccc} \mathrm{Mo}(\mathcal{C}) & \overset{\mathrm{s}}{\underset{\mathrm{t}}{\rightrightarrows}} & \mathrm{Ob}(\mathcal{C}) \\ {\scriptstyle \mathrm{Mo}(f)}\downarrow & & \downarrow{\scriptstyle \mathrm{Ob}(f)} \\ \mathrm{Mo}(\mathcal{D}) & \overset{\mathrm{s'}}{\underset{\mathrm{t'}}{\rightrightarrows}} & \mathrm{Ob}(\mathcal{D}) \end{array}$$

with $\mathrm{s}'(\mathrm{Mo}(f)(\alpha)) = \mathrm{Ob}(f)(\partial^-\alpha)$ and $\mathrm{t}'(\mathrm{Mo}(f)(\alpha)) = \mathrm{Ob}(f)(\partial^+\alpha)$

Hence it is in particular a morphism of graphs.

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The "mappings" Ob(f) and Mo(f) also make the following diagram commute

$$\begin{array}{ccc} \mathrm{Mo}(\mathcal{C}) & \xleftarrow{\ \mathrm{id}\ } & \mathrm{Ob}(\mathcal{C}) \\ {\scriptstyle \mathrm{Mo}(f)}\downarrow & & \downarrow{\scriptstyle \mathrm{Ob}(f)} \\ \mathrm{Mo}(\mathcal{D}) & \xleftarrow[\ \mathrm{id}'\ ]{} & \mathrm{Ob}(\mathcal{D}) \end{array}$$

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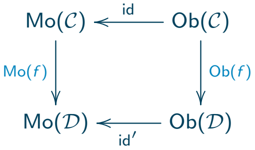
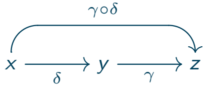
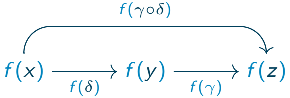


and satisfies $\mathrm{Mo}(f)(\gamma \circ \delta) = \mathrm{Mo}(f)(\gamma) \circ \mathrm{Mo}(f)(\delta)$





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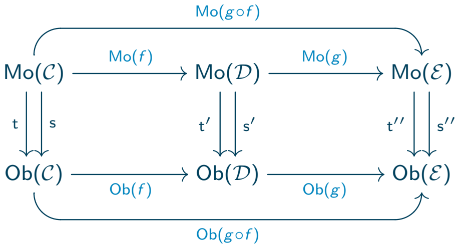

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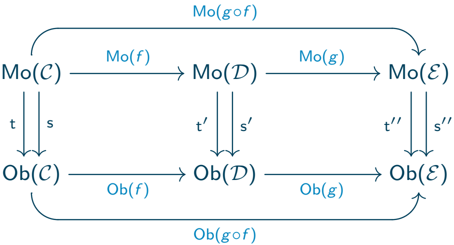

Hence the functors should be thought of as the morphisms of categories

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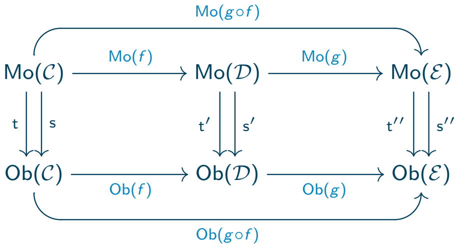

Hence the functors should be thought of as the morphisms of categories

The small categories and their funtors form a (large) category denoted by *Cat*

Connectedness

The overall idea of algebraic topology

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Problem: prove the topological spaces X and Y are *not* the same

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Problem: prove the topological spaces X and Y are *not* the same
Strategy: find a functor F defined over $\mathcal{Top}$ such that $F(X) \ncong F(Y)$

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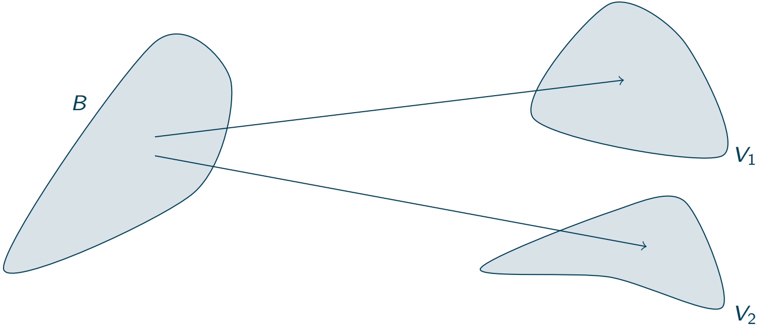
$$\mathcal{T}op \xrightarrow{\ \pi_0\ } \mathcal{S}et$$

$$\begin{array}{ccc} X & & \pi_0(X) \\ \downarrow f & \longmapsto & \pi_0(f) \downarrow \\ Y & & \pi_0(Y) \end{array}$$

An application

The continuous image of a connected space is connected

The image of the space B is entirely contained in a connected component of the space V.



The set of connected components

is a functorial construction

This situation is abstracted by classifying continuous maps from B to V according to which connected component (V_1 or V_2) the single connected components of B (namely B itself) is sent to. There are exactly two set theoretic maps from the singleton $\{B\}$ to the pair $\{V_1, V_2\}$ hence there is at most (in fact exactly) two kinds of continuous maps from B to V.

$$\{B\} \rightrightarrows \{V_1, V_2\}$$

In particular B and V are not homeomorphic.

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Let $\mathbb{S}^1 := \{z \in \mathbb{C} \mid |z| = 1\}$ be the Euclidean circle and suppose $\varphi : [0,1] \to \mathbb{S}^1$ is a homeomorphism.

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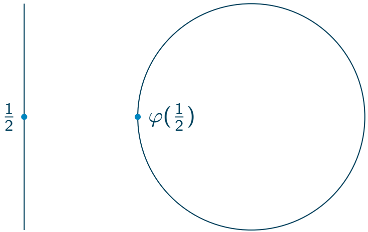
Then φ induces a homeomorphism

$$[0, \tfrac{1}{2}[\cup]\tfrac{1}{2}, 1] \ \to \ \mathbb{S}^1 \setminus \{\varphi(\tfrac{1}{2})\}$$

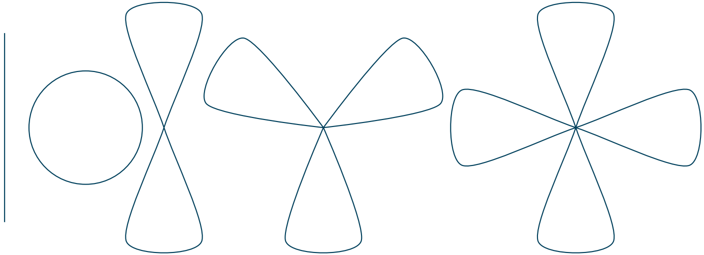
which does not exist!



Generalization

Bouquets of circles

These topological spaces are pairwise not homeomorphic. Why ?



Distributed Computing *through* Combinatorial Topology

Maurice Herlihy
Dmitry Kozlov
Sergio Rajsbaum

METRIC SPACES

Functor Terminology

Given a functor $f : \mathcal{C} \to \mathcal{D}$ and two objects x and y we have the mapping

$$\begin{aligned} f_{x,y} : \mathcal{C}[x, y] &\longrightarrow \mathcal{D}[\mathrm{Ob}(f)(x), \mathrm{Ob}(f)(y)] \\ \alpha &\longmapsto \mathrm{Mo}(f)(\alpha) \end{aligned}$$

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- f is an embedding when it is faithful and $\mathrm{Ob}(f)$ is one-to-one

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The actions of a monoid M over a set X are the functors from M to $\mathcal{S}et$ which sends the only element of M to X

Some full embeddings in $\mathcal{Cat}$

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Categories of Metric Spaces

Metric spaces

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Goal: turn any graph into metric space in a natural way.

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The length $\ell(\gamma)$ of a path $\gamma : [0, r] \to (X, d)$ is the least upper bound of the collection of sums

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The metric space (X, d) is a length space when the distance between two points $x, x' \in X$ is the following greatest lower bound

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The space is said to be geodesic when any two points are related by a geodesic path.

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Metric Spaces of Non-Positive Curvature, *M. R. Bridson, and A. Haefliger*, 1999

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Metric Graphs

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- The underlying set of the metric graph is $A \times]0,1[\, \sqcup V$
- Two points p, p' are said to be neighbours when there is an arrow a such that $p, p' \in \{a\} \times]0,1[\, \sqcup \{\partial^- a, \partial^+ a\}$

Distance between two neighbours

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- If $\partial^- a \neq \partial^+ a$ there is a canonical bijection

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An itinerary on $A\times]0,1[\ \sqcup V$ is a (finite) sequence $p_0,\ldots,p_q$ of points such that p_k and p_{k+1} are neighbours for $k \in \{0,\ldots,q-1\}$.

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The distance between two points p and p' of $A\times]0,1[\ \sqcup V$ is

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The metric graph associated with G is the metric space

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Open balls

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The open ball of radius $r < 1$ centered at the vertex v is the set

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That collection of open balls forms a basis of open sets.

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Every finite graph with weighted arrows (in $\mathbb{R}_+ \backslash \{0\}$) with can be embedded in $\mathbb{R}^3$.

LOCALLY ORDERED METRIC GRAPHS

Partially Ordered Spaces

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Topology and Order, L. Nachbin, 1965

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A partially ordered space (or pospace) is a topological space X together with a partial order $\sqsubseteq$ on (the underlying set of) X such that

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is a closed subset of $X \times X$.

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The underlying space of a pospace is Hausdorff.

Examples

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- The real line with standard topology and order.

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- The collection of compact subsets of a metric space equipped with the Hausdorff distance is a metric space.

$$d_H(K, K') = \sup\left\{d(x, K'), d(x', K) \mid x \in K; x' \in K'\right\}$$

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  The induced topological space ordered by inclusion is a pospace.
- Problem: there is no pospace on the circle whose collection of directed paths is

$$\{\rho(t)\cdot e^{i\theta(t)} \mid \rho,\theta : [0,r] \to \mathbb{R}_+ \text{ increasing}\}$$

Ordered Atlases

Ordered atlas

Ordered manifolds, invariant cone fields, and semigroups, J. D. Lawson, 1989

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Any subset of X inherits an ordered atlas from $\mathcal{U}$.

Ordered atlas

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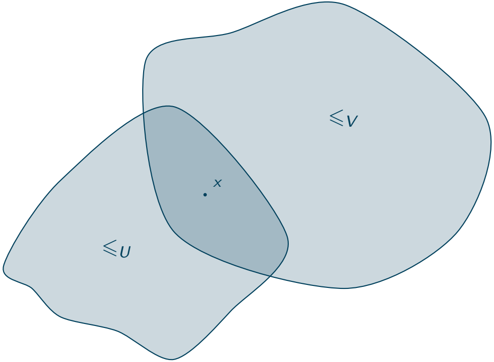

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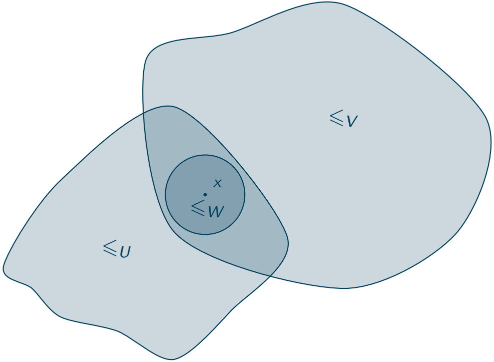

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A local pospace is a Hausdorff space togethter with an equivalence class of ordered atlas.

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Examples of equivalent atlases on $\mathbb{R}$

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- $\{(U, \sqsubseteq'_U) \mid U$ open subset of $\mathbb{R}\}$ where $x \sqsubseteq'_U y$ is any extension of $\sqsubseteq_U$.

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The locally ordered circle

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- $\{(A, \leqslant) \mid A \text{ open arc}\}$ where $\leqslant$ is the order induced by $\mathbb{R}$ and the restriction of the exponential map to an open subinterval of $\{t \in \mathbb{R} \mid e^{it} \in A\}$ of length at most 2π,

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Basic Properties

Morphisms

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An atlas morphism from $\mathcal{U}$ to $\mathcal{V}$ is a map f (between the underlying sets of $\mathcal{U}$ and $\mathcal{V}$) such that for all $x \in \text{dom}(f)$ there exists an ordered chart $U \in \mathcal{U}$ and an ordered chart $V \in \mathcal{V}$ such that $x \in U$ and f induces a pospace morphism from U to V (implicitly $f(U) \subseteq V$).

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A local pospace morphism defined over a locally ordered compact interval is called a directed path.

Pospaces as local pospaces

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A local pospace has no vortex.

Ordered Atlas on Metric Graphs

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- $\{a\}\times]0,r[$ for each arrow a such that $\partial^- a = v$
- $\{a\}\times]1-r,1[$ for each arrow a such that $\partial^+ a = v$

The partial order on B is characterized by the following constraints:

- each branch $\{a\}\times]1-r,1[$ and $\{a\}\times]0,r[$ inherits its order from $\mathbb{R}$
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We have $B \cap B' \neq \emptyset \;\Rightarrow\; B \cap B' \in \mathcal{B}$ and

$$\sqsubseteq_{B_{|_{B\cap B'}}} \quad = \quad \sqsubseteq_{B\cap B'} \quad = \quad \sqsubseteq_{B'_{|_{B\cap B'}}}$$

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The metric graph of $|G|$ thus becomes a local pospace.

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The locally ordered metric graph construction is functorial.